Mathematics 3113-005 Name (please print) ______________________

Exam II Form A

March 25, 2011

Instructions: Read the question carefully and make sure that you answer the question given. Give concise answers, but clearly indicate your reasoning. Most of the problems have rather short answers.

I. (8) (a) Define a *linear combination* of functions.

A linear combination of the functions $y_1, \ldots, y_n$ is a function that is a sum $c_1y_1 + c_2y_2 + \cdots c_ny_n$, where the c_i are numbers.

(b) State the Principle of Superposition for a DE of order n. Be sure to tell the requirements on the DE (that is, the hypotheses) needed for the Principle of Superposition to apply.

The Principle of Superposition states that for a homogeneous linear DE, any linear combination of solutions is a solution.

(c) Verify the Principle of Superposition for the first-order linear homogeneous DE $y' + P(x)y = 0$.

We must show that any linear combination of solutions is a solution. Suppose that y_1 and y_2 are any two solutions. They satisfy $y_1' + P(x)y_1 = 0$ and $y_2' + P(x)y_2 = 0$. Testing an arbitrary linear combination $Ay_1 + By_2$, we find

$$\begin{aligned}(Ay_1 + By_2)' + P(x)(Ay_1 + By_2) &= Ay_1' + By_2' + P(x)Ay_1 + P(x)By_2 \\ &= A\,(y_1' + P(x)y_1) + B\,(y_2' + P(x)y_2) = 0\ ,\end{aligned}$$

so $Ay_1 + By_2$ is also a solution.

II. (8) (a) Write a general solution of $y^{(4)} + 4y'' + 4y = 0$.

The characteristic polynomial is $\lambda^4 + 4\lambda^2 + 4 = (\lambda^2 + 2)^2$, so has roots $\pm\sqrt{2}i$ and $\pm\sqrt{2}i$. A general solution would be

$$c_1\cos(\sqrt{2}x) + c_2\sin(\sqrt{2}x) + c_3x\cos(\sqrt{2}x) + c_4x\sin(\sqrt{2}x)\ .$$

(b) The function $\cos(x) + 3x^2$ satisfies the DE $y^{(4)} + 4y'' + 4y = \cos(x) + 12x^2 + 24$. Find a general solution to this DE.

$$c_1\cos(\sqrt{2}x) + c_2\sin(\sqrt{2}x) + c_3\cos(\sqrt{2}x) + c_4x\sin(\sqrt{2}x + \cos(x) + 3x^2$$

III. (6) A certain mass-spring system has $m = 1$, $c = 2$, and $k = 2$, so can be modeled by the equation $x''+2x'+2x = 0$, where $x = x(t)$.

(a) Find a general solution.

The characteristic polynomial $\lambda^2+2\lambda+2$ has roots $-1\pm i$. A general solution is $c_1e^{-t}\cos(t)+c_2e^{-t}\sin(t)$.

(b) What value of c would make the system critically damped?

The characteristic polynomial $\lambda^2+c\lambda+2$ would have two equal real roots, so we would need $c^2-4\cdot1\cdot2 = c^2 - 8 = 0$, that is, $c = \sqrt{8} = 2\sqrt{2}$.

(c) Is the system in (a) (with $c = 2$) overdamped, critically damped, or underdamped?

Underdamped.

IV. Write the function $-3\cos(\sqrt{2}t) - 4\sin(\sqrt{2}t)$ in phase-angle form.
(5)

We have $A = -3$ and $B = -4$, so $B/A = 4/3$ and (A, B) lies in the third quadrant. Therefore the phase angle is $\alpha = \pi + \tan^{-1}(4/3)$. Since $C = \sqrt{A^2 + B^2} = 5$, the phase-angle form is $5\cos(\sqrt{2}t - \pi - \tan^{-1}(4/3))$.

V. Define what it means to say that a collection of functions $\{y_1, y_2, \ldots, y_n\}$ is *linearly independent.*
(3)

It means that $c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0$ for constants c_i only when all the c_i are 0.
or
It means that if $c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0$ for constants c_i, then all the c_i are 0.
or any of various other ways of stating this.

VI. For the following two DE's, a general solution of the associated homogeneous equation is shown (you do not
(7) need to show how to find it or check that it is a solution). Using this information, write a trial solution for the method of undetermined coefficients, but do *not* continue on to try to determine the exact coefficients.

(a) $y'' + 4y = \cos(2x)$, $y_c = c_1\cos(2x) + c_2\sin(2x)$.

The general form $P_m(x)e^{rx}\cos(kx)$ becomes just $1 \cdot e^{0x} \cdot \cos(2x)$, so the general form of the trial solution is $x^s(A\cos(2x) + B\sin(2x))$. The smallest power of s that will prevent any term from being a complementary function is $s = 1$. Therefore a trial solution is $Ax\cos(2x) + Bx\sin(2x)$.

(b) $y^{(4)} + 4y'' = x^2e^x\cos(2x)$, $y_c = c_1 + c_2x + c_3\cos(2x) + c_4\sin(2x)$.

The general form $P_m(x)e^{rx}\cos(kx)$ becomes $P_2(x)e^x \cdot \cos(2x)$, so the general form of the trial solution is $x^se^x((A_0 + A_1x + A_2x^2)\cos(2x) + (B_0 + B_1x + B_2x^2)\sin(2x))$. None of the terms is a complementary function, so $s = 0$. Therefore a trial solution is $e^x((A_0+A_1x+A_2x^2)\cos(2x)+(B_0+B_1x+B_2x^2)\sin(2x))$,

VII. Using the trial solution $y_p = Ax\cos(x) + Bx\sin(x)$, carry out the rest of the method of undetermined
(7) coefficients to find a particular solution of $y'' + y = \sin(x)$.

We have $y_p' = A\cos(x) + B\sin(x) - Ax\sin(x) + Bx\cos(x)$ and $y_p'' = -2A\sin(x) + 2B\cos(x) - Ax\cos(x) - Bx\sin(x)$, so $y_p'' + y_p = -2A\sin(x) + 2B\cos(x)$. To equal $\sin(x)$, we must have $A = -1/2$ and $B = 0$, giving $y_p = -x\cos(x)/2$.

VIII. (10) Recall that in the method of variation of parameters, one uses a trial solution of the form $y_p = u_1 y_1 + u_2 y_2$, and if we choose u_1 and u_2 to satisfy the two conditions $y_1 u_1' + y_2 u_2' = 0$ and $y_1' u_1' + y_2' u_2' = f(x)$, then y_p will be a particular solution. In this problem, we will use the method to find a particular solution to the DE $y'' - y = 4e^{-x}$. Some remarks:

(1) You do not need to give the theoretical derivation of the two equations involving u_1' and u_2'. We will simply apply them, with the correct choices of y_1 and y_2, to find a particular solution for this example.

(2) Please be careful to calculate accurately. Miscalculations are likely to make the problem become much more complicated. The correct integrals you need to calculate for this problem are easy ones.

(3) Yes, a solution of this DE can be found rather easily using the method of undetermined coefficients, but that is not relevant to this problem, which concerns the method of variation of parameters.

(a) Write the appropriate form of the trial solution $y_p = u_1 y_1 + u_2 y_2$ for $y'' - y = 4e^{-x}$ (that is, what functions would y_1 and y_2 be?)

The functions $y_1 = e^x$ and $y_2 = e^{-x}$ are linearly independent solutions of the associated homogeneous equation $y'' - y = 0$, so we may take $y_p = u_1 e^x + u_2 e^{-x}$.

(b) Write the appropriate version of the system of two equations $y_1 u_1' + y_2 u_2' = 0$ and $y_1' u_1' + y_2' u_2' = f(x)$ for $y'' - y = 4e^{-x}$.

The equations become

$$\begin{aligned} e^x u_1' + e^{-x} u_2' &= 0 \\ e^x u_1' - e^{-x} u_2' &= 4e^{-x} \end{aligned}$$

(c) Solve the system of two equations and use the results to find a particular solution of $y'' - y = 4e^{-x}$.

Subtracting the equations gives $2e^{-x} u_2' = -4e^{-x}$ and hence $u_2' = -2$. The first equation becomes

$$e^x u_1' - 2e^{-x} = 0 \ ,$$

giving $u_1' = 2e^{-2x}$. Integrating gives $u_1 = -e^{2x}$ and $u_2 = -2x$, so a particular solution is

$$y = (-e^{-2x}) \cdot e^x - 2x \cdot e^{-x} = -e^{-x} - 2xe^{-x} \ .$$

Checking (not required), we find

$$\begin{aligned} y' &= e^{-x} - 2e^{-x} + 2xe^{-x} = -e^{-x} + 2xe^{-x} \\ y'' &= e^{-x} + 2e^{-x} - 2xe^{-x} = 3e^{-x} - 2xe^{-x} \\ y'' - y &= 4e^{-x} \end{aligned}$$