Mathematics 4853-001 Name (please print) ______________________

Examination I

February 19, 2010

Instructions: Give concise answers, but clearly indicate your reasoning.

I. Use the ϵ-δ definition to prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x$ is continuous.
(6)

II. Use the ϵ-δ definition to prove that if the function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, then so is $3f$.
(6)

III. Give an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ that is continuous at every irrational number and is discontinuous
(5) at every rational number. You do *not* need to verify that it has this property, just tell how f is defined.

IV. Take as known the facts that
(6)

(i) The function $F\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $F((x,y)) = xy$ is continuous.

(ii) A function from $\mathbb{R}^m$ to $\mathbb{R}^n$ is continuous if and only if its coordinate functions are continuous.

(iii) A composition of continuous functions is continuous.

Use (i), (ii), and (iii) to give a quick proof that if f and g are continuous functions from $\mathbb{R}$ to $\mathbb{R}$, then so is their product $f \cdot g$ (defined by $(f \cdot g)(x) = f(x)\, g(x)$).

V. (a) Let $x \in \mathbb{R}^n$ and $\epsilon > 0$. Prove that if $z \in B(x, \epsilon)$, then $B(z, \epsilon - \| z - x \|) \subseteq B(x, \epsilon)$.
(10)

(b) Recall that our official definition of open sets in $\mathbb{R}^n$ was that a set U is open if and only if $\forall x \in U, \exists \epsilon > 0, B(x, \epsilon) \subseteq U$. Using this definition and part (a), prove that if U is a union of open balls in $\mathbb{R}^n$, then U is open.

VI. Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a function satisfying the ϵ-δ definition of continuity. Using the definition of open sets
(8) in $\mathbb{R}^n$, and so on, prove that if U is an open subset of $\mathbb{R}^n$, then the inverse image $f^{-1}(U)$ is open in $\mathbb{R}^m$.

VII. Let X be a set. Give a full, precise definition of a *topology on* X.
(8)

VIII. Recall that if $\mathbb{R}$ has the cofinite topology, then a subset U of $\mathbb{R}$ is open if and only if $\mathbb{R} - U$ is finite (or U
(6) is empty). Use DeMorgan's Law $\cup(X - S_\alpha) = X - (\cap S_\alpha)$ to show that if U and V are any two nonempty open sets in $\mathbb{R}$ with the cofinite topology, then $U \cap V$ is not empty.