On the matrices AB *and* BA

Darryl McCullough

University of Oklahoma

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One of the first things we learn about matrices in linear algebra is that AB need not equal BA.

For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} ,$$

but

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

So we can even have $AB \neq 0$ but $BA = 0$!

How different can AB and BA be? Can we even write *any* two $n \times n$ matrices X and Y as $X = AB$ and $Y = BA$?

No, AB and BA cannot be just any two matrices. They must have the same determinant, where for 2×2 matrices the determinant is defined by

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \ .$$

The determinant function has the remarkable property that $\det(AB) = \det(A)\det(B)$.

So we have

$$\begin{aligned} \det(AB) &= \det(A)\ \det(B) \\ &= \det(B)\ \det(A) \\ &= \det(BA) \end{aligned}$$

Are there other functions f for which $f(AB) = f(BA)$?

There is another function that satisfies $f(AB) = f(BA)$— the trace function, which is just the sum of the diagonal entries:

$$\text{tr}(A) = \text{tr}([a_{ij}]) = \sum_{i=1}^{n} a_{ii}$$

Unlike the determinant function, one does not usually have $\text{tr}(AB) = \text{tr}(A)\,\text{tr}(B)$.

But one always has $\text{tr}(AB) = \text{tr}(BA)$:

$$\begin{aligned}
\text{tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji} \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij} \\
&= \sum_{j=1}^{n} (BA)_{jj} \\
&= \text{tr}(BA)
\end{aligned}$$

Are there are any other functions that satisfy $f(AB) = f(BA)$?

Of course we can generate lots of silly examples using the trace and determinant, such as

$$f(AB) = \cos(23\,\det(AB)) - 7\,\mathrm{tr}(AB)\ .$$

In fact, just taking polynomial expressions in trace and determinant, we can get many polynomials in the matrix entries that have this property, e. g.

$$6\,\mathrm{tr}^2(A)\det(A) = 6(a+d)^2(ad-bc)\ .$$

What we are actually wondering is:

Are there polynomials p in the matrix entries such that $p(AB) = p(BA)$, other than polynomial expressions in the trace and determinant themselves?

The answer is yes. There is a source that gives both the trace and determinant, and others as well— the characteristic polynomial:

$$\mathrm{char}(A) = \det(\lambda I_n - A)$$

It is a polynomial in λ, with coefficients that are are polynomials in the entries of A.

For example, for a 3×3 matrix we have

$$\begin{aligned} &\mathrm{char}\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{bmatrix}\right) \\ &= \lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 \\ &+\Big(a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} \\ &-a_{13}a_{31} + a_{22}a_{33} - a_{23}a_{32}\Big)\lambda \\ &-(a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} \\ &-a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{12}a_{22}a_{31}) \\ &= \lambda^3 - \mathrm{tr}(A)\,\lambda^2 + p_2(A)\,\lambda - \det(A) \end{aligned}$$

In general, for an $n \times n$ matrix we have

$$\text{char}(A) = \lambda^n - \text{tr}(A)\lambda^{n-1} + p_2(A)\lambda^{n-2}$$
$$+ \cdots + (-1)^{n-1}p_{n-1}(A)\lambda + (-1)^n \det(A)$$

for certain polynomials p_i in the entries of A.

We should actually write $p_{n,i}$ for these polynomials, since their formulas depend on the size n of the matrix. And we can write $p_{n,1}(A) = \text{tr}(A)$ and $p_{n,n}(A) = \det(A)$.

So we wonder whether $\text{char}(AB) = \text{char}(BA)$. That would be the same as saying that $p_{n,i}(AB) = p_{n,i}(BA)$ for each of these polynomials.

The answer is yes:

Theorem: *If A and B are $n \times n$ matrices, then* char$(AB) =$ char(BA).

A beautiful proof of this was given in:

J. Schmid, A remark on characteristic polynomials, *Am. Math. Monthly,* 77 (1970), 998-999.

In fact, he proved a stronger result, that becomes the theorem above if we have $m = n$:

Theorem: *Let A be an $n \times m$ matrix and B an $m \times n$ matrix. Then*

$$\lambda^m \operatorname{char}(AB) = \lambda^n \operatorname{char}(BA)$$

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proof (J. Schmid): Put

$$C = \begin{bmatrix} \lambda I_n & A \\ B & I_m \end{bmatrix}, D = \begin{bmatrix} I_n & 0 \\ -B & \lambda I_m \end{bmatrix}$$

Then we have

$$CD = \begin{bmatrix} \lambda I_n - AB & \lambda A \\ 0 & \lambda I_m \end{bmatrix}, DC = \begin{bmatrix} \lambda I_n & A \\ 0 & \lambda I_m - BA \end{bmatrix}$$

So

$$\begin{aligned} \lambda^m \operatorname{char}(AB) &= \det(\lambda I_m) \det(\lambda I_n - AB) \\ &= \det(CD) \\ &= \det(DC) \\ &= \det(\lambda I_n) \det(\lambda I_m - BA) \\ &= \lambda^n \operatorname{char}(BA) \end{aligned}$$

So, have we now found all the f's with $f(AB) = f(BA)$?

Yes!

Every polynomial p in the matrix entries that satisfies $p(AB) = p(BA)$ can be written as a polynomial in the $p_{n,i}$.

Consider first the case of diagonal matrices, where the entries are the eigenvalues. Any p with $p(AB) = p(BA)$ is a similarity invariant, so gives the same values if we permute the diagonal entries. Therefore it is a symmetric polynomial in the eigenvalues. The polynomials 1, $p_{n,1}$, $p_{n,2}, \ldots,$ $p_{n,n}$ are the elementary symmetric polynomials in the eigenvalues, so any symmetric polynomial in the eigenvalues can be written (uniquely) as a polynomial in them, say $p = P(1, p_{n,1}, \ldots, p_{n,n})$, on diagonal matrices. Since p is invariant under similarity, it equals P on all the set of all conjugates of diagonal matrices with distinct nonzero eigenvalues, which form an open subset of $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$. Since p and P are polynomials, this implies that $p = P$ on all of $M_n(\mathbb{R})$.

A final question: If $p_{n,i}(X) = p_{n,i}(Y)$ for all the polynomials, does this ensure that we can write $X = AB$ and $Y = BA$ for some A and B?

No, there are easy examples that show this is not enough, such as

$$X = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

X and Y have the same trace and determinant (i. e. $p_{2,1}(X) = p_{2,1}(Y)$ and $p_{2,2}(X) = p_{2,2}(Y)$), but if $AB = I$ then A and B are inverses, and $BA = I$ as well.

There are many such examples for larger n. The condition that $p_{n,i}(X) = p_{n,i}(Y)$ for all i is equivalent to X and Y having the same eigenvalues, which is much weaker than being able to write $X = AB$ and $Y = BA$ (which is equivalent to similarity when X and Y are nonsingular).