Math 4853 homework

42. (4/2) Let $X \times Y$ be a product of topological spaces. Prove that for each $x_0 \in X$, the subspace $\{x_0\} \times Y$ is homeomorphic to Y. Show, in fact, that the restriction π of the projection function $\pi_Y \colon X \times Y \to Y$ is a homeomorphism. Hint: Let $j \colon Y \to \{x_0\} \times Y$ be defined by $j(y) = (x_0, y)$. Observe that j is an inverse function to π, hence π is bijective. Give a simple reason why π is continuous, and apply a theorem to show that j is continuous. (Of course, the same kind of arguments would show that each $X \times \{y_0\}$ is homeomorphic to X.)

43. (4/14) (a) Let $B \subseteq A \subseteq X$. Prove that B is closed in the subspace topology on A if and only if there exists a closed subset $C \subseteq X$ such that $B = C \cap A$.
(b) Prove that if A is a closed subset of X and B is a closed subset of Y, then $A \times B$ is a closed subset of $X \times Y$. Hint: Find a simple description of $X \times Y - A \times B$.
(c) Let $f \colon X \to Y$ be a function. Prove that f is continuous if and only if for every closed subset $C \subseteq Y$, the inverse image $f^{-1}(C)$ is closed in X.

44. (4/14) Let $S \subset X$.
(a) Prove that $x \in \overline{S}$ if and only if every open set containing x contains a point of S.
(b) Prove that S is closed if and only if $S = \overline{S}$.
(c) Prove that $\overline{S} = \cap\{\, A \subseteq X \mid \text{A is closed and } S \subseteq A \,\}$.
(d) Let $f \colon X \to Y$ be continuous. Prove that $f(\overline{S}) \subseteq \overline{f(S)}$.
(e) Give an example of a continuous surjective function $f \colon X \to Y$ and a subset $S \subset X$ such that $f(\overline{S}) \neq \overline{f(S)}$.

45. (4/14) Let X be $\mathbb{R}$ with the lower-limit topology, and let A be the subspace $[0, 1]$ of X. Give an example of a continuous unbounded function from A to $\mathbb{R}$.

46. (4/14) Let $X = \{1/n \mid n \in \mathbb{N}\} \cup \{0\}$, a subspace of $\mathbb{R}$. Prove that every continuous function $f \colon X \to \mathbb{R}$ is bounded, by considering the open set $V = (f(0) - 1, f(0) + 1)$.

47. (4/14) Let X be a topological space. Prove that if X is compact, then every continuous function $f \colon X \to \mathbb{R}$ is bounded. Use the open cover $\{V_n\}_{n \in \mathbb{N}}$ of $\mathbb{R}$, where $V_n = (-n, n)$.

48. (4/14) For any set X, the *cofinite topology* on X is the topology in which a set is open if and only if it is either empty or has finite complement. Prove that any set X with the cofinite topology is compact.

49. (4/14) Prove that if A is a compact subset of $\mathbb{R}$, then A is bounded (i. e. A lies in some interval $[-M, M]$).

50. (4/14) Prove that if A is a compact subset of $\mathbb{R}$, then A is closed. (Hint: It seems easiest to argue the contrapositive: if A is not closed then it is not compact. If A is not closed, then $A \neq \overline{A} = A \cup A'$, so there is some limit point x_0 of A that is not contained in A. Then...)