Mathematics 4853-001 Name (please print) ______________________

Examination II

March 26, 2010

Instructions: Give concise answers, but clearly indicate your reasoning in any arguments that you give. You do not need to verify commonly used facts, such as that the composition of continuous functions is continuous (unless demonstrating that you have the ability to verify a fact is the point of the problem).

**I.** (7) Using a Cantor-style argument, prove that the product

$$P = \{0,1,2\} \times \{0,1,2\} \times \{0,1,2\} \times \cdots = \{(a_1, a_2, a_3, \ldots) \mid a_i \in \{0,1,2\}\}$$

of countably many copies of the set $\{0,1,2\}$ is uncountable.

**II.** (7) Let $P = \{0,1,2\} \times \{0,1,2\} \times \{0,1,2\} \times \cdots = \{(a_1, a_2, a_3, \ldots) \mid a_i \in \{0,1,2\}\}$, and let $A \subset P$ be defined by $A = \{(a_1, a_2, \ldots) \in P \mid \exists N, \forall n > N, a_n = 0\}$. Using the facts that products of finitely many countable sets are countable and unions of countably many countable sets are countable [where in this context, "countable" means "finite or countably infinite"], prove that $A$ is countable.

**III.** (10) Let $X$ be a set and let $\mathcal{A}$ be a collection of subsets of $X$.

(a) Define what it means to say that $\mathcal{A}$ is a *basis.*

(b) If $\mathcal{A}$ is a basis, define the *topology generated by* $\mathcal{A}$.

(c) Define what it means to say that $\mathcal{A}$ is a *sub-basis.*

(d) If $\mathcal{A}$ is a sub-basis, define the *topology generated by* $\mathcal{A}$.

**IV.** (10) (a) State the Basis Recognition Theorem.

(b) Use the Basis Recognition Theorem to prove that if $\mathcal{B}$ is a basis for a topology on $X$, and $A \subseteq X$, then $\{B \cap A \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on $A$.

**V.** (8) Let $X$ be the set of real numbers. There is a topology on $X$ defined by $\mathcal{U} = \{\emptyset, X\} \cup \{(a, \infty) \mid a \in X\}$, that is, a nonempty set is open if and only if it is empty, is $X$, or is an open ray to $\infty$ (you do not need to check that $\mathcal{U}$ is a topology).

(a) Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is continuous (where as usual, $\mathbb{R}$ means the real numbers with the standard topology), then $f$ is also continuous when its codomain is given the topology $\mathcal{U}$.

(b) Give a counterexample to the converse of (a).

**VI.** (10) Let $X$ and $Y$ be topological spaces.

(a) Define what it means to say that $h\colon X \to Y$ is a *homeomorphism.*

(b) Verify that if $X$ is homeomorphic to $Y$ and $Y$ is homeomorphic to $Z$, then $X$ is homeomorphic to $Z$. You may assume the known fact that for bijections, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ [proof: $(g \circ f)^{-1}(z) = x \Leftrightarrow (g \circ f)(x) = z \Leftrightarrow g(f(x)) = z \Leftrightarrow g^{-1}(z) = f(x) \Leftrightarrow f^{-1}(g^{-1}(z)) = x \Leftrightarrow (f^{-1} \circ g^{-1})(z) = x$].

(c) Let $\mathcal{L}$ be the lower-limit topology on $\mathbb{R}$. Show that the identity function from $(\mathbb{R}, \mathcal{L})$ to $\mathbb{R}$ is not a homeomorphism.