Mathematics 4853-001 Name (please print) ______________________

Examination I

February 19, 2010

Instructions: Give concise answers, but clearly indicate your reasoning.

**I.** (6) Use the $\epsilon$-$\delta$ definition to prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x$ is continuous.

Fix $x_0 \in \mathbb{R}$ and let $\epsilon > 0$. Put $\delta = \epsilon/3$. If $|x - x_0| < \delta$, then $|3x - 3x_0| = 3|x - x_0| < 3\delta = 3\epsilon/3 = \epsilon$.

**II.** (6) Use the $\epsilon$-$\delta$ definition to prove that if the function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, then so is $3f$.

Fix $x_0 \in \mathbb{R}$ and let $\epsilon > 0$. Since $f$ is continuous, there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon/3$. If $|x - x_0| < \delta$, then $|3f(x) - 3f(x_0)| = 3|f(x) - f(x_0)| < 3\epsilon/3 = \epsilon$.

**III.** (5) Give an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ that is continuous at every irrational number and is discontinuous at every rational number. You do *not* need to verify that it has this property, just tell how $f$ is defined.

Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1/q & \text{if } x \text{ is rational and } x = \pm\dfrac{p}{q} \text{ in lowest terms with } p \geq 0 \text{ and } q > 0 \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

**IV.** (6) Take as known the facts that

(i) The function $F\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $F((x,y)) = xy$ is continuous.

(ii) A function from $\mathbb{R}^m$ to $\mathbb{R}^n$ is continuous if and only if its coordinate functions are continuous.

(iii) A composition of continuous functions is continuous.

Use (i), (ii), and (iii) to give a quick proof that if $f$ and $g$ are continuous functions from $\mathbb{R}$ to $\mathbb{R}$, then so is their product $f \cdot g$ (defined by $(f \cdot g)(x) = f(x)\, g(x)$).

Define $G\colon \mathbb{R} \to \mathbb{R}^2$ by $G(x) = (f(x), g(x))$. Since the coordinate functions $f$ and $g$ of $G$ are continuous, so is $G$. Therefore the composition $F \circ G\colon \mathbb{R} \to \mathbb{R}$ is continuous. But $(F \circ G)(x) = F(G(x)) = F((f(x), g(x))) = f(x)\, g(x)$, that is, $F \circ G = f \cdot g$.

**V.** (10) (a) Let $x \in \mathbb{R}^n$ and $\epsilon > 0$. Prove that if $z \in B(x, \epsilon)$, then $B(z, \epsilon - \| z - x \|) \subseteq B(x, \epsilon)$.

(b) Recall that our official definition of open sets in $\mathbb{R}^n$ was that a set $U$ is open if and only if $\forall x \in U, \exists \epsilon > 0, B(x, \epsilon) \subseteq U$. Using this definition and part (a), prove that if $U$ is a union of open balls in $\mathbb{R}^n$, then $U$ is open.

For (a), assume that $z \in B(x, \epsilon)$. Suppose $y \in B(z, \epsilon - \|z - x\|)$. Then $\| y - x \| = \| y - z + z - x \| \leq \| y - z \| + \| z - x \| < \epsilon - \| z - x \| + \| z - x \| = \epsilon$, so $y \in B(x, \epsilon)$.

For (b), suppose $U = \cup_{\alpha \in \mathcal{A}} B(x_\alpha, \epsilon_\alpha)$. Let $x \in U$. Then $x \in B(x_\beta, \epsilon_\beta)$ for some $\beta \in \mathcal{A}$. By part (a), $B(x, \epsilon_\beta - \| x - x_\beta \|) \subseteq B(x_\beta, \epsilon_\beta) \subseteq U$. Therefore $U$ is open.

**VI.** Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a function satisfying the $\epsilon$-$\delta$ definition of continuity. Using the definition of open sets
(8) in $\mathbb{R}^n$, and so on, prove that if $U$ is an open subset of $\mathbb{R}^n$, then the inverse image $f^{-1}(U)$ is open in $\mathbb{R}^m$.

Assume that $U$ is open in $\mathbb{R}^n$. Let $x \in f^{-1}(U)$. Then $f(x) \in U$. Since $U$ is open, there exists $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq U$. Since $f$ is continuous in the $\epsilon$-$\delta$ definition, there exists $\delta > 0$ such that if $\| z - x \| < \delta$, then $\| f(z) - f(x) \| < \epsilon$. This says that if $z \in B(x, \delta)$, $f(z) \in B(f(x), \epsilon) \subset U$, and consequently $z \in f^{-1}(U)$. That is, $B(x, \delta) \in f^{-1}(U)$.

**VII.** Let $X$ be a set. Give a full, precise definition of a *topology on $X$*.
(8)
A topology on $X$ is a collection $\mathcal{U}$ of subsets of $X$ such that

1. $X \in \mathcal{U}$ and the empty subset $\emptyset \in \mathcal{U}$.
2. If $\{U_\alpha\}_{\alpha \in \mathcal{A}} \subseteq \mathcal{U}$, then the union $\cup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{U}$.
3. If $\{U_i\}_{1 \le i \le n} \subseteq \mathcal{U}$, then the intersection $\cap_{1 \le i \le n} U_i \in \mathcal{U}$.

**VIII.** Recall that if $\mathbb{R}$ has the cofinite topology, then a subset $U$ of $\mathbb{R}$ is open if and only if $\mathbb{R} - U$ is finite (or $U$
(6) is empty). Use DeMorgan's Law $\cup(X - S_\alpha) = X - (\cap S_\alpha)$ to show that if $U$ and $V$ are any two nonempty open sets in $\mathbb{R}$ with the cofinite topology, then $U \cap V$ is not empty.

Suppose you have two nonempty open sets $U$ and $V$. Then, $\mathbb{R} - (U \cap V) = (\mathbb{R} - U) \cup (\mathbb{R} - V)$. Since $\mathbb{R} - U$ and $\mathbb{R} - V$ are finite, so is $(\mathbb{R} - U) \cup (\mathbb{R} - V)$. Therefore $\mathbb{R} - (U \cap V) = (\mathbb{R} - U) \cup (\mathbb{R} - V)$ is not $\mathbb{R}$, so $U \cap V$ is not empty.