Mathematics 4853-001 Name (please print) ____________________

Final Examination

May 12, 2010

Instructions: Give concise answers, but clearly indicate your reasoning in any arguments that you give. You do not need to verify commonly used facts, unless demonstrating that you have the ability to verify a fact is the point of the problem.

I. Let X be a set. Give a full, precise definition of a *topology on* X.
(4)

A topology on X is a collection $\mathcal{U}$ of subsets of X such that

1. $X \in \mathcal{U}$ and the empty subset $\emptyset \in \mathcal{U}$.
2. If $\{U_\alpha\}_{\alpha\in\mathcal{A}} \subseteq \mathcal{U}$, then the union $\cup_{\alpha\in\mathcal{A}} U_\alpha \in \mathcal{U}$.
3. If $\{U_i\}_{1\leq i\leq n} \subseteq \mathcal{U}$, then the intersection $\cap_{1\leq i\leq n} U_i \in \mathcal{U}$.

II. Let X be a topological space.
(6)

(a) Let U be a subset of X. Suppose that for every $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$. Prove that U is open.

For each $x \in U$, choose an open set U_x with $x \in U_x \subseteq U$. Then $U = \cup_{x\in U}\{x\} \subseteq \cup_{x\in U} U_x \subseteq U$, which implies that $U = \cup_{x\in U} U_x$. Since a union of open sets is open, U is open.

(b) Let Y be a topological space, and let $f\colon X \to Y$ be a function. Suppose that for every $x \in X$ and every neighborhood U of $f(x)$, there exists a neighborhood V of x in X such that $f(V) \subseteq U$. Prove that f is continuous.

Let U be open in Y. Let $x \in f^{-1}(U)$, so $f(x) \in U$. By hypothesis, there exists a neighborhood V_x of x such that $f(V_x) \subseteq U$. That is, $x \in V_x \subseteq f^{-1}(U)$. By part (a), this shows that $f^{-1}(U)$ is open. Therefore f is continuous.

III. Let S be any set with at least two elements. Using a Cantor-style argument, prove that the product
(5)

$$P = S \times S \times S \times \cdots = \{(s_1, s_2, s_3, \ldots) \mid s_i \in S\}$$

of countably many copies of S is uncountable. [Hint: Choose two different elements s and t of S. If $\phi\colon \mathbb{N} \to P$ is any function, use s and t to construct an element $p \in P$ that is not equal to any $\phi(n)$.]

Let $\phi\colon \mathbb{N} \to P$ be any function. Choose two different elements s and t in S. For each n, write $\phi(n) = (s_{n,1}, s_{n,2}, s_{n,3}, \ldots)$, and put $s_n = s$ if $s_{n,n} \neq s$, and put $s_n = t$ if $s_{n,n} = s$. Then the element $p = (s_1, s_2, s_3, \ldots) \in P$ differs from $\phi(n)$ in the n^{th} coordinate, so $p \neq \phi(n)$ for any n. Therefore ϕ is not surjective. Since there is no surjective function from $\mathbb{N}$ to P, there cannot be any bijection between them, so P is not countable.

IV. Take as known the fact that if (X, d) is a metric space, then d is continuous. Prove that if $f\colon Z \to X$
(5) and $g\colon Z \to X$ are continuous, then the function $h\colon Z \times Z \to \mathbb{R}$ defined by $h((z_1, z_2)) = d(f(z_1), g(z_2))$ is continuous. [Hint: What is the domain of d?]

Consider the function $H\colon Z \times Z \to X \times X$ defined by $H((z_1, z_2)) = (f(z_1), g(z_2))$. Its coordinate functions are $f \circ \pi_1$, where $\pi_1\colon Z \times Z \to Z$ is projection to the first factor, and $g \circ \pi_2$, where $\pi_2\colon Z \times Z \to Z$ is projection to the second factor. Since these coordinate functions are continuous, the Mapping Into Products Theorem ensures that H is continuous. Therefore $h = d \circ H$ is also continuous.

V. (6) (a) State the Basis Recognition Theorem.

Let X be a topological space and let $\mathcal{B}$ be a collection of subsets of X. Suppose that
(i) the elements of $\mathcal{B}$ are open in X, and
(ii) for every open set U in X and every $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
Then $\mathcal{B}$ is a basis that generates the topology on X.

(b) Let X be a topological space, and let $\mathcal{B}$ be a basis that generates the topology of X. Let $\mathcal{C}$ be a collection of open subsets of X. Suppose that for every *basic* open set $B \in \mathcal{B}$ and every $x \in B$, there exists an element $C \in \mathcal{C}$ such that $x \in C \subseteq B$. Prove that $\mathcal{C}$ is a basis that generates the topology of X.

We will apply the Basis Recognition Theorem.

(i) The sets in $\mathcal{C}$ are open.

(ii) Let U be open in X, and let $x \in U$. Since $\mathcal{B}$ is a basis for the topology on X, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. By hypothesis, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq B \subseteq U$.

By the Basis Recognition Theorem, $\mathcal{C}$ is a basis for the topology on X.

VI. (5) Let X and Y be two topological spaces, each with the cofinite topology. Let $\phi\colon X \to Y$ be a bijection. Prove that ϕ is a homeomorphism.

By hypothesis, ϕ is a bijection. To prove that ϕ is continuous, start with U be open in Y. The inverse image of the empty set is empty, hence is open, so we may assume that U is not empty. Then $U = Y - F$ for some finite set. We have $\phi^{-1}(U) = \phi^{-1}(Y - F) = X - \phi^{-1}(F)$. Since ϕ is a bijection, $\phi^{-1}(F)$ is also finite, so $\phi^{-1}(U)$ is open. Therefore ϕ is continuous. Since $\phi^{-1}\colon Y \to X$ is also a bijection, the same argument shows that ϕ^{-1} is continuous. Therefore ϕ is a homeomorphism.

VII. (6) Let X be the union of the usual line $\mathbb{R}$ with one additional point p which is not in $\mathbb{R}$. Define a topology on X by the rule that U is open if and only if $p \in U$ and $U \cap \mathbb{R}$ is open (possibly empty) in the standard topology on $\mathbb{R}$. (You do not need to check that this defines a topology.)

(a) Prove that no sequence of points in $\mathbb{R}$ converges to p.

Let $\{r_n\}$ be a sequence of points in $\mathbb{R}$. The set $\{p\}$ is open, since it intersects $\mathbb{R}$ in the empty set, and $\{p\}$ contains no r_n. Therefore $\{r_n\}$ does not converge to p.

(b) Prove that the constant sequence $\{p_n\}$ for which every $p_n = p$ converges to each point of $\mathbb{R}$.

Let $x_0 \in \mathbb{R}$. Let U be any neighborhood of x_0. By definition, $p \in U$, so for all $n \geq 1$, $p_n \in U$. Therefore $\{p_n\} \to x_0$.

VIII. (5) Let $f\colon X \to Y$ be a continuous, surjective function. Prove that if X is compact, then Y is compact.

Let $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ be an open cover of Y. Then $\{f^{-1}(U_\alpha)\}_{\alpha\in\mathcal{A}}$ is an open cover of X, which has a finite subcover $\{f^{-1}(U_{\alpha_i})\}_{1\leq i\leq k}$. Since $Y = f(X) = f(\cup_{1\leq i\leq k} f^{-1}(U_{\alpha_i})) = \cup_{1\leq i\leq k} f(f^{-1}(U_{\alpha_i})) = \cup_{1\leq i\leq k} U_{\alpha_i}$, $\{U_{\alpha_i}\}_{1\leq i\leq k}$ is a finite subcover of $\{U_\alpha\}_{\alpha\in\mathcal{A}}$.

IX. (5) Let X be a compact topological space. Let $f\colon X \to \mathbb{R}$ be a continuous function. Suppose that $f(x) > 0$ for every $x \in X$. Prove that there exists a *positive* number δ such that for every $x \in X$, $\delta \le f(x)$. [Hint: One approach uses the collection $\{(1/n, n)\}$ of open sets in $\mathbb{R}$. Another approach uses the fact every continuous real-valued function on a compact space is bounded.]

Consider the collection $\{(1/n, n)\}$ of open sets in $\mathbb{R}$. Since $f(X) \subseteq (0, \infty)$, $\{f^{-1}((1/n, n))\}$ is an open cover of X. Therefore there is a finite subcover $\{f^{-1}((1/n_i, n_i))\}_{1\le i\le k}$. Taking N to be the maximum of the n_i, we have $f(X) \subseteq (1/N, N)$, so $\delta = 1/N$ is a possible choice for δ.

Alternatively, suppose that no such δ exists. Define $g\colon X \to \mathbb{R}$ by $g(x) = 1/f(x)$. It is well-defined and continous. For each $n \in \mathbb{N}$, there exists $x_n \in X$ such that $f(x_n) < 1/n$, for if not then $1/n$ would be an acceptable choice of δ, so $g(x_n) > n$. Therefore g is continuous and unbounded, contradicting the fact that X is compact.

X. (6) Let (X, d) be a metric space, and let $x_0 \in X$. For $\epsilon > 0$, define $C(x, \epsilon) = \{z \in X \mid d(x_0, z) \le \epsilon\}$. It is called the *closed ball* of radius ϵ centered at x_0.

(a) Prove that $C(x_0, \epsilon)$ is a closed subset of X.

One proof: Suppose that $z \notin C(x_0, \epsilon)$. Then $d(x_0, z) - \epsilon > 0$. We claim that $B(z, d(x_0, z) - \epsilon)$ contains no point of $C(x_0, \epsilon)$, and hence the complement of $C(x_0, \epsilon)$ is open. Suppose for contradiction that $y \in B(z, d(x_0, z) - \epsilon) \cap C(x_0, \epsilon)$. Then $d(x_0, z) \le d(x_0, y) + d(y, z) < \epsilon + (d(x_0, z) - \epsilon) = d(x_0, z)$, a contradiction.

A nicer proof: Let $f\colon X \to \mathbb{R}$ be defined by $f(x) = d(x_0, x)$. It is continuous, since $f = d \circ i$, where $i\colon X \to X \times X$ is $i(x) = (x_0, x)$, and we know that i and d are continuous. Since $C(x, \epsilon) = f^{-1}([0, \epsilon])$, $C(x, \epsilon)$ is closed.

(b) Give an example showing that sometimes the closure $\overline{B(x_0, \epsilon)}$ of $B(x_0, \epsilon)$ in X is not equal to $C(x_0, \epsilon)$.

Among many possible examples, let $X = \{0, 1\}$ with the usual distance in $\mathbb{R}$. In X, $B(0, 1) = \{0\}$, which is closed, so $\overline{B(0, 1)} = \{0\}$. But $C(0, 1) = \{0, 1\}$.

XI. (6) Let $X = C([0, 1])$, the set of continuous functions from $[0, 1]$ to $\mathbb{R}$. For $f, g \in X$, define $d(f, g) = \int_0^1 |f(x) - g(x)|\,dx$ and $\rho(f, g) = \max_{x\in[0,1]}\{|f(x) - g(x)|\}$. These are metrics (do not verify this). Let f_n be the function $f_n(x) = x^n$, and let 0 be the zero function. Using the definition of convergence, prove that:

(a) In the d-topology, $\{f_n\}$ converges to 0.

We compute that $d(f_n, 0) = \int_0^1 x^n\,dx = 1/(n+1)$. Now, given a neigbhorhood U of 0, there exists $\epsilon > 0$ such that $B_d(0, \epsilon) \subseteq U$. For every $n > 1/\epsilon$, $f_n \in B(0, 1/n) \subseteq B(0, \epsilon) \subseteq U$.

(b) In the ρ-topology, $\{f_n\}$ does not converge to 0.

For each $x \in [0, 1]$, $|f_n(x) - 0| \le 1$, so $\rho(f_n, 0) \le 1$. On the other hand, $|f_n(1) - 0| = 1$, so $\rho(f_n, 0) \ge 1$. Therefore $\rho(f_n, 0) = 1$. So $B_\rho(0, 1/2)$ contains no f_n, and therefore $\{f_n\}$ does not converge to 0 in the ρ-topology.

XII. (6) (a) Define for a metric space (X, d), define what it means to say that a sequence $\{x_n\}$ in X is *Cauchy,* and define what it means to say that (X, d) is *complete.*

$\{x_n\}$ is Cauchy when for every $\epsilon > 0$, there exists N so that if $m, n > N$, then $d(x_m, x_n) < \epsilon$.

(b) Prove that if $\{x_n\}$ converges, say $\{x_n\} \to x_0$, then $\{x_n\}$ is Cauchy.

Given $\epsilon > 0$, there exists N so that if $n > N$, then $x_n \in B_d(x_0, \epsilon/2)$. For $m, n > N$, $d(x_m, x_n) \leq d(x_m, x_0) + d(x_0, x_n) < \epsilon/2 + \epsilon/2 = \epsilon$.

XIII. Let $\{f_n\}$ be a sequence of functions from $[0, 1]$ to $\mathbb{R}$.
(10)

(a) Define what it means to say that $\{f_n\}$ converges to $f\colon [0, 1] \to \mathbb{R}$ *uniformly.*

It means that for every $\epsilon > 0$, there exists N such that if $n \geq N$, then for every $x \in [0, 1]$, $|f_n(x) - f(x)| < \delta$.

(b) Prove that if $\{f_n\} \to f$ uniformly, then $\{f_n\} \to f$ pointwise.

Fix $x_0 \in [0, 1]$, and let $\epsilon > 0$. Since $\{f_n\} \to f$ uniformly, there exists N so that if $n \geq N$, then for every $x \in [0, 1]$, $\| f_n(x) - f(x) \| < \epsilon$. In particular, if $n \geq N$, then $\| f_n(x_0) - f(x_0) \| < \epsilon$. Therefore $\{f(x_0)\} \to f(x_0)$.

(c) State without proof an example of continuous functions $f_n\colon [0, 1] \to \mathbb{R}$ and another function f such that $\{f_n\}$ converges to f pointwise, but not uniformly.

$f_n(x) = x^n$ and $f(x) = 0$ for $0 \leq x < 1$ and $f(1) = 1$.

(d) State the Uniform Convergence Theorem.

If $\{f_n\}$ converges uniformly to f, and each f_n is continuous, then f is continuous.