Mathematics 3113-005 Name (please print) ______________________

Exam III Form A

April 29, 2011

Instructions: Read the question carefully and make sure that you answer the question given. Give concise answers, but clearly indicate your reasoning. Most of the problems have rather short answers.

I. (14) Making use of the tables when needed, calculate the following Laplace transforms and inverse Laplace transforms:

(i) $\mathcal{L}\left(t^{6/5} - e^{-3t}\right)$

(ii) $\mathcal{L}(4\cos^2(5t))$ (you may need the identity $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$)

(iii) $\mathcal{L}(4\cosh^2(2t))$

(iv) $\mathcal{L}(x'' + 3x' - 5x)$, if $x(0) = 2$, $x'(0) = 3$ (give the answer in terms of $X(s)$, the Laplace transform of $x(t)$).

(v) $f(t)$ if $F(s) = \dfrac{11 + s}{s^2 + 7}$

II. (4) For the matrix $P = \begin{bmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{bmatrix}$, one of the eigenvalues is -2, and an eigenvector associated to -2 is $\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$. Use this information to write one specific solution of the first-order homogeneous linear system $X' = PX$.

III. (12) For a certain homogeneous first-order linear system $X' = PX$ of three equations in three unknown functions, three linearly independent solutions are $X_1 = e^{2t}\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $X_2 = e^{4t}\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$, and $X_3 = e^{-3t}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(a) Write a 3×3 matrix whose determinant is the Wronskian of these three solutions, and calculate the determinant.

(b) Write a general solution for the system, and use Gauss-Jordan elimination to solve the initial value problem $X' = PX$, $x_1(0) = 1$, $x_2(0) = 0$, $x_3(0) = 5$ (that is, find the specific solutions x_1, x_2, and x_3 that satisfy the IVP).

IV. (6) (a) Give a specific example of three nonzero 2×2 matrices A, B, and C for which $AB = AC$ but $B \neq C$.

(b) Show that if A, B, and C are 2×2 matrices for which $AB = AC$ and $\det(A) \neq 0$, then $B = C$.

V. (3) Define an *eigenvalue* of a matrix A, and define an *eigenvector* associated to that eigenvalue. You may use the version of the definitions given in class, or the version given in the book, or any equivalent statement.

VI. (5) Let P be the matrix $\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$.

(a) Find the eigenvalues of P.

(b) For the *larger* of the two eigenvalues, find an associated eigenvector.

VII. (7) (a) Write the definition of $\mathcal{L}(f(t))$ as an integral.

(b) As you know, for $a \geq 0$ the step function $u_a(t)$ is defined to be 0 for $0 \leq t < a$ and 1 for $a \leq t$. Use the definition to calculate that $\mathcal{L}(u_a(t)) = \dfrac{e^{-as}}{s}$ for $s > 0$. Write the calculation correctly using limits, not treating infinity as a number.

Formulas for the Laplace Transform

$\mathcal{L}(t^n) = \dfrac{n!}{s^{n+1}}$

$\mathcal{L}(t^\alpha) = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, where $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$

$\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$

$\mathcal{L}(\cos(at)) = \dfrac{s}{s^2+a^2}$

$\mathcal{L}(\sin(at)) = \dfrac{a}{s^2+a^2}$

$\mathcal{L}(\cosh(at)) = \dfrac{s}{s^2-a^2}$

$\mathcal{L}(\sinh(at)) = \dfrac{a}{s^2-a^2}$

$\mathcal{L}\left(\dfrac{1}{2a}\, t \sin(at)\right) = \dfrac{s}{(s^2+a^2)^2}$

$\mathcal{L}\left(\dfrac{1}{2a^3}\, (\sin(at) - at\cos(at))\right) = \dfrac{1}{(s^2+a^2)^2}$

$\mathcal{L}(u_a(t)) = \dfrac{e^{-as}}{s}$, where $u_a(r) = 0$ for $r < a$ and $u_a(r) = 1$ for $r > a$

$\mathcal{L}(f^{(n)}(t)) = s^n\, \mathcal{L}(f(t)) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$

$\mathcal{L}\left(\int_0^t f(\tau)\, d\tau\right) = \dfrac{1}{s}\, \mathcal{L}(f(t))$

$\mathcal{L}(e^{at} f(t)) = F(s-a)$

$\mathcal{L}(t^n f(t)) = (-1)^n \dfrac{d^n}{ds^n} F(s)$

$\mathcal{L}\left(\dfrac{f(t)}{t}\right) = \int_s^\infty F(\sigma)\, d\sigma$

$\mathcal{L}((f * g)(t)) = F(s)\, G(s)$, where $(f * g)(t) = \int_0^t f(\sigma) g(t-\sigma)\, d\sigma$

$\mathcal{L}(f(t)) = \dfrac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt$, if $f(t+p) = f(t)$ for all t

$\mathcal{L}(u_a(t) f(t-a)) = e^{-as} F(s)$

$\mathcal{L}(u(t-a) f(t-a)) = e^{-as} F(s)$, where $u(r) = 0$ for $r < 0$ and $u(r) = 1$ for $r > 0$

$\mathcal{L}(\delta_a(t)) = e^{-as}$, where $\delta_a(t)$ is the Dirac δ function (this is often written using $\delta(t-a)$ to mean $\delta_a(t)$)