Mathematics 3113-005 Name (please print) ____________________

Exam I Form B

February 18, 2011

Instructions: Give concise answers, but clearly indicate your reasoning. Most of the problems have rather short answers, so if you find yourself involved in a lengthy calculation, it might be a good idea to move on and come back to that problem if you have time.

I. (16) For each of the following first-order DE's, carry out a substitution to put the DE into a form that can be solved by either separation of variables or the method for linear equations, and simplify. If it is separable, write it as an equality of a differential of v and a differential of x (that is, up to the step where you are about to integrate both sides), and if it is linear, find the integrating factor and multiply through to make the left-hand side a derivative. In either case, *do not continue on from there to find the solution.*

(a) $4xy' + 8y = \sqrt{y}/x$

The DE is Bernoulli with $n = 1/2$, so we will substitute $v = y^{1-1/2} = \sqrt{y}$. We have $v' = y'/2\sqrt{y}$, so $y' = 2\sqrt{y}v' = 2vv'$. Substituting, we have

$$\begin{aligned} 4xy' + 8y &= \sqrt{y}/x \\ 8xvv' + 8v^2 &= v/x \\ v' + v/x &= \frac{1}{8x^2} \end{aligned}$$

which is linear. An integrating factor is $\rho(x) = e^{\int \frac{1}{x}} = e^{\ln(x)} = x$, and multiplying through by $\rho(x)$ gives

$$xv' + v = \frac{1}{8x}\ .$$

(b) $xy' = y + \sqrt{x^2 - y^2}$

It is not Bernoulli and does not involve a linear expression in x and y, so let's see if it is homogeneous.

$$xy' = y + \sqrt{x^2 - y^2}$$

$$y' = \frac{y}{x} + \frac{\sqrt{x^2 - y^2}}{x} = \frac{y}{x} + \sqrt{\frac{x^2 - y^2}{x^2}} = \frac{y}{x} + \sqrt{1 - \frac{y^2}{x^2}} = \frac{y}{x} + \sqrt{1 - \left(\frac{y}{x}\right)^2}$$

So it is homogeneous. Putting $v = \dfrac{y}{x}$, we have $y = xv$, $y' = v + xv'$, and we substitute to obtain

$$\begin{aligned} v + xv' &= v + \sqrt{1 - v^2} \\ x\frac{dv}{dx} &= \sqrt{1 - v^2} \\ \frac{1}{\sqrt{1 - v^2}}\, dv &= \frac{1}{x}\, dx\ . \end{aligned}$$

II. (5) (a) Give a definition of an (ordinary) *differential equation.*

An equation involving an unknown function and its derivatives. [Any reasonable definition is acceptable, as long as it mentions that the equation involves derivatives and indicates that the unknowns (that is, the solutions) are functions.]

(b) Define the *order* of a differential equation.

It is the highest order of derivative of the unknown function that appears.

(c) Give the general form (not a specific example) of a *first-order initial value problem* (of the kind that appears in the first Existence and Uniqueness Theorem).

$y' = F(x, y),\ y(a) = b$

III. (4) For the first-order linear homogeneous DE $y' + P(x)y = 0$, verify that if y_1 and y_2 are solutions, then so is $Ay_1 + By_2$ for any constants A and B.

Since y_1 and y_2 are solutions, $y_1' + P(x)y_1 = 0$ and $y_2' + P(x)y_2 = 0$. Testing $Ay_1 + By_2$, we find

$$\begin{aligned}(Ay_1 + By_2)' + P(x)(Ay_1 + By_2) &= Ay_1' + By_2' + P(x)Ay_1 + P(x)By_2 \\ &= A\,(y_1' + P(x)y_1) + B\,(y_2' + P(x)y_2) = 0\ .\end{aligned}$$

IV. (10) A small lake contains 100 billion liters of water, contaminated with a pollutant at 1% concentration. Starting at time $t = 0$, four billion liters per day of water with the pollutant at 5% concentration enters, mixes instantaneously with all the water in the lake, and the same amount of mixed water leaves. We want to know how many days it will take until the level of contamination reachs 3%. Let $x(t)$ denote the volume of pollutant in the lake, in billions of liters.

(a) What is $x(0)$? In terms of $x(t)$, what does the problem ask us to find?

$x(0) = (0.01) \cdot 100 = 1$. The problem asks us to find t_0 for which $x(t_0) = 3$.

(b) Write an expression for the volume of pollutant that enters between t and $t + \Delta t$.

$(0.05) \cdot 4 = 0.2$ billion liters of pollutant enter per day. During a time interval Δt, $0.2\,\Delta t$ billion liters enter.

(c) Write an expression for the approximate volume of pollutant that leaves between t and $t + \Delta t$.

The concentration at time t is $x(t)/100$, so during a time interval Δt, approximately $3 \cdot (x(t)/100) \cdot \Delta t$ leaves.

(d) Use (b), (c), and first-semester calculus to set up a first-order linear initial value problem that models the problem, but *do not* go on to solve it or attempt to find the answer to the problem.

From (b) and (c), $\Delta x \approx 0.2\,\Delta t - 3 \cdot (x(t)/100)\,\Delta t$, so $\dfrac{\Delta x}{\Delta t} \approx 0.2 - 3x(t)/100$. Taking the limit as $\Delta t \to 0$, we obtain $\dfrac{dx}{dt} = 0.2 - 3x/100$. The initial value problem is

$$x' + 3x/100 = 0.2, \quad x(0) = 1\ .$$

V. (5) Show that the substitution $v = \ln(y)$ transforms the differential equation $dy/dx + P(x)y = Q(x)y\ln(y)$ into the linear equation $dv/dx + P(x) = Q(x)v(x)$.

When $v = \ln(y)$, $v' = y'/y$ so $y' = yv'$. Substituting, we find that

$$\begin{aligned} yv' + P(x)y &= Q(x)yv \\ v' + P(x) &= Q(x)v \end{aligned}$$

VI. (10) The differential equation in the following initial value problem is separable. Use the method of separation of variables to find the solutions, and check your solution.

$$y' = 4e^{2x-y},\ y(1) = 2\ .$$

Hint: To get started, recall that $e^{2x-y} = e^{2x}e^{-y}$.

Separating variables, we calculate that

$$\begin{aligned} \frac{dy}{dx} &= 4e^{2x}e^{-y} \\ e^y dy &= 4e^{2x}dx \\ e^y &= 2e^{2x} + C \\ y &= \ln(2e^{2x} + C)\ . \end{aligned}$$

Using the initial condition,

$$\begin{aligned} 2 &= \ln(2e^2 + C) \\ e^2 &= 2e^2 + C \\ C &= -e^2 \\ y &= \ln(2e^{2x} - e^2)\ . \end{aligned}$$

To check that y works in the DE, we have $y' = \dfrac{1}{2e^{2x} - e^2} \cdot 4e^{2x}$. Since $e^{-y} = \dfrac{1}{e^{\ln(2e^{2x}-e^2)}} = \dfrac{1}{2e^{2x} - e^2}$, y' is exactly $e^{-y} \cdot 4e^{2x} = 4e^{2x-y}$. For the initial condition, we have $y(1) = \ln(2e^2 - e^2) = \ln(e^2) = 2$

VII. (4) (a) From the viewpoint of existence and uniqueness, what is noteworthy about the solutions of the Torricelli equation $\dfrac{dV}{dt} = -k\sqrt{y}$?

The solutions passing through points on the x-axis are not unique.

(b) From the viewpoint of existence and uniqueness, what is noteworthy about the nonzero solutions of the DE $y' = y^2$?

None of them exists on the *entire* real line.