Mathematics 4853-001 Name (please print) ______________________

Examination II

March 26, 2010

Instructions: Give concise answers, but clearly indicate your reasoning in any arguments that you give. You do not need to verify commonly used facts, such as that the composition of continuous functions is continuous (unless demonstrating that you have the ability to verify a fact is the point of the problem).

I. (7) Using a Cantor-style argument, prove that the product

$$P = \{0,1,2\} \times \{0,1,2\} \times \{0,1,2\} \times \cdots = \{(a_1, a_2, a_3, \ldots) \mid a_i \in \{0,1,2\}\}$$

of countably many copies of the set $\{0,1,2\}$ is uncountable.

Let $\phi\colon \mathbb{N} \to P$ be any function. For each n, write $\phi(n) = (a_{n,1}, a_{n,2}, a_{n,3}, \ldots)$ and [one possible choice is to] put $b_n = 0$ if $a_{n,n} = 1$ or $a_{n,n} = 2$, and put $b_n = 2$ if $a_{n,n} = 0$. Then $b = (b_1, b_2, b_3, \ldots) \in P$ and b differs from $\phi(n)$ in the n^{th} coordinate, so $b \neq \phi(n)$ for any n. Therefore ϕ is not surjective. Since there is no surjective function from $\mathbb{N}$ to P, there cannot be any bijection between them, so P is not countable.

[Remark: one can deduce this result immediately from the fact that P contains the subset $\prod_{i=1}^{\infty}\{0,2\}$. Since every subset of a countable set is countable, and we proved in homework that the latter set is uncountable, P cannot be countable.]

II. (7) Let $P = \{0,1,2\} \times \{0,1,2\} \times \{0,1,2\} \times \cdots = \{(a_1, a_2, a_3, \ldots) \mid a_i \in \{0,1,2\}\}$, and let $A \subset P$ be defined by $A = \{(a_1, a_2, \ldots) \in P \mid \exists N, \forall n > N, a_n = 0\}$. Using the facts that products of finitely many countable sets are countable and unions of countably many countable sets are countable [where in this context, "countable" means "finite or countably infinite"], prove that A is countable.

For each N, put $A_N = \{(a_1, a_2, \ldots) \in P \mid \forall n > N, a_n = 0\}$. For each N, there is a bijection ϕ_N from A_N to the finite set $\displaystyle\prod_{i=1}^{N}\{0,1,2\}$ given by $\phi_N((a_1, a_2, \ldots, a_N, 0, 0, 0, \ldots)) = (a_1, a_2, \ldots, a_N)$. Therefore each A_N is finite. Since $A = \cup_{N\in\mathbb{N}} A_N$, A is a union of countably many countable sets, so A is countable.

III. (10) Let X be a set and let $\mathcal{A}$ be a collection of subsets of X.

(a) Define what it means to say that $\mathcal{A}$ is a *basis.*

It means that $\cup_{B\in\mathcal{A}} B = X$ and for all $B_1, B_2 \in \mathcal{A}$ and for all $x \in B_1 \cap B_2$, there exists $B \in \mathcal{A}$ such that $x \in B \subseteq B_1 \cap B_2$.

(b) If $\mathcal{A}$ is a basis, define the *topology generated by* $\mathcal{A}$.

$\{U \subseteq X \mid \forall x \in U, \exists B \in \mathcal{A}, x \in B \subseteq U\}$.

(c) Define what it means to say that $\mathcal{A}$ is a *sub-basis.*

It means that $\cup_{S\in\mathcal{A}} S = X$.

(d) If $\mathcal{A}$ is a sub-basis, define the *topology generated by* $\mathcal{A}$.

Let $\mathcal{B} = \{S_1 \cap S_2 \cap \cdots \cap S_n \mid S_i \in \mathcal{A}\}$. It is a basis. The topology generated by $\mathcal{A}$ is defined to be the topology generated by $\mathcal{B}$.

IV. (10) (a) State the Basis Recognition Theorem.

Let X be a topological space and let $\mathcal{B}$ be a collection of subsets of X. Suppose that (i) the elements of $\mathcal{B}$ are open in X, and (ii) for every open set U in X and every $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Then $\mathcal{B}$ is a basis that generates the topology on X.

(b) Use the Basis Recognition Theorem to prove that if $\mathcal{B}$ is a basis for a topology on X, and $A \subseteq X$, then $\{B \cap A \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on A.

We need only verify the hypotheses of the Basis Recognition Theorem.

(i) Since $\mathcal{B}$ is a basis for the topology on X, each element B of $\mathcal{B}$ is open in X and hence each $B \cap A$ is open in (the subspace topology of) A.

(ii) Let U be open in A, and let $a \in U$. There exists V open in X such that $U = V \cap A$. Since $a \in V$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq V$. Intersecting with A, we have $a \in B \cap A \subseteq V \cap A = U$.

V. (8) Let X be the set of real numbers. There is a topology on X defined by $\mathcal{U} = \{\emptyset, X\} \cup \{(a, \infty) \mid a \in X\}$, that is, a nonempty set is open if and only if it is empty, is X, or is an open ray to ∞ (you do not need to check that $\mathcal{U}$ is a topology).

(a) Prove that if $f: \mathbb{R} \to \mathbb{R}$ is continuous (where as usual, $\mathbb{R}$ means the real numbers with the standard topology), then f is also continuous when its codomain is given the topology $\mathcal{U}$.

Let U be open in X. Then U is either empty, X, or of the form (a, ∞), and in any case U is also open in the standard topology. Since $f: \mathbb{R} \to \mathbb{R}$ is continuous, $f^{-1}(U)$ is open in $\mathbb{R}$.

(b) Give a counterexample to the converse of (a).

Define $f: \mathbb{R} \to X$ by $f(x) = 1$ if $x > 0$ and $f(x) = 0$ if $x \leq 0$. Then $f^{-1}((a, \infty))$ is equal to $\emptyset$ if $1 \leq a$, to $(0, \infty)$ if $0 \leq a < 1$, and to $\mathbb{R}$ if $a < 0$, so in any case $f^{-1}((a, \infty))$ is open in $\mathbb{R}$. But f is not continuous as a function from $\mathbb{R}$ to $\mathbb{R}$ (for example, $f^{-1}((-1, 1/2)) = (-\infty, 0]$ is not open).

VI. (10) Let X and Y be topological spaces.

(a) Define what it means to say that $h: X \to Y$ is a *homeomorphism.*

It means that h is a continuous bijection whose inverse function $h^{-1}: Y \to X$ is also continuous. [Or one can say it means h is a bijection and U is open in X if and only if $h(U)$ is open in Y.]

(b) Verify that if X is homeomorphic to Y and Y is homeomorphic to Z, then X is homeomorphic to Z. You may assume the known fact that for bijections, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ [proof: $(g \circ f)^{-1}(z) = x \Leftrightarrow (g \circ f)(x) = z \Leftrightarrow g(f(x)) = z \Leftrightarrow g^{-1}(z) = f(x) \Leftrightarrow f^{-1}(g^{-1}(z)) = x \Leftrightarrow (f^{-1} \circ g^{-1})(z) = x$].

Let $f: X \to Y$ and $g: Y \to Z$ be homeomorphisms. They are bijections, so their composition $g \circ f: X \to Z$ is a bijection. Since f and g are continuous, so is their composition $g \circ f$. Each of f^{-1} and g^{-1} is continuous, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, so $(g \circ f)^{-1}$ is a composition of continuous functions and therefore is continuous.

(c) Let $\mathcal{L}$ be the lower-limit topology on $\mathbb{R}$. Show that the identity function from $(\mathbb{R}, \mathcal{L})$ to $\mathbb{R}$ is not a homeomorphism.

Let $I: (\mathbb{R}, \mathcal{L}) \to \mathbb{R}$ be the identity map. The set $[0, 1)$ is open in $(\mathbb{R}, \mathcal{L})$, but $(I^{-1})^{-1}([0, 1)) = [0, 1)$ is not open in $\mathbb{R}$. Therefore I^{-1} is not continuous.