Mathematics 3113-005 Name (please print) ____________________

Final Exam Form B

May 12, 2011

Instructions: Read the question carefully and make sure that you answer the question given. Give concise answers, but clearly indicate your reasoning. Most of the problems have rather short answers.

I. (12) Making use of the tables when needed, calculate the following Laplace transforms and inverse Laplace transform:

(i) $\mathcal{L}\left(\dfrac{e^t - e^{-t}}{t}\right)$ (remember that $\ln(a) - \ln(b) = \ln(\frac{a}{b})$, and $\lim_{b\to\infty} \dfrac{b-1}{b+1} = 1$)

Using the general formula $\mathcal{L}\left(\dfrac{f(t)}{t}\right) = \int_s^\infty F(u)\,du$, we have

$$\mathcal{L}\left(\frac{e^t - e^{-t}}{t}\right) = \int_s^\infty \mathcal{L}(e^t - e^{-t})(u)\,du = \int_s^\infty \frac{1}{u-1} - \frac{1}{u+1}\,du = \lim_{b\to\infty}\int_s^b \frac{1}{u-1} - \frac{1}{u+1}\,du$$
$$= \lim_{b\to\infty} \ln(u-1) - \ln(u+1)\Big|_s^b = \lim_{b\to\infty} \ln\left(\frac{u-1}{u+1}\right)\Big|_s^b$$
$$= \lim_{b\to\infty} \ln\left(\frac{b-1}{b+1}\right) - \ln\left(\frac{s-1}{s+1}\right) = \ln(1) - \ln\left(\frac{s-1}{s+1}\right) = \ln\left(\frac{s+1}{s-1}\right)$$

(ii) $\mathcal{L}(t\sin(t))$

Using $\mathcal{L}(t^n f(t)) = (-1)^n \dfrac{d^n}{ds^n} F(s)$, we have

$$\mathcal{L}(t\sin(t)) = -\frac{d}{ds}\,\mathcal{L}(\sin(t)) = -\frac{d}{ds}\frac{1}{1+s^2} = \frac{2s}{(1+s^2)^2}$$

(iii) $f(t)$, if $F(s) = \dfrac{2s}{s^2+4s+13}$

We have

$$F(s) = \frac{2s}{s^2+4s+13} = \frac{2s}{s^2+4s+4+9} = \frac{2s}{(s+2)^2+9}$$
$$= \frac{2(s+2)-4}{(s+2)^2+9} = 2\frac{(s+2)}{(s+2)^2+9} - \frac{4}{3}\frac{3}{(s+2)^2+9}\ .$$

Since $\mathcal{L}(2\cos(3t) - \frac{4}{3}\sin(3t)) = 2\dfrac{s}{s^2+9} - \dfrac{4}{3}\dfrac{3}{s^2+9}$, the general formula $\mathcal{L}(e^{at}f(t)) = F(s-a)$ tells us that the inverse transform of $2\dfrac{(s+2)}{(s+2)^2+9} - \dfrac{4}{3}\dfrac{3}{(s+2)^2+9}$ is $e^{-2t}(2\cos(3t) - \frac{4}{3}\sin(3t))$.

II. (5) For the rational function $\dfrac{\lambda^3+1}{(\lambda^2-9)^2(\lambda^2+9)^2}$, write the sum of partial fractions with unknown coefficients that would be used in the method of partial fractions, but *do not* go on to solve for the coefficients.

Fully factored, the denominator is $(\lambda+3)^2(\lambda-3)^2(\lambda^2+9)^2$, so the partial fraction decomposition is

$$\frac{A_1}{\lambda+3} + \frac{A_2}{(\lambda+3)^2} + \frac{B_1}{\lambda-3} + \frac{B_2}{(\lambda-3)^2} + \frac{C_1\lambda + D_1}{\lambda^2+9} + \frac{C_2\lambda + D_2}{(\lambda^2+9)^2}$$

III. (8) Use an integrating factor to solve the linear IVP

$$y' = (1 - y)\cos(x), \quad y(\pi) = 3 .$$

You will want to start by putting the DE into the standard form for a first-order linear DE.

In standard form, the equation is

$$y' + y\cos(x) = \cos(x) \ ,$$

so an integrating factor is $\mu(x) = e^{\int \cos(x)\,dx} = e^{\sin(x)}$. Multiplying through by this and integrating, we have

$$\begin{aligned} e^{\sin(x)}y' + ye^{\sin(x)}\cos(x) &= e^{\sin(x)}\cos(x) \\ (e^{\sin(x)}y)' &= e^{\sin(x)}\cos(x) \\ e^{\sin(x)}y &= e^{\sin(x)} + C \\ y &= 1 + Ce^{-\sin(x)} \\ 3 = y(\pi) &= 1 + Ce^{-\sin(\pi)} = 1 + C \\ C &= 2 \\ y &= 1 + 2e^{-\sin(x)} \end{aligned}$$

IV. (4) Define an *eigenvalue* of a matrix A, and define an *eigenvector* associated to that eigenvalue. You may use the version of the definitions given in class, or the version given in the book, or any equivalent statement.

An *eigenvalue* of A is a number λ such that $\det(A - \lambda I) = 0$, or equivalently such that $A\vec{v} = \lambda\vec{v}$ for some nonzero vector $\vec{v}$.

An *eigenvector* associated to the eigenvalue λ is a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$.

V. (5) (a) Define a *linear combination* of functions.

A linear combination of the functions $y_1, \ldots, y_n$ is a function that is a sum $c_1y_1 + c_2y_2 + \cdots c_ny_n$, where the c_i are numbers.

(b) State the Principle of Superposition for a DE of order n. Be sure to tell the requirements on the DE (that is, the hypotheses) needed for the Principle of Superposition to apply.

The Principle of Superposition states that for a homogeneous linear DE, any linear combination of solutions is a solution.

VI. (8) For the matrix $\begin{bmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{bmatrix}$, the eigenvalues are 3, 1, and -2. An eigenvector associated to 3 is $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, an eigenvector associated to 1 is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, and an eigenvector associated to -2 is $\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$.

(a) Write a general solution to the system $X' = \begin{bmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{bmatrix} X$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} + c_2 e^{t} + 3c_3 e^{-2t} \\ -c_1 e^{3t} - c_2 e^{t} - 2c_3 e^{-2t} \\ c_2 e^{t} + 2c_3 e^{-2t} \end{bmatrix}$$

(b) Write a set of linear equations whose solutions are the unknown coefficients in the general solution if the initial values are $x_1(-1) = -3$, $x_2(-1) = 0$, and $x_3(-1) = 3$.

$$\begin{aligned} c_1 e^{-3} + c_2 e^{-1} + 3c_3 e^{2} &= 3 \\ -c_1 e^{-3} - c_2 e^{-1} - 2c_3 e^{2} &= -3 \\ + c_2 e^{-1} + 2c_3 e^{2} &= 0 \end{aligned}$$

(c) Write an augmented matrix which would be the first step in using Gauss-Jordan elimination to solve the system in part (b), but *do not* continue with the process or attempt to find the unknown coefficients or the solution to the differential equation.

$$\left[\begin{array}{ccc|c} e^{-3} & e^{-1} & 3e^{2} & -3 \\ -e^{-3} & -e^{-1} & -2e^{2} & 0 \\ 0 & e^{-1} & 2e^{2} & 3 \end{array}\right]$$

VII. (6) (a) Give a specific example of three nonzero 2×2 matrices A, B, and C for which $AB = AC$ but $B \neq C$.

There are many possible examples, such as

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(b) Show that if A, B, and C are 2×2 matrices for which $AB = AC$ and $\det(A) \neq 0$, then $B = C$.

When $\det(A) \neq 0$, A has an inverse matrix A^{-1}. So we can multiply by A^{-1} to get $A^{-1}AB = A^{-1}AC$, that is, $IB = IC$, so $B = C$.

VIII. In this problem, we will solve the IVP $x'' - 4x' - 5x = 0$, $x(0) = x'(0) = 1$.
(8)

(a) Use the characteristic polynomial to write down a general solution with coefficients c_1 and c_2.

The characteristic polynomial is $\lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$, so the roots are -1 and 5. Therefore a general solution is $x(t) = c_1 e^{-t} + c_2 e^{5t}$.

(b) Use the initial conditions to write down a system of two linear equations that c_1 and c_2 must satisfy.

We have $x'(t) = -c_1 e^{-t} + 5c_2 e^{5t}$, so $x(0) = c_1 + c_2$ and $x'(0) = -c_1 + 5c_2$. A system of linear equations for c_1 and c_2 is

$$\begin{aligned} c_1 + \ \ c_2 &= 1 \\ -c_1 + 5c_2 &= 1 \end{aligned}$$

(c) Use Gauss-Jordan elimination to solve the system of two linear equations, and *write the solution of the initial value problem.*

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 5 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 6 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1/3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 2/3 \\ 0 & 1 & 1/3 \end{array}\right],$$

so the solution to the inital value problem is $x(t) = 2e^{-t}/3 + e^{5t}/3$.

IX. In this problem, we will solve the initial value problem $x'' - 4x' - 5x = 0$, $x(0) = x'(0) = 1$.
(10)

(a) Apply the Laplace transform to change the problem to an algebra equation for $X(s)$, the Laplace transform of $x(t)$. Solve it for $X(s)$ to obtain an expression giving $X(s)$ as a rational function of s.

$$\begin{gathered} s^2 x(S) - sx(0) - x'(0) - 4(sX(s) - x(0)) - 5X(s) = 0 \\ s^2 X(S) - s - 1 - 4sX(s) + 4 - 5X(s) = 0 \\ (s^2 - 4s - 5)X(s) = s - 3 \\ X(s) = \frac{s-3}{s^2 - 4s - 5} \end{gathered}$$

(b) Write $X(s)$ as a sum of partial fractions, with unknown coefficients, and find the coefficients.

$$\begin{gathered} X(s) = \frac{s-3}{s^2 - 4s - 5} = \frac{s-3}{(s+1)(s-5)} = \frac{A}{s+1} + \frac{B}{s-5} \\ s - 3 = A(s-5) + B(s+1) \end{gathered}$$

For $s = -1$, this says that $-6A = -4$, so $A = 2/3$. For $s = 5$, it says that $6B = 2$, so $B = 1/3$. Therefore

$$X(s) = \frac{2/3}{s+1} + \frac{1/3}{s-5}$$

(c) Apply the inverse transform to find the solution $x(t)$.

Since the Laplace transform of e^{at} is $\dfrac{1}{s-a}$, we have $x(t) = 2e^{-t}/3 + e^{5t}/3$.

X. In this problem, we will solve the differential equation $x'' - 4x' - 5x = 0$.
(14)

(a) Rewrite the DE $x'' - 4x' - 5x = 0$ as an equivalent first-order system with unknown functions x and $y = x'$ (or you may write $x_1 = x$ and $x_2 = x'$ if you prefer to use that notation).

Putting $y = x'$, we have $y' = x'' = 4x' + 5x = 5x + 4y$, so an equivalent system is

$$\begin{aligned} x' &= y \\ y' &= 5x + 4y \end{aligned}$$

(b) Write the system in the form $X' = PX$, where P is a 2×2 matrix and $X = \begin{bmatrix} x \\ y \end{bmatrix}$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(c) Find the eigenvalues of P.

We want $0 = \det(P - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 5 & 4-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 5 = (\lambda+1)(\lambda-5)$, so the eigenvalues are -1 and 5.

(d) An eigenvector associated to one of the eigenvalues is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Find an eigenvector associated to the other eigenvalue.

Since $P\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector associated to the eigenvalue $-!$.

For the eigenvalue 5, we want $0 = (P - 5I)X = \begin{bmatrix} -5 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, which says $5a = b$, so we may take the eigenvector to be $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

(e) Use the eigenvalues and eigenvectors to write two solutions of $X' = PX$, and use them to write a general solution for X (its top function x will be a general solution $x(t)$ for the DE $x'' - 4x' - 5x = 0$, although not necessarily written in exactly the same way as the general solution found by other methods, and its bottom function y should be $x'(t)$).

Two solutions are $e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $e^{5t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$, so a general solution is

$$c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + c_2 e^{5t} \\ -c_1 e^{-t} + 5c_2 e^{5t} \end{bmatrix}$$

Formulas list

Cramer's Rule: If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$, then the solutions of the linear system

$$\begin{aligned} ax + by &= e \\ cs + dy &= f \end{aligned}$$

are $x = \begin{vmatrix} e & b \\ f & d \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and $y = \begin{vmatrix} a & e \\ c & f \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

$\mathcal{L}(t^n) = \dfrac{n!}{s^{n+1}}$

$\mathcal{L}(t^\alpha) = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, where $\Gamma(x) = \displaystyle\int_0^\infty e^{-t} t^{x-1}\, dt$

$\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$

$\mathcal{L}(\cos(at)) = \dfrac{s}{s^2+a^2}$

$\mathcal{L}(\sin(at)) = \dfrac{a}{s^2+a^2}$

$\mathcal{L}(\cosh(at)) = \dfrac{s}{s^2-a^2}$

$\mathcal{L}(\sinh(at)) = \dfrac{a}{s^2-a^2}$

$\mathcal{L}\left(\dfrac{1}{2a}\, t\sin(at)\right) = \dfrac{s}{(s^2+a^2)^2}$

$\mathcal{L}\left(\dfrac{1}{2a^3}\,(\sin(at) - at\cos(at))\right) = \dfrac{1}{(s^2+a^2)^2}$

$\mathcal{L}(u_a(t)) = \dfrac{e^{-as}}{s}$, where $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$

$\mathcal{L}(f^{(n)}(t)) = s^n\, \mathcal{L}(f(t)) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$

$\mathcal{L}\left(\displaystyle\int_0^t f(\tau)\, d\tau\right) = \dfrac{1}{s}\, \mathcal{L}(f(t))$

$\mathcal{L}(e^{at} f(t)) = F(s-a)$

$\mathcal{L}(t^n f(t)) = (-1)^n \dfrac{d^n}{ds^n} F(s)$

$\mathcal{L}\left(\dfrac{f(t)}{t}\right) = \displaystyle\int_s^\infty F(\sigma)\, d\sigma$

$\mathcal{L}((f * g)(t)) = F(s)\, G(s)$, where $(f * g)(t) = \displaystyle\int_0^t f(\sigma) g(t-\sigma)\, d\sigma$

$\mathcal{L}(u_a(t) f(t-a)) = e^{-as} F(s)$