SUBADDITIVITY OF TEST IDEALS AND DIAGONAL F-REGULARITY

by

Daniel Smolkin

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The dissertation of **Daniel Smolkin**

has been approved by the following supervisory committee members:

Karl Schwede,	Chair(s)	**April 22, 2019** Date Approved
Anurag Singh,	Member	**April 22, 2019** Date Approved
Srikanth Iyengar,	Member	**April 22, 2019** Date Approved
Tommaso DeFernex,	Member	**April 22, 2019** Date Approved
Daniel Hernández,	Member	**April 22, 2019** Date Approved

by **Davar Khoshnevisan**, Chair/Dean of

the Department/College/School of **Mathematics**

and by **David Kieda**, Dean of The Graduate School.

Abstract

Test ideals are an important object of study in positive characteristic commutative algebra. The test ideals of a regular ring in positive characteristic enjoy a property known as *subadditivity*. This subadditivity property has numerous important applications, such as Ein–Lazarsfeld–Smith's argument showing a uniform boundedness in the growth of symbolic powers of ideals in regular rings.

In [Tak06], Takagi finds a subadditivity formula for test ideals in the non-regular setting that uses the Jacobian ideal $\mathrm{Jac}(R)$ as a correction term. Both Takagi's proof, as well as the original proof of subadditivity in [HY03], uses the classical perspective on test ideals as annihilators of tight closure.

In this thesis, we use the more modern perspective of test ideals described in [HT04] and [Sch10] to find a new subadditivity formula for so-called "big" or "non-finitistic" test ideals, which are conjecturally the same as ordinary test ideals; this conjecture is known to hold for graded rings and for $\mathbb{Q}$-Gorenstein rings. In particular, we use the theory of *Cartier algebras* introduced in [Sch11]. We introduce a new set of Cartier algebras, called the *diagonal Cartier algebras*, that measure the failure of R to be smooth. These Cartier algebras appear as correction terms in various versions of our subadditivity formula.

This thesis is organized as follows. In Chapter 1 we review the basic terminology of test ideals and Cartier algebras, as well as the process of "reduction modulo p." In Chapter 2 we introduce our subadditivity formula (Theorem 2.2.11) and show this formula yields sharper containments than Takagi's subadditivity formula (Theorem 2.3.1). Chapter 3 is devoted to showing that our subadditivity formula yields sharper containments than (the mod p reduction of) a subadditivity formula for multiplier ideals found by Eisenstein in [Eis10] (Theorem 3.2.2). To get there, we generalize earlier constructions of test ideals and multiplier ideals with good restriction properties (Definition 3.1.3, Definition 3.1.22) found in [Sch09, Tak10, Eis10]. In Chapter 4 we use our new subadditivity formula to generalize the symbolic power containment formulas of [ELS01, HH02] (Theorem 4.2.4, Proposition 4.3.5, Theorem 4.4.1). Finally, in Chapter 5 we provide a combinatorial description of the diagonal Cartier algebras of toric rings (Theorem 5.0.4). We briefly examine the singularities of rings with large diagonal Cartier algebras in Section 5.1.

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Notation and Symbols

$\mathscr{k}$	An F-finite field of characteristic $p > 0$
$F^e_* R$	Restriction of scalars of R via the e-th iterate of Frobenius
$\tau(R)$	Test ideal of a ring R
$\tau(R, \mathscr{C})$	Test ideal of a Cartier algebra, $\mathscr{C}$, on a ring R
$\tau(R, \mathscr{C}, \prod_i \mathfrak{a}_i^{t_i})$	Test ideal of a Cartier R-algebra $\mathscr{C}$, ideals $\mathfrak{a}_i \subseteq R$, and exponents $t_i > 0$
$\mathscr{C}^R$	The full Cartier algebra on a ring R
$\mathscr{C}_e$, $e \in \mathbb{N}$	The degree-e piece of a Cartier algebra $\mathscr{C}$
$\mathscr{C}_\gamma$, $\gamma \in R$	The localization of a Cartier algebra $\mathscr{C}$ at $\gamma \in R$.
$\mathscr{C}\|_{R/I}$	Restriction of a Cartier algebra $\mathscr{C}$ on R to R/I.
$\mathscr{C}^{I\circlearrowleft}$	Subalgebra of maps in $\mathscr{C}$ compatible with an ideal I
$\underline{\mathfrak{a}}^{\lceil t(p^e-1)\rceil}$	Shorthand for $\prod_i \mathfrak{a}_i^{\lceil t_i(p^e-1)\rceil}$
$\mathscr{C}^{\prod_i \mathfrak{a}_i^{t_i}}$	The Cartier algebra given in degree e by $\mathscr{C}_e \cdot F^e_* \underline{\mathfrak{a}}^{\lceil t(p^e-1)\rceil}$
$R^{\otimes_{\mathscr{k}} n}$	Shorthand for $R \otimes_{\mathscr{k}} R \otimes_{\mathscr{k}} \cdots \otimes_{\mathscr{k}} R$ (n-fold tensor product of R over $\mathscr{k}$)
μ_n	The multiplication map $R^{\otimes_{\mathscr{k}} n} \to R$, i.e. $\mu_n(x_1 \otimes_{\mathscr{k}} \cdots \otimes_{\mathscr{k}} x_n) = x_1 \cdots x_n$
μ	Shorthand for μ_2
I_Δ	The kernel of μ_2, i.e. the ideal defining the diagonal in $\mathrm{Spec}(R \otimes_{\mathscr{k}} R)$
$\mathscr{D}^{(n)}(R)$	The n-th diagonal Cartier algebra on a ring R.
$\mathrm{Jac}(S/R)$	The Jacobian ideal of an extension $R \subseteq S$
$\mathrm{Jac}(R)$	Shorthand for $\mathrm{Jac}(R/\mathscr{k})$, when R is a $\mathscr{k}$-algebra
$\tau_I(R, \mathscr{C})$	Test ideal of a Cartier algebra $\mathscr{C}$ along an ideal I
$\mathscr{J}_X(A, Z)$	Multiplier ideal of the pair (A, Z) along a subscheme $X \subseteq A$
$\mathscr{K}(X)$	Fraction field sheaf of an integral scheme X
$\mathrm{Cl}(R)$	Divisor class group of R
$\overline{I}$	The integral closure of an ideal I
$X(\Sigma)$	The toric variety associated to a fan Σ
$\Sigma(1)$	The set of rays (1-dimensional cones) of a fan Σ
$\sigma^\vee$	The dual of a cone σ
v_ρ	The primitive generator of a ray ρ
P_D	The polytope associated to a divisor D on a toric variety
$\mathrm{int}(P)$	The interior of a set P
π_a	The map $k[x_1^{\pm 1/q}, \ldots, x_n^{\pm 1/q}] \to k[x_1^{\pm}, \ldots, x_n^{\pm}]$ associated to $a \in \frac{1}{q}\mathbb{Z}^n$
$s(R)$	The F-signature of a ring R
$s(\mathscr{C})$	The F-signature of a Cartier algebra $\mathscr{C}$

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Chapter 1

Preliminaries on F-singularities

Я ответа не нашёл,
Убедили—я пошёл.

С. В. Михалков

In this section, we provide the basic definitions and results from the theory of test ideals and Cartier algebras. A more complete account of these theories may be found in the surveys, [ST12, BS13]. We also discuss the process of reducing objects (such as rings, modules, and morphisms) from characteristic 0 to positive characteristic.

1.1 Test Ideals and Cartier Algebras

Let R be a Noetherian ring of characteristic $p > 0$. We let $F\colon R \to R$ denote the Frobenius map and let F^e denote the e-th iterate of F. In particular, $F^e(x) = x^{p^e}$ for all $x \in R$. We define $F^e_* R$ to be the R-module given by restriction of scalars via F^e. In other words, $F^e_* R := \{F^e_* r \mid r \in R\}$ as a set, and the R-module structure on $F^e_* R$ is given by $sF^e_* r = F^e_* s^{p^e} r$ for all $r, s \in R$. We give $\operatorname{Hom}_R(F^e_* R, R)$ an $F^e_* R$ module structure defined by pre-multiplication. That is, we define:

$$(F^e_* r \cdot \varphi)(F^e_* x) := \varphi(F^e_*(rx))$$

Similarly, we define a right R-module structure on $\operatorname{Hom}_R(F^e_* R, R)$ by setting $\varphi \cdot r := F^e_* r \cdot \varphi$. The left R-module structure on $\operatorname{Hom}_R(F^e_* R, R)$ is given by post-multiplication, namely $r \cdot \varphi(F^e_* x) := r\varphi(F^e_* x)$. Note that $r \cdot \varphi = \varphi \cdot r^{p^e}$ for all r and φ. Note also that the Frobenius map gives us a map of left R-modules, $F^e\colon R \to F^e_* R$, defined as $x \mapsto F^e_* x^{p^e}$.

Remark 1.1.1. In the literature, people often use the notation R^{1/p^e} instead of $F^e_* R$. If R is a domain, we can identify R^{1/p^e} with the set of (p^e)-th roots of elements of R in a fixed algebraic closure of $\operatorname{frac}(R)$. Even if R is just reduced, the Frobenius map is injective and we have $R \subseteq R^{1/p^e}$.

Definition 1.1.2. A ring R is said to be F-finite if $F^e_* R$ is a finitely-generated R-module for some (equivalently, all) $e > 0$.

For instance, perfect fields are F-finite. Further, any algebra essentially of finite type over an F-finite ring is F-finite.

Global Setting. We will assume in this paper that all of our rings are F-finite and Noetherian.

Perhaps the central idea in the study of so-called "F-singularities" is that the non-regularity of a ring R in positive characteristic can be understood by studying the modules $F^e_* R$. This theory was initiated by the following result of Kunz.

Theorem 1.1.3 ([Kun69])**.** *Let R be Noetherian a ring of positive characteristic. Then R is regular if and only if $F^e_* R$ is a flat R-module for some (equivalently, all) $e > 0$.*

Next we define *test ideals*, which feature prominently throughout this document.

Definition 1.1.4. Let R be a reduced ring of positive characteristic. The *test ideal* of R, denoted $\tau(R)$, is the unique, smallest ideal J not contained in any minimal prime of R such that $\varphi(F^e_* J) \subseteq J$ for all e and all $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$.

Remark 1.1.5. The condition that $\tau(R)$ is not contained in any minimal prime is important for avoiding trivialities. Indeed, a simple calculation shows $\varphi(F^e_* \mathfrak{p}) \subseteq \mathfrak{p}$ for all $e > 0$, all morphisms $\varphi \colon F^e_* R \to R$, and all minimal primes $\mathfrak{p} \subseteq R$. Thus, without that condition, the test ideal of a ring would always be contained in the set of nilpotents of R.

Proving this ideal exists is non-trivial. It should be noted that this definition is technically considered the "big" or "non-finitistic" in the literature. Test ideals were classically defined as "uniform annihilators" of tight closure. The perspective on test ideals given above is due to Schwede [Sch10], which was in turn inspired by the work of Lyubeznik–Smith [LS01] and Hara–Takagi [HT04].

Test ideals can be used to measure the (ir)regularity of a ring. Indeed, whenever R is regular, we have $\tau(R) = R$. The converse is not true, however. Thus, we say that a ring R is F-*regular* if $\tau(R) = R$. One philosophy in the study of F-singularities is that rings with milder singularities have larger test ideals.

This is analogous to the role played by *multiplier ideals* in characteristic-0 birational geometry. Hara and Smith formalized this analogy by showing that, whenever R is $\mathbb{Q}$-Gorenstein and finite type over a field of characteristic 0, the multiplier ideal $\mathscr{J}(R)$ reduces to $\tau(R \mod p)$ modulo p sufficiently large, in the sense of Section 1.2. However, birational geometers have long considered multiplier ideals of *pairs* (X, Z) where X is a variety and Z is a formal $\mathbb{R}$-linear combination of subschemes of R. This leads one to ask whether we can construct a corresponding test ideal associated to a ring R and a formal product of formal powers ideals.

We can construct such variants of the test ideal by restricting the set of maps φ under consideration. For instance, consider a pair $(R, \mathfrak{a}^t)$ where $\mathfrak{a}$ is an ideal of R and $t \geq 0$ is a real number. We define the test ideal of this data as follows.

Definition 1.1.6 ([Sch10])**.** Let R be reduced, $\mathfrak{a} \subseteq R$ an ideal not contained in any minimal prime of R, and t a positive real number. Then we define $\tau(R, \mathfrak{a}^t)$ to be the unique, minimal ideal J not contained in any minimal prime of R such that $\varphi(F^e_* J) \subseteq J$ for all e and all $\varphi \in F^e_* \left(\mathfrak{a}^{\lceil t(p^e-1)\rceil}\right) \operatorname{Hom}_R(F^e_* R, R)$.

Note that we don't actually define the power $\mathfrak{a}^t$ for arbitrary real numbers t. However, this formal notation for the test ideal of a pair turns out to be useful. Indeed, the following lemma shows that the notation $\mathfrak{a}^t$ is meaningful:

Lemma 1.1.7 (Unambiguity of the exponent)**.** *Let R be reduced, $\mathfrak{a} \subseteq R$ an ideal not contained in a minimal prime of R, and t a positive real number. Then $\tau(R, (\mathfrak{a}^n)^t) = \tau(R, \mathfrak{a}^{nt})$ for all integers $n \geq 0$.*

When R is regular, the test ideals on R enjoy the following property known as *subadditivity*.

Theorem 1.1.8 ([HY03])**.** *Let R be a regular ring of positive characteristic, $\mathfrak{a}, \mathfrak{b} \subseteq R$ ideals, and s, t positive real numbers. Then $\tau(R, \mathfrak{a}^s\mathfrak{b}^t) \subseteq \tau(R, \mathfrak{a}^s)\tau(R, \mathfrak{b}^t)$.*

We will discuss a variant of this subadditivity property that holds more generally in Chapter 2.

People have also considered test ideals of triples $(R, \mathfrak{a}^t, \Delta)$, where one further restricts the maps φ under consideration using a divisor Δ on $\operatorname{Spec} R$. This leads one to ask, more generally, for which sets of maps $F^e_* R \to R$ can one define the test ideal of that set? The answer is that this set of maps must form a *Cartier algebra*.

Definition 1.1.9 ([Sch11])**.** A *Cartier algebra* on R is an additive Abelian group $\mathscr{C} = \bigoplus_e \mathscr{C}_e$, with $\mathscr{C}_e \subseteq \operatorname{Hom}_R(F^e_* R, R)$ for all e, that is closed under multiplication on the left and right by elements in R and closed under composition. In other words, given $\varphi_1, \varphi_2 \in \mathscr{C}_e$, $\psi \in \mathscr{C}_d$, and $r \in R$, we have: $\varphi_1 + \varphi_2 \in \mathscr{C}_e$, $r \cdot \varphi \in \mathscr{C}_e$, $\varphi \cdot r \in \mathscr{C}_e$, and $\varphi \cdot \psi := \varphi \circ (F^e_* \psi) \in \mathscr{C}_{d+e}$. By convention, we also assume that $\mathscr{C}_0 = R$. The *full Cartier algebra* on R is the Cartier algebra $\mathscr{C}^R := \bigoplus_{e \geq 0} \operatorname{Hom}_R(F^e_* R, R)$.

Here, $F^e_*\psi$ denotes the map $F^{e+d}_*R \to F^e_*R$ given by

$$(F^e_*\psi)\left(F^{e+d}_*x\right) := F^e_*\psi(F^d_*x)$$

We note that Cartier algebras are typically not commutative rings. Further, they're not necessarily R-algebras, in the sense that R is typically not in the center of a given Cartier algebra.

Let $\Psi = \sum_i \psi_i \in \mathscr{C}$, where the sum is finite, each ψ_i is nonzero, and $\psi_i \in \mathscr{C}_{e_i}$ for each i. Then R has a natural left $\mathscr{C}$-module structure given by

$$\Psi \cdot r := \sum_i \psi_i\left(F^{e_i}_* r\right)$$

for each $r \in R$. Further, we say Ψ has *minimal degree* e_0 if $e_0 = \min_i\{e_i\}$. The reader should be warned that this terminology is not standard.

If $\mathscr{C}$ is a Cartier algebra on a reduced ring R, then the test ideal $\tau(R,\mathscr{C})$, if it exists, is defined to be the smallest ideal $J \subseteq R$ not contained in any minimal prime of R such that $\varphi(F^e_*J) \subseteq J$ for all e and for all $\varphi \in \mathscr{C}_e$. Schwede showed in [Sch11] that these test ideals exist whenever $\mathscr{C}$ is *non-degenerate*[1]. This is a very weak condition. For instance, if R is a domain, then a Cartier algebra $\mathscr{C}$ on R is non-degenerate whenever $\mathscr{C}_e \neq 0$ for some $e > 0$.

Remark 1.1.10. Blickle generalized the definition of Cartier R-algebras so that they're not always contained in the full Cartier algebra $\mathscr{C}^R$, and developed a theory of modules over these algebras, called Cartier modules, in [Bli13]. Given a fixed Cartier algebra $\mathscr{C}$, he and Stäbler showed how to think of τ as an additive subfunctor of the forgetful functor from $\mathscr{C}$-mod to R-mod [BS19]. We will not need this level of generality in this thesis, so we omit further discussion of Blickle–Stäbler's beautiful theory, and all of our Cartier algebras will be subalgebras of $\mathscr{C}^R$.

We can also define the test ideal $\tau(R,\mathscr{C},\prod_i \mathfrak{a}_i^{t_i})$ for any nonzero ideals $\mathfrak{a}_i \subseteq R$ and real numbers $t_i \geq 0$: this is the smallest ideal $J \subseteq R$ not contained in any minimal prime of R such that $\varphi(F^e_*J) \subseteq J$ for all e and for all $\varphi \in F^e_*\left(\prod_i \mathfrak{a}_i^{\lceil t_i(p^e-1)\rceil}\right)\mathscr{C}_e$. As before, this product $\prod_i \mathfrak{a}_i^{t_i}$ is just formal notation.

Notation 1.1.11. Given a Cartier algebra $\mathscr{C}$ on R, a collection of nonzero ideals $\mathfrak{a}_i \subseteq R$, and rational numbers $t_i \geq 0$, we define

$$\mathscr{C}^{R,\prod_i \mathfrak{a}_i^{t_i}} := \bigoplus_{e\geq 0} F^e_*\left(\prod_i \mathfrak{a}_i^{\lceil t_i(p^e-1)\rceil}\right)\mathscr{C}_e.$$

[1] A map $\varphi\colon F^e_*R \to R$ is called non-degenerate if $\varphi(F^e_*R)R_\eta \neq 0$ for all minimal primes $\eta \in \operatorname{Spec} R$. A Cartier algebra $\mathscr{C}$ is called non-degenerate if $\mathscr{C}_e$ contains a non-degenerate map for some $e > 0$.

Further, we define $\tau(R, \prod_i \mathfrak{a}_i^{t_i}) := \tau(R, \mathscr{C}^R, \prod_i \mathfrak{a}_i^{t_i})$. In other words, when we omit $\mathscr{C}$ from the test ideal notation, $\mathscr{C}$ is understood to be the full Cartier algebra $\mathscr{C}^R$. Note that $\tau(R, \mathscr{C}, \prod_i \mathfrak{a}_i^{t_i}) = \tau\left(R, \mathscr{C}^{R, \prod_i \mathfrak{a}_i^{t_i}}\right)$. We will stick to the first notation.

Theorem 1.1.12 (*Cf.* [Sch11])**.** *Let R be reduced, $\mathscr{C}$ a non-degenerate Cartier algebra on R, and $c \in \tau(R, \mathscr{C})$ an element not in any minimal prime of R. Then*

$$\tau(R, \mathscr{C}) = \sum_{e \geq 0} \sum_{\varphi \in \mathscr{C}_e} \varphi\left(F_*^e c\right).$$

Given any map $\varphi \colon F_*^e R \to R$ and any multiplicative set $W \subseteq R$, we get an induced map $W^{-1}\varphi \colon W^{-1}F_*^e R \to W^{-1}R$. Note that $W^{-1}F_*^e R = F_*^e\left(W^{-1}R\right)$. Thus, any Cartier algebra $\mathscr{C}$ on R induces a Cartier algebra $W^{-1}\mathscr{C}$ on $W^{-1}R$.

Notation 1.1.13. Let $\mathscr{C}$ be a Cartier algebra on R and $\gamma \in R$. Then $\mathscr{C}$ induces a Cartier algebra on the localization R_γ. We denote this induced Cartier algebra by $\mathscr{C}_\gamma$.

An important notion in the study of test ideals is that of *compatibility* of ideals and Cartier algebras.

Definition 1.1.14. Let R be a reduced ring, J an ideal of R, and let $\varphi \colon F_*^e R \to R$. We say that J is *compatible* with φ, or *φ-compatible*, or that φ is compatible with J, if $\varphi(F_*^e J) \subseteq J$. Let $\mathscr{D}$ be a Cartier algebra on R. Similarly, we say that J is compatible with $\mathscr{D}$, or $\mathscr{D}$-compatible, or that $\mathscr{D}$ is compatible with J, if J is compatible with each map in $\mathscr{D}$. Thus, $\tau(R, \mathscr{D})$ can be defined as the smallest $\mathscr{D}$-compatible ideal not contained in a minimal prime of R.

Finally, we collect some well-known and useful properties of test ideals.

Lemma 1.1.15 (*Cf.* [HH94, HY03, Sch11])**.** *Let R be a reduced F-finite ring.*

(a) (Monotonicity) Let $\mathscr{C} \subseteq \mathscr{D}$ be two non-degenerate Cartier algebras on R. Then $\tau(R, \mathscr{C}) \subseteq \tau(R, \mathscr{D})$.

(b) (Unambiguity) Let R be reduced, $\mathfrak{a} \subseteq R$ an ideal not contained in a minimal prime of R, and t a positive real number, and $\mathscr{C}$ a Cartier algebra on R. Then $\tau(R, \mathscr{C}, (\mathfrak{a}^n)^t) = \tau(R, \mathscr{C}, \mathfrak{a}^{nt})$ for all integers $n \geq 0$.

(c) (Integral closures) Let $\mathfrak{a}_i$ be a collection of ideals of R and $t_i \geq 0$ a collection of rational numbers. For each i, let $\mathfrak{b}_i = \overline{\mathfrak{a}}_i$ be the integral closure of $\mathfrak{a}_i$. Then $\tau(R, \prod_i \mathfrak{a}_i^{t_i}) = \tau(R, \prod_i \mathfrak{b}_i^{t_i})$.

(d) *(Localization) Let $W \subseteq R$ be a multiplicative set consisting of regular elements. Then* $W^{-1}\tau(R, \mathscr{C}) = \tau(W^{-1}R, W^{-1}\mathscr{C})$.

(e) *The Cartier algebra $\mathscr{C}^R$ is non-degenerate.*

1.2 Reduction Modulo p

Here we briefly review the process of reducing rings and schemes modulo p, which we need for Section 3.2. Our reference is [HH99, Chapter 2]. See also [Tak13, Section 2] for another succinct treatment of this material.

Let R be an algebra essentially of finite type over a field ℓ of characteristic 0. We can find a finitely generated $\mathbb{Z}$-subalgebra B of ℓ and a B-subalgebra R_B of R such that the inclusion $R_B \subseteq R$ induces an isomorphism $R = R_B \otimes_B \ell$. This algebra B is called *descent datum* for R, and the algebra R_B is called a *model* for R over B. For instance, if

$$R = \frac{\mathbb{C}[x_1, \dots, x_n]}{(f_1, \dots, f_m)},$$

we can choose B to be the $\mathbb{Z}$-subalgebra of $\mathbb{C}$ generated by all of the coefficients of all of the polynomials f_i, and we can set

$$R_B = \frac{B[x_1, \dots, x_n]}{(f_1, \dots, f_m)}.$$

For any maximal ideal $\mathfrak{m} \subseteq B$, the residue field $B/\mathfrak{m}$ will have positive characteristic p. Then $R_{\kappa(\mathfrak{m})} := R_B \otimes_B B/\mathfrak{m}$ is a *mod p reduction* of R. More specifically, we call $R_{\kappa(\mathfrak{m})}$ the *mod p reduction of R at* $\mathfrak{m}$. For any finitely generated R-module M, we can find a model M_B for M over B in the same manner. The choice of descent datum B is not unique. However, given two descent data B and B', we can find a third descent datum C containing B and B', such that $R_B \otimes_B C = R_{B'} \otimes_{B'} C$. Given any descent datum B, many properties of R will be preserved by $R_{\kappa(\mathfrak{m})}$ for $\mathfrak{m}$ sufficiently general, that is, for all $\mathfrak{m}$ in a dense open subset of $\operatorname{MaxSpec} B$. For instance, if R is regular, then so is $R_{\kappa(\mathfrak{m})}$ for all $\mathfrak{m}$ sufficiently general.

Note that, for all $N \in \mathbb{Z}$, the subset of $\mathfrak{m}$ in $\operatorname{MaxSpec} B$ such that $\operatorname{char} \kappa(\mathfrak{m}) > N$ forms a dense open set. Thus we have $\operatorname{char} R_{\kappa(\mathfrak{m})} \gg 0$ for $\mathfrak{m}$ sufficiently general.

It will be useful for us later to note that reduction modulo p commutes with tensor products, in the following sense. If B is a descent datum for R, then B is also a descent datum for $R \otimes_\ell R$, and we have $R_B \otimes_B R_B = (R \otimes_\ell R)_B$. Then we compute:

$$\begin{aligned} R_{\kappa(\mathfrak{m})} \otimes_{\kappa(\mathfrak{m})} R_{\kappa(\mathfrak{m})} &= R_B \otimes_B \kappa(\mathfrak{m}) \otimes_{\kappa(\mathfrak{m})} R_B \otimes_B \kappa(\mathfrak{m}) \\ &= R_B \otimes_B R_B \otimes_B \kappa(\mathfrak{m}) = (R \otimes_\ell R)_{\kappa(\mathfrak{m})} \end{aligned}$$

An important tool in the study of reduction modulo p is the *generic freeness lemma* [HH99, (2.1.4)]. By this lemma, we can always enlarge our choice of descent datum B to ensure that, for any finite collection $\{M_i\}$ of finitely generated R-modules, the models $(M_i)_B$ will be free B-modules.

Given any map $\phi\colon M \to M'$ of finitely generated R-modules, we can find a suitable descent datum B so that $\phi(M_B) \subseteq M'_B$. Then we say $\phi_B \coloneqq \phi\big|_{M_B}$ is a model for ϕ, and we have $\phi_B \otimes_B \ell = \phi$. Given a bounded exact sequence of finitely generated R-modules, we can choose our descent datum B so that the models of these maps over B form an exact sequence of B-modules.

Similarly, given any scheme X of finite type over ℓ we can find a descent datum $B \subseteq \ell$ and a B-scheme X_B of finite type such that $X = X_B \times_{\operatorname{Spec} B} \operatorname{Spec} \ell$. We can perform a similar construction for coherent sheaves on X, morphisms $f\colon Y \to X$, and divisors on X. Given any closed point $\mu \in \operatorname{Spec} B$, the residue field $\kappa(\mu)$ will have positive characteristic. We call the fiber $X_{\kappa(\mu)} \coloneqq (X_B)_\mu$ a *mod* p *reduction* of X, where $p = \operatorname{char}(\kappa(\mu))$. Given a morphism of k-schemes $f\colon Y \to X$, we get an induced morphism $f_{\kappa(\mu)}\colon Y_{\kappa(\mu)} \to X_{\kappa(\mu)}$. If f is projective, then so is $f_{\kappa(\mu)}$ for all μ sufficiently general. If K_X is the canonical divisor of X, then $(K_X)_{\kappa(\mu)}$ is the canonical divisor of $X_{\kappa(\mu)}$ for all μ sufficiently general.

Chapter 2

Subadditivity and Diagonal Cartier Algebras

I think it's bad to have too good a memory if you want to be a mathematician.

Sir Andrew Wiles

In this chapter we discuss the subadditivity property of test ideals and introduce the main characters of this thesis, the *Diagonal Cartier algebras*, denoted $\mathscr{D}^{(n)}(R)$. As we explain in Section 2.1, the original subadditivity formula of test ideals holds only in the regular case. These diagonal Cartier algebras are our main tools for writing new subadditivity formulas for $\mathcal{k}$-algebras essentially of finite type, where $\mathcal{k}$ is a perfect field of positive characteristic, and they may be thought of as correction terms that account for the singularity of the $\mathcal{k}$-algebra in question. In Section 2.2, we explain the derivation of these new subadditivity formulas. Finally, in Section 2.3, we show that the subadditivity containments of the preceding section are sharper than the subadditivity formulas found by Takagi in [Tak06].

2.1 Introduction

The subadditivity theorem for test ideals says the following:

Theorem 2.1.1 ([HY03]). *Let R be a complete regular local*[1] *ring of positive characteristic, and let $\mathfrak{a}, \mathfrak{b} \subseteq R$ be ideals. Then $\tau(R, \mathfrak{a}^s\mathfrak{b}^t) \subseteq \tau(R, \mathfrak{a}^s)\tau(R, \mathfrak{b}^t)$.*

As (big) test ideals commute with localization and completion, the result follows immediately for all Noetherian regular rings R. This theorem was first shown in the characteristic 0 context (for multiplier ideals) by Demailly–Ein–Lazarsfeld [DEL00]. Subsequently, Hara and Yoshida defined the test ideal of the data $(R, \mathfrak{a}^s\mathfrak{b}^t)$ and proved subadditivity in an analogous way [HY03]. The argument goes as follows. For simplicity we assume $s = t = 1$.

[1] We define local rings to be *Noetherian* rings with unique maximal ideal.

Let ℓ be the residue field of R. Let $T = R\widehat{\otimes}_\ell R$. One starts by showing

$$\tau(T, (\mathfrak{a} \otimes_\ell \mathfrak{b})T) \subseteq (\tau(R, \mathfrak{a}) \otimes_\ell \tau(R, \mathfrak{b}))T.$$

Applying the multiplication map $\mu\colon T \to R$, induced by $\mu(x \otimes y) = xy$, we get $\mu(\tau(T, (\mathfrak{a} \otimes_\ell \mathfrak{b})T)) \subseteq \tau(R, \mathfrak{a})\tau(R, \mathfrak{b})$. By the Cohen structure theorem, we know T is smooth over ℓ, and so the kernel of μ is generated by a regular sequence. We conclude the proof using the following restriction theorem:

Theorem 2.1.2 ([HY03]). *Let $(R, \mathfrak{m})$ be a normal complete $\mathbb{Q}$-Gorenstein local ring of positive characteristic and let $x \in \mathfrak{m}$ be a non-zerodivisor of R. Let $S = R/xR$ and suppose S is normal. Then for any ideal $\mathfrak{a} \subseteq R$ we have $\tau(S, \mathfrak{a}S) \subseteq \tau(R, \mathfrak{a})S$.*

Evidently, the place where this proof breaks down for the non-regular case is in this restriction theorem. Our approach is to come up with a new restriction theorem of (big) test ideals using the language of Cartier algebras. Namely, we find a Cartier algebra $\mathscr{D}^{(2)}$ on A so that

$$\tau(A, \mathscr{D}^{(2)}, \mu(\mathfrak{c})) \subseteq \mu(\tau(A \otimes_{\Bbbk} A, \mathfrak{c})),$$

whenever A is a ring essentially of finite type over a perfect field $\Bbbk$ and $\mathfrak{c} \subseteq A \otimes_{\Bbbk} A$ is an ideal, where $\mu\colon R \otimes_{\Bbbk} R \to R$ is the multiplication map. This Cartier algebra $\mathscr{D}^{(2)}$ is a correction term that witnesses the non-smoothness of R. The fact that $\mathscr{D}^{(2)}$ is trivial (in the sense that it equals $\mathscr{C}^R$) when R is smooth will follow quickly from Kunz's theorem.

Earlier work of Takagi [Tak06] shows that, when R is equidimensional, one can use the Jacobian ideal $\operatorname{Jac}(R)$ as a correction term as well, in that one has $\operatorname{Jac}(R)\tau(R, \mathfrak{a}\mathfrak{b}) \subseteq \tau(R, \mathfrak{a})\tau(R, \mathfrak{b})$. We show in Theorem 2.3.1 that our subadditivity formula is sharper than Takagi's, namely

$$\operatorname{Jac}(R)\tau(R, \mathfrak{a}\mathfrak{b}) \subseteq \tau(R, \mathscr{D}^{(2)}, \mathfrak{a}\mathfrak{b}).$$

An important application of the subadditivity formula of multiplier/test ideals is Ein–Lazarsfeld–Smith's celebrated theorem on the *uniform* equivalence of the adic and symbolic topologies of ideals in regular rings. We show in Chapter 4 that this new subadditivity formula can be used to apply Ein–Lazarsfeld–Smith's method in broader generality, thereby finding new rings with the *uniform symbolic topology property*.

2.2 Diagonal Cartier Algebras

Let R be a Noetherian ring in characteristic p and $I \subseteq R$ an ideal. Then we have $F^e_*(R/I) = F^e_*R/F^e_*I$. Thus, each map $\varphi\colon F^e_*R \to R$ satisfying $\varphi(F^e_*I) \subseteq I$ induces a map

$$\overline{\varphi}\colon F^e_*(R/I) \to R/I.$$

We can do something similar for Cartier algebras:

Definition 2.2.1. Let $\mathscr{D}$ be a Cartier algebra on R compatible with an ideal $I \subseteq R$. We define the *restriction of $\mathscr{D}$ to R/I*, denoted $\mathscr{D}|_{R/I}$, to be the Cartier algebra $\bigoplus_{e\geq 0} \mathscr{D}_e|_{R/I}$, where

$$\mathscr{D}_e|_{R/I} \coloneqq \{\overline{\varphi}\colon F^e_*(R/I) \to R/I \mid \varphi \in \mathscr{D}_e\}.$$

Proposition 2.2.2. *Let $\mathscr{D}$ be a Cartier algebra on R compatible with an ideal $I \subseteq R$. Then $\mathscr{D}|_{R/I}$ is a Cartier algebra on R/I.*

Proof. Let $\varphi_1, \varphi_2 \in \mathscr{D}_d|_{R/I}$, $\psi \in \mathscr{D}_e|_{R/I}$, and $r \in R/I$. Then there exist some $\varphi_1', \varphi_2' \in \mathscr{D}_d$ with $\overline{\varphi_1'} = \varphi_1$ and $\overline{\varphi_2'} = \varphi_2$. As $\mathscr{D}$ is a Cartier algebra, we have $\varphi_1' + \varphi_2' \in \mathscr{D}_d$, and we see that

$$\varphi_1 + \varphi_2 = \overline{\varphi_1' + \varphi_2'} \in \mathscr{D}_d|_{R/I}.$$

A similar argument shows that $r\psi, \psi r \in \mathscr{D}_e|_{R/I}$ and $\varphi_1 \circ F^d_*\psi \in \mathscr{D}_{d+e}|_{R/I}$. It follows from the definitions that $\mathscr{D}_0|_{R/I} = R/I$ provided that $\mathscr{D}_0 = R$. □

We define another useful operation on Cartier algebras.

Definition 2.2.3. Let $\mathscr{C}$ be a Cartier algebra on R and let $I \subseteq R$ be an ideal. We define the *subalgebra of maps compatible with I*, denoted $\mathscr{C}^{I\circlearrowleft}$, to be the set of maps $\bigoplus_{e\geq 0} \mathscr{C}_e^{I\circlearrowleft}$, where

$$\mathscr{C}_e^{I\circlearrowleft} \coloneqq \{\varphi \mid \varphi \in \mathscr{C}_e,\ \varphi(F^e_* I) \subseteq I\}.$$

Proposition 2.2.4. *Let $\mathscr{C}$ be a Cartier algebra on R and $I \subseteq R$ an ideal. Then $\mathscr{C}^{I\circlearrowleft}$ is a Cartier algebra.*

Proof. Suppose $\varphi \in \mathscr{C}_e$ and $\psi \in \mathscr{C}_d$ are two maps satisfying $\varphi\left(F^e_* I\right) \subseteq I$ and $\psi\left(F^d_* I\right) \subseteq I$. Then it's clear we have $x\varphi(F^e_* I) \subseteq I$ and $\varphi(F^e_* xI) \subseteq I$ for all $x \in R$. It's also clear that $\varphi(F^e_* I) + \psi(F^d_* I) \subseteq I$. Further,

$$\varphi \circ F^e_*\psi\left(F^{e+d}_* I\right) \subseteq \varphi\left(F^e_* I\right) \subseteq I.$$

Finally, note that $\mathscr{C}_0^{I\circlearrowleft} = R$ whenever $\mathscr{C}_0 = R$. □

The next lemma will be the key ingredient in proving our new subadditivity formula.

Lemma 2.2.5. *For any reduced ring R, Cartier algebra $\mathscr{C}$ on R, and radical ideal $I \subseteq R$ we have*

$$\tau\left(R/I, \mathscr{C}^{I\circlearrowleft}|_{R/I}\right) \subseteq \tau\left(R, \mathscr{C}\right) R/I,$$

provided that the right-hand side is not contained in any minimal prime of R/I and the test ideal on the left, $\tau\left(R/I, \mathscr{C}^{I\circlearrowleft}|_{R/I}\right)$, exists.

Proof. Let $\varphi \in \mathscr{C}_e|_{R/I}$. By definition there exists a lifting $\widehat{\varphi} \in \mathscr{C}_e$, so that this diagram commutes:

$$\begin{array}{ccc} F^e_* R & \xrightarrow{\widehat{\varphi}} & R \\ {\scriptstyle F^e_*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ F^e_*(R/I) & \xrightarrow{\varphi} & R/I \end{array}$$

This means that

$$\varphi\left(F^e_* \tau\left(R, \mathscr{C}\right) R/I\right) = \widehat{\varphi}\left(F^e_* \tau\left(R, \mathscr{C}\right)\right) R/I$$

By definition of $\tau\left(R, \mathscr{C}\right)$, we see the right hand side is contained in $\tau\left(R, \mathscr{C}\right) R/I$. Then we are done by the minimality of $\tau\left(R/I, \mathscr{C}|_{R/I}\right)$. Note that R/I is reduced as I is radical. □

Note that the above lemma does not apply when $\mathscr{C}$ is compatible with I, for then we would have $\tau(R, \mathscr{C}) \subseteq I$. This motivates Definition 3.1.3, *Cf.* Proposition 3.1.14.

Proposition 2.2.6 (Restriction theorem of test ideals)**.** *For all reduced rings R, Cartier algebras $\mathscr{C}$ on R, formal products $\prod_i \mathfrak{a}_i^{t_i}$ of ideals on R, and radical ideals $I \subseteq R$, we have*

$$\tau\left(R/I, \mathscr{C}^{I\circlearrowleft}|_{R/I}, \prod_i (\mathfrak{a}_i R/I)^{t_i}\right) \subseteq \tau\left(R, \mathscr{C}, \prod_i \mathfrak{a}_i^{t_i}\right) R/I,$$

provided that the right-hand side contains a regular element of R/I and the test ideal on the left-hand side exists.

Proof. Let $\mathscr{D}$ be a Cartier algebra on R compatible with I. Then we have

$$\left(\mathscr{D}|_{R/I}\right)^{\prod_i (\mathfrak{a}_i R/I)^{t_i}} \subseteq \left(\mathscr{D}^{\prod_i \mathfrak{a}_i^{t_i}}\right)\Big|_{R/I}.$$

It follows that

$$\tau\left(R/I, \mathscr{C}^{I\circlearrowleft}|_{R/I}, \prod_i (\mathfrak{a}_i R/I)^{t_i}\right) \subseteq \tau\left(R/I, \left(\mathscr{C}^{I\circlearrowleft}\right)^{\prod_i \mathfrak{a}_i^{t_i}}\Big|_{R/I}\right),$$

by Lemma 1.1.15. Similarly, we note that

$$\left(\mathscr{C}^{I\circlearrowleft}\right)^{\prod_i \mathfrak{a}_i^{t_i}} \subseteq \left(\mathscr{C}^{\prod_i \mathfrak{a}_i^{t_i}}\right)^{I\circlearrowleft}.$$

Then we get

$$\tau\left(R/I, \left(\mathscr{C}^{I\circlearrowleft}\right)^{\prod_i \mathfrak{a}_i^{t_i}}\Big|_{R/I}\right) \subseteq \tau\left(R/I, \left(\mathscr{C}^{\prod_i \mathfrak{a}_i^{t_i}}\right)^{I\circlearrowleft}\Big|_{R/I}\right) \subseteq \tau\left(R, \mathscr{C}, \prod_i \mathfrak{a}_i^{t_i}\right) R/I,$$

where the second containment follows from Lemma 2.2.5. □

We obtain our subadditivity formula by applying Proposition 2.2.6 to the cases where we consider the ideal $I_\Delta := \ker(R \otimes_k R \xrightarrow{\mu} R)$, or more generally, $\ker(R^{\otimes_{\mathscr{k}} n} \to R)$. First, we introduce some notation which will be used throughout the rest of this thesis:

Notation 2.2.7 (Diagonal Cartier algebras)**.** Let R be a $\mathscr{k}$-algebra essentially of finite type, where $\mathscr{k}$ is a field of positive characteristic.

- $\mu_n \colon R^{\otimes_{\mathscr{k}} n} \to R$ is the map given by $x_1 \otimes \cdots \otimes x_n \mapsto x_1 \cdots x_n$. We will use the abbreviation $\mu := \mu_2$.
- $I_\Delta \subseteq R \otimes_{\mathscr{k}} R$ denotes the kernel of μ_2. Geometrically, I_Δ cuts out the diagonal embedding $\operatorname{Spec} R \subseteq \operatorname{Spec} R \times_{\operatorname{Spec} \mathscr{k}} \operatorname{Spec} R$. Note that we can express I_Δ in terms of generators as $I_\Delta = \langle x \otimes 1 - 1 \otimes x \mid x \in R \rangle$.
- We let $\mathscr{C}^{R \otimes_{\mathscr{k}} R,\, I_\Delta \circlearrowleft} := (\mathscr{C}^{R \otimes_{\mathscr{k}} R})^{I_\Delta \circlearrowleft}$ denote the Cartier algebra on $R \otimes_{\mathscr{k}} R$ of all maps compatible with I_Δ. We say that such maps are *compatible with the diagonal*.
- We define the *n-th diagonal Cartier algebra on R* to be

$$\mathscr{D}^{(n)}(R) := \left(\mathscr{C}^{R^{\otimes_{\mathscr{k}} n}}\right)^{\ker(\mu_n) \circlearrowleft} \Big|_{(R^{\otimes_{\mathscr{k}} n})/\ker(\mu_n)}.$$

  If the ring R is understood from context, we will denote this Cartier algebra simply as $\mathscr{D}^{(n)}$.

Remark 2.2.8. In particular, $\mathscr{D}_e^{(n)}(R)$ is the set of maps $\varphi \colon F^e_* R \to R$ that admit a lifting $\widehat{\varphi}$ on the tensor product $R^{\otimes_{\mathscr{k}} n}$:

$$\begin{array}{ccc} F^e_*\left(R^{\otimes_{\mathscr{k}} n}\right) & \overset{\widehat{\varphi}}{\dashrightarrow} & R^{\otimes_{\mathscr{k}} n} \\ {\scriptstyle F^e_*\mu_n}\big\downarrow & & \big\downarrow{\scriptstyle \mu_n} \\ F^e_* R & \xrightarrow{\ \varphi\ } & R \end{array}$$

We now have the necessary definitions to state our new subadditivity theorem. Before we may proceed, however, we must recall a general fact about modules:

Lemma 2.2.9. *Let R and S be commutative algebras over a field $\mathscr{k}$. Let M and N be R-modules and let U and V be S-modules. Suppose also that M and U are finitely presented over their respective rings. Then the canonical map:*

$$\Theta \colon \operatorname{Hom}_R(M, N) \otimes_{\mathscr{k}} \operatorname{Hom}_S(U, V) \to \operatorname{Hom}_{R \otimes_{\mathscr{k}} S}(M \otimes_{\mathscr{k}} U, N \otimes_{\mathscr{k}} V)$$

is an isomorphism.

Proof. We have the following chain of natural isomorphisms:

$$\begin{aligned}
\operatorname{Hom}_R(M,N)\otimes_{\mathscr{k}}\operatorname{Hom}_S(U,V) &\cong \operatorname{Hom}_R\left(M, N\otimes_{\mathscr{k}}\operatorname{Hom}_S(U,V)\right) && (2.1)\\
&\cong \operatorname{Hom}_R\left(M,\operatorname{Hom}_S(U,N\otimes_{\mathscr{k}} V)\right) && (2.2)\\
&\cong \operatorname{Hom}_R\left(M,\operatorname{Hom}_S(U,\operatorname{Hom}_{R\otimes_{\mathscr{k}} S}(R\otimes_{\mathscr{k}} S, N\otimes_{\mathscr{k}} V))\right) && (2.3)\\
&\cong \operatorname{Hom}_R\left(M,\operatorname{Hom}_{R\otimes_{\mathscr{k}} S}\left(R\otimes_{\mathscr{k}} U, N\otimes_{\mathscr{k}} V\right)\right) && (2.4)\\
&\cong \operatorname{Hom}_{R\otimes_{\mathscr{k}} S}\left(M\otimes_R R\otimes_{\mathscr{k}} U, N\otimes_{\mathscr{k}} V\right) && (2.5)\\
&\cong \operatorname{Hom}_{R\otimes_{\mathscr{k}} S}\left(M\otimes_{\mathscr{k}} U, N\otimes_{\mathscr{k}} V\right) && (2.6)
\end{aligned}$$

The isomorphism in (2.1) follows from the facts that M is finitely presented and $\operatorname{Hom}_S(U,V)$ is a flat $\mathscr{k}$-module (*Cf.* [Lan05, Chapter XVI, Exercise 11]). The isomorphism in (2.2) follows by the same argument. The isomorphisms in (2.4) and (2.5) follow from Hom-Tensor adjunction. □

Corollary 2.2.10. *Let $\mathscr{k}$ be an F-finite field of characteristic p and let R be a $\mathscr{k}$-algebra essentially of finite type. Then we have a natural inclusion,*

$$\operatorname{Hom}_{R^{\otimes n}}\left(F^e_*\left(R^{\otimes_{\mathscr{k}} n}\right), R^{\otimes_{\mathscr{k}} n}\right) \subseteq \operatorname{Hom}_R(F^e_* R, R)^{\otimes_{\mathscr{k}} n},$$

which is an isomorphism if $\mathscr{k}$ is perfect.

Proof. Note that we have a canonical surjection

$$F^e_* R\otimes_{\mathscr{k}}\cdots\otimes_{\mathscr{k}} F^e_* R \to F^e_* R\otimes_{F^e_*\mathscr{k}}\cdots\otimes_{F^e_*\mathscr{k}} F^e_* R = F^e_*\left(R\otimes_{\mathscr{k}}\cdots\otimes_{\mathscr{k}} R\right).$$

If $\mathscr{k}$ happens to be perfect, then $F^e_*\mathscr{k} = \mathscr{k}$ as $\mathscr{k}$-vector spaces, and the above map is an isomorphism. Thus we have an inclusion

$$\operatorname{Hom}_{R^{\otimes n}}\left(F^e_*\left(R^{\otimes_{\mathscr{k}} n}\right), R^{\otimes_{\mathscr{k}} n}\right) \subseteq \operatorname{Hom}_{R^{\otimes n}}\left((F^e_* R)^{\otimes_{\mathscr{k}} n}, R^{\otimes_{\mathscr{k}} n}\right).$$

As R is F-finite and Noetherian, we can repeatedly use Lemma 2.2.9 to see that

$$\operatorname{Hom}_{R^{\otimes n}}\left((F^e_* R)^{\otimes_{\mathscr{k}} n}, R^{\otimes_{\mathscr{k}} n}\right) = \operatorname{Hom}_R(F^e_* R, R)^{\otimes_{\mathscr{k}} n}.$$

□

Theorem 2.2.11 (Subadditivity using $\mathscr{D}^{(n)}$)**.** *Let $\mathscr{k}$ be a perfect field of positive characteristic and let R be a reduced $\mathscr{k}$-algebra essentially of finite type. Then*

$$\tau\left(R, \mathscr{D}^{(n)}, \prod_{i=1}^n \mathfrak{a}_i^{t_i}\right) \subseteq \prod_{i=1}^n \tau(R, \mathfrak{a}_i^{t_i})$$

for all ideals $\mathfrak{a}_i \subseteq R$ and real numbers $t_i \geq 0$, provided that none of the ideals $\mathfrak{a}_i$ is contained in a minimal prime of R.

Proof. From Corollary 2.2.10 it's easy to see that

$$\bigotimes_{i=1}^{n}{}_{\mathscr{k}}\, \tau(R, \mathfrak{a}_i^{t_i})$$

is compatible with every map in

$$\operatorname{Hom}_{R^{\otimes n}}\left(F_*^e\left(R^{\otimes_{\mathscr{k}} n}\right), R^{\otimes_{\mathscr{k}} n}\right) F_*^e\left(\bigotimes_{i=1}^{n}{}_{\mathscr{k}}\, \mathfrak{a}_i^{\lceil t_i(p^e-1)\rceil}\right).$$

For all $1 \le i, j \le n$, we set $A_{i,j} = \mathfrak{a}_i$ when $i = j$ and $A_{i,j} = R$ when $i \neq j$. It follows that

$$\tau\left(R^{\otimes_{\mathscr{k}} n}, \prod_{i=1}^{n}(A_{1,i} \otimes \cdots \otimes A_{n,i})^{t_i}\right) \subseteq \bigotimes_{i=1}^{n}{}_{\mathscr{k}}\, \tau(R, \mathfrak{a}_i^{t_i})$$

by the minimality of the test ideal on the left. If we apply μ_n to the above equation we get

$$\mu_n\left(\tau\left(R^{\otimes_{\mathscr{k}} n}, \prod_{i=1}^{n}(A_{1,i} \otimes \cdots \otimes A_{n,i})^{t_i}\right)\right) \subseteq \prod_{i=1}^{n}\tau(R, \mathfrak{a}_i^{t_i})$$

Let R° denote the complement in R of the union of the minimal primes of R. By Proposition 2.2.6, we're done if we can show that the left-hand side above contains an element of R° and that $\mathscr{D}^{(n)}(R)$ is non-degenerate. For this, we will need to use our assumption that $\mathscr{k}$ is perfect. Since R is reduced, we know that R_η is regular for all minimal primes $\eta \in \operatorname{Spec} R$. The singular locus of $\operatorname{Spec} R$ is Zariski-closed, so it follows by prime avoidance that the defining ideal of the singular locus has non-trivial intersection with R°. In other words, there exists an element $f \in R^\circ$ such that R_f is regular. For each i, let $f_i \in \mathfrak{a}_i \cap R^\circ$. Then, setting $g_i = f \cdot f_i$, we see that each g_i is an element of $\mathfrak{a}_i \cap R^\circ$ such that R_{g_i} is regular. As we know $(R^{\otimes_{\mathscr{k}} n})_{g_1 \otimes \cdots \otimes g_n} = R_{g_1} \otimes \cdots \otimes R_{g_n}$, we see that

$$\begin{aligned}
&\tau\left(R^{\otimes_{\mathscr{k}} n}, \prod_{i=1}^{n}(A_{1,i} \otimes \cdots \otimes A_{n,i})^{t_i}\right)(R^{\otimes_{\mathscr{k}} n})_{g_1 \otimes \cdots \otimes g_n} \\
=&\tau\left(\bigotimes_{i=1}^{n}{}_{\mathscr{k}}\, R_{g_i}, \prod_{i=1}^{n}(A_{1,i}R_{g_1} \otimes \cdots\cdots \otimes A_{n,i}R_{g_n})^{t_i}\right) \\
=&\tau\left(\bigotimes_{i=1}^{n}{}_{\mathscr{k}}\, R_{g_i}\right) \\
=&R_{g_1} \otimes_{\mathscr{k}} \cdots \otimes_{\mathscr{k}} R_{g_n}
\end{aligned}$$

where the last equality follows from the strong F-regularity of $R_{g_1} \otimes_{\mathscr{k}} \cdots \otimes_{\mathscr{k}} R_{g_n}$, which in turn follows from the regularity of this ring[2]. Thus, we have

$$(g_1 \otimes \cdots \otimes g_n)^m \in \tau\left(R^{\otimes_{\mathscr{k}} n}, \prod_{i=1}^{n}(A_{1,i} \otimes \cdots \otimes A_{n,i})^{t_i}\right),$$

[2]Note that this relies on the fact that $\mathscr{k}$ is perfect. Indeed, $R \otimes_{\mathscr{k}} R$ need not be regular if $\mathscr{k}$ is not perfect, see [Zha]. On the other hand, since $\mathscr{k}$ is perfect, we know that each R_{g_i} is in fact *smooth*. Tensor products of smooth $\mathscr{k}$ algebras are still smooth, and therefore regular.

for some m, and so $(g_1 \cdots g_n)^m \in \mu_n\Big(\tau\left(R^{\otimes_{\Bbbk} n}, \prod_{i=1}^n \left(A_{1,i} \otimes \cdots \otimes A_{n,i}\right)^{t_i}\right)\Big)$.

It remains to check that $\mathscr{D}^{(n)}(R)$ is non-degenerate. Choose, as before, an element $f \in R^\circ$ such that R_f is regular, and let $\varphi \in \mathscr{C}^R$ be a non-degenerate map. As $(R_f)^{\otimes_{\Bbbk} n}$ is regular and F-finite, we have $F^e_*\left((R_f)^{\otimes_{\Bbbk} n}\right)$ is a projective $(R_f)^{\otimes_{\Bbbk} n}$-module, by Theorem 1.1.3. It follows that there exists some map

$$\widehat{\varphi} \in \operatorname{Hom}_{R^{\otimes n}}\left(F^e_*((R_f)^{\otimes_{\Bbbk} n}), (R_f)^{\otimes_{\Bbbk} n}\right)_{f\otimes\cdots\otimes f}$$

such that the diagram,

$$\begin{array}{ccc} F^e_*\left((R_f)^{\otimes_{\Bbbk} n}\right) & \overset{\widehat{\varphi}}{\dashrightarrow} & (R_f)^{\otimes_{\Bbbk} n} \\ {\scriptstyle F^e_*\mu_n}\downarrow & & \downarrow{\scriptstyle \mu_n} \\ F^e_* R_f & \xrightarrow{\quad\varphi\quad} & R_f \end{array}$$

commutes. Thus there exists some N with $(f \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} f)^N \widehat{\varphi} \in \operatorname{Hom}_{R^{\otimes_{\Bbbk} n}}(F^e_*(R^{\otimes_{\Bbbk} n}), R^{\otimes_{\Bbbk} n})$. We see that $f^{n \cdot N}\varphi$ is a non-degenerate element of $\mathscr{D}^{(n)}_e(R)$, as desired. □

Remark 2.2.12. A consequence of Theorem 2.3.1 (*Cf.* Remark 2.3.4) is that, in fact, $f^{n-1}\varphi \in \mathscr{D}^{(n)}_e(R)$ in the above proof.

Remark 2.2.13. One can alternatively define $\widetilde{\mathscr{D}}^{(n)}(R)$, in degree e, as the set of maps $\varphi\colon F^e_* R \to R$ admitting a lifting $\widehat{\varphi}\colon (F^e_* R)^{\otimes_{\Bbbk} n} \to R^{\otimes_{\Bbbk} n}$. By Corollary 2.2.10, $\widetilde{\mathscr{D}}^{(n)}(R)$ is *a priori* larger than $\mathscr{D}^{(n)}(R)$. The above argument suggests this might be the correct notion to consider in case $\Bbbk$ is not perfect. Indeed, if R is regular and $\Bbbk$ is not perfect, then we still have $(F^e_* R)^{\otimes_{\Bbbk} n}$ is a projective $R^{\otimes_{\Bbbk} n}$-module, even though $F^e_*\left(R^{\otimes_{\Bbbk} n}\right)$ may not be. The downside of this approach is that it's not apparent whether $\widetilde{\mathscr{D}}^{(n)}(R)$ is closed under composition. In any case, the Cartier subalgebra of $\mathscr{C}^R$ generated by elements of $\widetilde{\mathscr{D}}^{(n)}(R)$ is non-degenerate even when $\Bbbk$ is not perfect.

2.3 Comparison with Takagi's Subadditivity Theorem

Our next order of business is showing that our new subadditivity formula is sharper than the one found in [Tak06]. The following is our main result of this chapter:

Theorem 2.3.1 (Comparison with Takagi's subadditivity)**.** *Let $\Bbbk$ be a perfect field of positive characteristic and let R be a $\Bbbk$-algebra essentially of finite type. Suppose also that R is equidimensional and reduced. Then we have* $\operatorname{Jac}(R)\mathscr{C}^R \subseteq \mathscr{D}^{(2)}(R)$. *In particular,*

$$\operatorname{Jac}(R)\tau\left(R, \prod_i \mathfrak{a}_i^{t_i}\right) \subseteq \tau\left(R, \mathscr{D}^{(2)}, \prod_i \mathfrak{a}_i^{t_i}\right)$$

for all formal products of ideals $\prod \mathfrak{a}_i^{t_i}$ such that no $\mathfrak{a}_i$ is contained in a minimal prime of R.

The key ingredient in proving this theorem is a result of Hochster and Huneke that we can find Noether normalizations of equidimensional $\mathscr{k}$ algebras that interact well with Frobenius. We state this result here for the sake of completeness.

Definition 2.3.2. Let S be a finitely generated algebra over a Noetherian ring R. We define the *Jacobian ideal* of the extension, denoted $\mathrm{Jac}(S/R)$, to be the 0th Fitting ideal[3] of the module of Kähler differentials $\Omega_{S/R}$. In particular, if R is a field and $S = R[x_1, \dots, x_n]/(f_1, \dots, f_m)$ is reduced and equidimensional[4] of dimension d, then $\mathrm{Jac}(S) = \mathrm{Jac}(S/R)$ is the ideal generated by the $d \times d$ minors of the *Jacobian matrix*, $[\frac{\partial f_i}{\partial x_j}]_{i,j}$.

Theorem 2.3.3 ([HH02, Theorem 3.4], *Cf.* [HH99, Corollary 1.5.4])**.** *Let R be a geometrically reduced equidimensional ring essentially of finite type over an infinite field K of characteristic $p > 0$, and let $c \in \mathrm{Jac}(R)$. Then there exists a regular ring $A \subseteq R$, depending on c, such that $cR^{1/q} \subseteq A^{1/q}[R]$ and $A^{1/q}[R] \cong A^{1/q} \otimes_A R$ for all $q = p^e > 0$.*

Proof of Theorem 2.3.1. For any multiplicative subset $W \subseteq R$, one checks that

$$W^{-1}(\mathrm{Jac}(R)\mathscr{C}^R) = \mathrm{Jac}(W^{-1}R)\mathscr{C}^{W^{-1}R}$$

and $W^{-1}\mathscr{D}^{(2)}(R) \subseteq \mathscr{D}^{(2)}(W^{-1}R)$. Thus we may assume that R is a finitely generated $\mathscr{k}$-algebra.

Next, we reduce to the case that $\mathscr{k}$ is infinite. Suppose that $\mathscr{k}$ is finite, let t be an indeterminate over $\mathscr{k}$, and let $L = \mathscr{k}(t^{1/p^\infty})$ be the perfection of $\mathscr{k}(t)$. Set $R_L = R \otimes_{\mathscr{k}} L$ and suppose $\mathrm{Jac}(R_L/L)\mathscr{C}^{R_L} \subseteq \mathscr{D}^{(2)}(R_L)$. Let $e \geq 0$ and set $q = p^e$. As L is perfect, any map $\varphi \in \mathscr{C}_e^R$ induces a map,

$$\varphi_L := (R \otimes_{\mathscr{k}} L)^{1/q} = R^{1/q} \otimes_{\mathscr{k}} L \to R \otimes_{\mathscr{k}} L,$$

in $\mathscr{C}_e^{R_L}$. Further, any $x \in \mathrm{Jac}\, R$ gives an element $x \otimes_{\mathscr{k}} 1 \in \mathrm{Jac}(R_L/L)$. By assumption, we have a lifting,

$$\begin{array}{ccc} R_L^{1/q} \otimes_L R_L^{1/q} & \xrightarrow{\widehat{\varphi}} & R_L \otimes_L R_L \\ \downarrow{\scriptstyle \mu_L^{1/q}} & & \downarrow{\scriptstyle \mu_L} \\ R_L^{1/q} & \xrightarrow{(x\otimes 1)\cdot\varphi_L} & R_L \end{array} \tag{2.7}$$

Observe that the inclusion $\mathscr{k} \subseteq L$ splits as a map of $\mathscr{k}$-modules. Indeed, we can write $\mathscr{k}\left(t^{1/p^\infty}\right)$ as the direct limit,

$$\mathscr{k}\left(t^{1/p^\infty}\right) = \varinjlim \left(\mathscr{k}(t) \xrightarrow{F} \mathscr{k}(t) \xrightarrow{F} \mathscr{k}(t) \xrightarrow{F} \cdots\right)$$

[3] See [Eis95, Chapter 20] for exposition on Fitting ideals.
[4] Meaning, $\dim(S/\eta)$ is the same for all minimal primes η of S.

where F is the Frobenius map. Define the $\Bbbk$-linear map $\sigma : \Bbbk(t) \to \Bbbk$ as follows: any element of $\Bbbk(t)$ can be written as $t^i \frac{f}{g}$ for some $i \in \mathbb{Z}$ and some $f, g \in \Bbbk[t]$. Then we set

$$\sigma\left(t^i \frac{f}{g}\right) = \begin{cases} 0, & i \neq 0 \\ \frac{f(0)}{g(0)}, & i = 0 \end{cases}$$

As $\sigma = \sigma \circ F$, this induces a map $\widehat{\sigma}\colon \Bbbk\left(t^{1/p^\infty}\right) \to \Bbbk$. Note that $\widehat{\sigma}\left(t^{i/p^\infty}\right) = 0$ for all $i \neq 0$ and that $\widehat{\sigma}$ acts as the identity on $\Bbbk$. We consider $\Bbbk$ to be an L-module via $\widehat{\sigma}$. Then all we have to do is apply the functor $- \otimes_L \Bbbk$ to the diagram in equation 2.7; this shows that $\widehat{\varphi} \otimes_L \Bbbk$ is a lifting of $x\varphi$ to $\mathscr{C}_e^{R \otimes_{\Bbbk} R}$, as desired.

Now assume that $\Bbbk$ is infinite. Let $x \in \operatorname{Jac}(R)$ and let $\varphi \in \mathscr{C}_e^R$. As $\Bbbk$ is perfect and R is reduced, R is in fact geometrically reduced. Then, since $\Bbbk$ is infinite, we can use Theorem 2.3.3 to say there exists a Noether normalization $A \subseteq R$ such that $xR^{1/q} \subseteq A^{1/q}[R]$ and $A^{1/q}[R] \cong A^{1/q} \otimes_A R$. As A is a polynomial ring, we have $A^{1/q}$ is a free A-module, and so $A^{1/q} \otimes_A S$ is a free S-module for any A-algebra S. In particular, we have that $A^{1/q}[R] \otimes_{\Bbbk} R^{1/q} \cong A^{1/q} \otimes_A R \otimes_{\Bbbk} R^{1/q}$ is a free $R \otimes_{\Bbbk} R^{1/q}$-module.

Further, the usual multiplication map

$$\mu\colon R^{1/q} \otimes_{\Bbbk} R^{1/q} \to R^{1/q}$$

induces an $R^{1/q} \otimes_{\Bbbk} R^{1/q}$-module structure on $R^{1/q}$. By definition we have that μ is $R^{1/q} \otimes_{\Bbbk} R^{1/q}$-linear, and in particular $R \otimes_{\Bbbk} R^{1/q}$-linear. As $A^{1/q}[R] \otimes_{\Bbbk} R^{1/q}$ and $R \otimes_{\Bbbk} R^{1/q}$ are contained in $R^{1/q} \otimes_{\Bbbk} R^{1/q}$, the map μ restricts to $R \otimes_{\Bbbk} R^{1/q}$-linear maps,

$$\mu\colon A^{1/q}[R] \otimes_{\Bbbk} R^{1/q} \to R^{1/q}$$
$$\mu\colon R \otimes_{\Bbbk} R^{1/q} \to R^{1/q}.$$

It follows that there exists an $R \otimes_{\Bbbk} R^{1/q}$-linear (and, *a fortiori*, $R \otimes_{\Bbbk} R$-linear) map,

$$\Psi\colon A^{1/q}[R] \otimes_{\Bbbk} R^{1/q} \to R \otimes_{\Bbbk} R^{1/q},$$

making the following diagram commute:

$$\begin{array}{ccc} A^{1/q}[R] \otimes_{\Bbbk} R^{1/q} & \overset{\Psi}{\dashrightarrow} & R \otimes_{\Bbbk} R^{1/q} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ R^{1/q} & \xrightarrow{\text{id}} & R^{1/q} \end{array}$$

The fact that φ is R-linear means that the diagram

$$\begin{array}{ccc} R\otimes_{\Bbbk} R^{1/q} & \xrightarrow{1\otimes\varphi} & R\otimes_{\Bbbk} R \\ \Big\downarrow{\scriptstyle\mu} & & \Big\downarrow{\scriptstyle\mu} \\ R^{1/q} & \xrightarrow{\quad\varphi\quad} & R \end{array}$$

commutes. The map $1\otimes\varphi$ is $R\otimes_{\Bbbk} R$-linear as φ is R-linear. Finally, we have a commuting diagram

$$\begin{array}{ccc} R^{1/q}\otimes_{\Bbbk} R^{1/q} & \xrightarrow{x\otimes 1\cdot -} & A^{1/q}[R]\otimes_{\Bbbk} R^{1/q} \\ \Big\downarrow{\scriptstyle\mu} & & \Big\downarrow{\scriptstyle\mu} \\ R^{1/q} & \xrightarrow{\quad x\cdot -\quad} & R^{1/q} \end{array}$$

where the horizontal maps are given by multiplication. Putting these three diagrams together, we have a commutative diagram,

$$\begin{array}{ccccccc} R^{1/q}\otimes_{\Bbbk} R^{1/q} & \xrightarrow{x\otimes 1\cdot -} & A^{1/q}[R]\otimes_{\Bbbk} R^{1/q} & \xrightarrow{\Psi} & R\otimes_{\Bbbk} R^{1/q} & \xrightarrow{1\otimes\varphi} & R\otimes_{\Bbbk} R \\ \Big\downarrow{\scriptstyle\mu} & & \Big\downarrow{\scriptstyle\mu} & & \Big\downarrow{\scriptstyle\mu} & & \Big\downarrow{\scriptstyle\mu} \\ R^{1/q} & \xrightarrow{\quad x\cdot -\quad} & R^{1/q} & \xrightarrow{\quad \mathrm{id}\quad} & R^{1/q} & \xrightarrow{\quad\varphi\quad} & R \end{array}$$

where each of the maps in the top row is $R\otimes_{\Bbbk} R$-linear. This proves the first assertion. The second assertion follows from 1.1.12. $\square$

Remark 2.3.4. A similar argument can be used to show that, in fact, $\mathrm{Jac}(R)^a\mathscr{D}^{(b)}(R)\subseteq\mathscr{D}^{(a+b)}(R)$ for all $a,b>0$ in the above set-up. Note that $\mathscr{D}^{(1)}(R)$ is always just $\mathscr{C}^R$.

Chapter 3

Comparison to Eisenstein's Subadditivity Theorem

Умный в гору не пойдёт,
Умный гору обойдёт.

С. В. Михалков

In this chapter we introduce two new definitions. The first is a notion of test ideals "along a closed subscheme." This is a generalization of Takagi's *generalized test ideal along* I [Tak10] and Schwede's *big test ideal outside* I [Sch09]; see also [BSTZ10]. We also define a similar generalization of Takagi's adjoint ideal, called the multiplier ideal along $\operatorname{Spec}(R/I)$. In Section 3.2, we show that the expected reduction theorem holds: the multiplier ideal along $\operatorname{Spec}(R/I)$ reduces to the test ideal along I mod $p \gg 0$. This generalizes slightly the results of [Tak08] and [Tak13], which respectively deal with the cases where our subspace is a divisor and our ambient space is regular. We use this result to show that our subadditivity formula for test ideals is sharper than the one obtained by reducing [Eis10, Theorem 6.5] mod $p \gg 0$.

Remark 3.0.1. In this chapter, we will make heavy use of the floor and ceiling functions, $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$. It will be helpful to keep in mind the following inequalities: for any $a, b \in \mathbb{R}$ we have

$$\lfloor a \rfloor + \lfloor b \rfloor \leq \lfloor a + b \rfloor \tag{3.1}$$

$$\lfloor a \rfloor - \lfloor b \rfloor \geq \lfloor a - b \rfloor \tag{3.2}$$

and, similarly,

$$\lceil a \rceil + \lceil b \rceil \geq \lceil a + b \rceil \tag{3.3}$$

$$\lceil a \rceil - \lceil b \rceil \leq \lceil a - b \rceil. \tag{3.4}$$

3.1 Test Ideals along a Closed Subscheme

In this section we generalize the constructions of test ideals *along an ideal* I given in [Tak10] and [BSTZ10]. This is a test ideal with good restriction properties, analogously with the adjoint ideals of algebraic geometry. We make this analogy rigorous in Section 3.2.

For the rest of this section, we will be working in the following setting.

Setting 3.1.1. R is a Noetherian F-finite domain, $I \subseteq R$ is a prime ideal, and $\mathscr{C}$ a nonzero Cartier algebra on R such that I is compatible with $\mathscr{C}$. We assume there exists a number $e > 0$ and a map $\psi \in \mathscr{C}_e$ with $\psi(F^e_* R) \nsubseteq I$.

Remark 3.1.2. We suspect that the constructions in this section can be done just as well in the setting where I is radical and unmixed, that is, where I is an intersection of different prime ideals of a fixed height. However, for our current purposes it suffices to work in the setting where I is prime.

Definition 3.1.3 (Test ideal along an ideal)**.** Let $R, I, \mathscr{C}$ be as in Setting 3.1.1. Then we define *the test ideal of* $\mathscr{C}$ *along* I, denoted $\tau_I(R, \mathscr{C})$, to be the unique smallest ideal of R not contained in I that is compatible with $\mathscr{C}$. We also call this the *test ideal of* $\mathscr{C}$ *along the closed subscheme* $\operatorname{Spec}(R/I)$.

The proof that $\tau_I(R, \mathscr{C})$ exists is a standard though technical argument. We save it for Section 3.1.1. For now, we just note some examples of interest where the conditions of Setting 3.1.1 are satisfied.

Example 3.1.4. Suppose R is a domain essentially of finite type over a perfect field $\Bbbk$. Then $I = I_\Delta \subseteq R \otimes_\Bbbk R$ and $\mathscr{C} = \mathscr{C}^{R\otimes_\Bbbk R,\, I_\Delta \circlearrowleft}$ satisfy Setting 3.1.1. Indeed, we have that $\mathscr{C}$ is compatible with I by construction, so we just need to check that there exists $e > 0$ and $\varphi \in \mathscr{C}_e$ with $\varphi(R) \nsubseteq I$. This is equivalent to checking that $\mathscr{D}^{(2)}_e(R) \neq 0$ for some $e > 0$, which follows, for instance, from Theorem 2.3.1. See also the end of the proof of Theorem 2.2.11, where we show $\mathscr{D}^{(n)}$ is non-degenerate in greater generality.

Notation 3.1.5. Let $\mathfrak{a}_i$ be a collection of ideals and t_i a set of non-negative real numbers. Then we denote, for all e,

$$\underline{\mathfrak{a}}^{\lceil t(p^e-1)\rceil} := \prod_i \mathfrak{a}_i^{\lceil t_i(p^e-1)\rceil}.$$

Notation 3.1.6. Work in Setting 3.1.1. If $\mathscr{C} = \mathscr{C}^{R, \prod_i \mathfrak{a}_i^{t_i}}$ for some ideals $\mathfrak{a}_i$ and non-negative numbers t_i, then we denote

$$\tau_I\left(R, \prod_i \mathfrak{a}_i^{t_i}\right) := \tau_I(R, \mathscr{C})$$

Example 3.1.7. Suppose R_I is regular, and $\dim R_I = c$. Let $\{\mathfrak{b}_i\}$ be a set of ideals, none of which is contained in I, and let $\{t_i\}$ be some collection of non-negative rational numbers. Then $I^c \prod_i \mathfrak{b}_i^{t_i} \cdot \mathscr{C}^R$ satisfies the conditions of Setting 3.1.1. In this case, $\tau_I\left(R, I^c \prod_i \mathfrak{b}_i^{t_i}\right)$ is what Takagi calls $\widetilde{\tau}_I\left(R, \prod \mathfrak{b}_i^{t_i}\right)$. The Cartier algebras $\mathscr{C}^{R, I^{(c)} \prod_i b_i^{t_i}}$ and $\mathscr{C}^{R, \overline{I^c} \prod_i b_i^{t_i}}$ satisfy the conditions of Setting 3.1.1 as well.

More generally, let $\{\mathfrak{a}_i\}$ be any collection of ideals and $\{t_i\}$ any collection of non-negative rational numbers, satisfying the following: for each i, suppose there exists some natural number n_i with $\mathfrak{a}_i R_I = I^{n_i} R_I$, and suppose $\sum_i t_i n_i = c$. Further, for all i such that $\mathfrak{a}_i \subseteq I$, we assume the denominator of t_i is not divisible by p. Then $\mathscr{C}^{R, \prod_i \mathfrak{a}_i^{t_i}}$ satisfies the conditions of Setting 3.1.1. To see this, we need to check that

(a) $\mathscr{C}^{R, \prod_i \mathfrak{a}_i^{t_i}}$ is compatible with I, and

(b) There exist $e > 0$ and $\psi \in F^e_*\left(\underline{\mathfrak{a}}^{\lceil t(p^e-1)\rceil}\right) \cdot \mathrm{Hom}_R(F^e_* R, R)$ with $\psi(F^e_* R) \not\subseteq I$.

To show condition (a), let $x \in \underline{\mathfrak{a}}^{\lceil t(p^e-1)\rceil}$ and let $\varphi \in \mathrm{Hom}_R(F^e_* R, R)$. We wish to show that $\varphi(F^e_* x I) \subseteq I$. Recall that I is prime. Whether or not $\varphi(F^e_* x I)$ is contained in I is not affected by localizing at I, so we localize at I. Then $I R_I$ is generated by c elements, since R_I is regular, and $x \in I^{cp^e - c} R_I$, since

$$\underline{\mathfrak{a}}^{\lceil t(p^e-1)\rceil} R_I = \prod_i I^{n_i \lceil t_i(p^e-1)\rceil} \subseteq \prod_i I^{\lceil n_i t_i (p^e-1)\rceil} \subseteq I^{\lceil \sum_i n_i t_i (p^e-1)\rceil} = I^{cp^e-c} R_I.$$

Thus $xIR_I \subseteq I^{cp^e-c+1} R_I \subseteq I^{[p^e]} R_I$. This shows that (a) is satisfied. To see condition (b) is satisfied, we notice again that this question can be checked locally at I. As R_I is regular local and F-finite, it follows from Theorem 1.1.3 that $F^e_* R_I$ is a free R_I module for all e. Then it follows that $\varphi(F^e_* x) \in I$ for all $\varphi \in \mathrm{Hom}_{R_I}(F^e_* R_I, R_I)$ if and only if $x \in I^{[p^e]} R_I$. As the denominator of each t_i is not divisible by p, we know there exists some e so that $t_i(p^e - 1)$ is an integer for all i. For this e, we have $\underline{\mathfrak{a}}^{\lceil t(p^e-1)\rceil} R_I = I^{cp^e-c} \not\subseteq I^{[p^e]}$, so we're done.

3.1.1 Proof that Test Ideals along Closed Subschemes Exist

In this section, we show there's a notion of test elements for test ideals along closed subschemes, working in Setting 3.1.1. Consequently, these test ideal exist. We remark that these proofs are essentially the same as those in [Sch09, §6]. The salient difference between our setting and the one in [Sch09] is that here we're not assuming that I is an F-pure center of R. Instead, we're just assuming that $\mathscr{C}$ is compatible with I and that $\mathscr{C}|_{R/I}$ is non-degenerate; namely, we assume that there exist an integer $e > 0$ and a map $\psi \in \mathscr{C}_e$ such that $\psi\left(F^e_* R\right) \not\subseteq I$.

Lemma 3.1.8. *Work in Setting 3.1.1. There exists some $\gamma \in R \smallsetminus I$ such that*

(a) All proper ideals of R_γ compatible with $\mathscr{C}_\gamma$ are contained in IR_γ, and

(b) The Cartier algebra $\mathscr{C}_\gamma$ is F-pure

Proof. Let $\pi\colon R \to R/I$ be the quotient map. It follows from our assumptions in Setting 3.1.1 that $\mathscr{C}|_{R/I}$ is a non-degenerate Cartier algebra on R/I, so $\tau\left(R/I, \mathscr{C}|_{R/I}\right)$ is well-defined (and, in particular, nonzero) [Sch11]. Choose $\gamma_1 \in R$ so that $\pi(\gamma_1) \in \tau\left(R/I, \mathscr{C}|_{R/I}\right)$ and $\pi(\gamma_1) \neq 0$. Then all proper ideals of R compatible with $\mathscr{C}_{\gamma_1}$ are contained in IR_{γ_1}. Indeed, we have the following diagram for all $\varphi \in \mathscr{C}_{\gamma_1}$:

$$\begin{array}{ccc} F^e_* R_{\gamma_1} & \xrightarrow{\varphi} & R_{\gamma_1} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ F^e_*(R/I)_{\gamma_1} & \xrightarrow{\overline{\varphi}} & (R/I)_{\gamma_1} \end{array}$$

If $\varphi(J) \subseteq J$, then $\overline{\varphi}(\pi(J)) \subseteq \pi(J)$. Note that as φ runs through all maps in $\mathscr{C}_{\gamma_1}$, $\overline{\varphi}$ will run through all maps in $\left(\mathscr{C}|_{R/I}\right)_{\gamma_1}$. So if J is a proper ideal of R_{γ_1} compatible with $\mathscr{C}_{\gamma_1}$, then $\pi(J)$ is compatible with $(\mathscr{C}|_{R/I})_{\gamma_1}$. But we have that $\tau(R/I, (\mathscr{C}|_{R/I})_{\gamma_1}) = \tau(R/I, \mathscr{C}|_{R/I})(R/I)_{\gamma_1} = (R/I)_{\gamma_1}$, so $(\mathscr{C}|_{R/I})_{\gamma_1}$ is F-regular. This means that $\pi(J) = 0$, meaning $J \subseteq IR_{\gamma_1}$.

We note that all proper ideals of $R_{\gamma_1\gamma_2}$ compatible with $\mathscr{C}_{\gamma_1\gamma_2}$ are contained in $IR_{\gamma_1\gamma_2}$, for all $\gamma_2 \in R \smallsetminus I$. So choose $e > 0$ and $\psi \in \mathscr{C}_e$ to be some map whose image is not contained in I and let $\gamma_2 \in \psi(F^e_* R) \smallsetminus I$. Then the element $\gamma = \gamma_1\gamma_2$ satisfies the conclusion of the lemma. □

Proposition 3.1.9 (*Cf.* [Sch09, Lemma 6.12])**.** *Work in Setting 3.1.1. There exists an element $\gamma \in R \smallsetminus I$ such that, for all $d \in R \smallsetminus I$, there exists an integer m and a map $\Psi \in \mathscr{C}$ of minimal degree greater than 0 such that $\gamma^m = \Psi \cdot d$.*

Proof. Choose γ as in Lemma 3.1.8. It suffices to prove that

$$J := \sum_{e>0} \sum_{\varphi \in \mathscr{C}_e} \varphi(F^e_* d) R_\gamma = R_\gamma$$

for all $d \in R \smallsetminus I$. By definition of γ, it suffices to show that J is compatible with $\mathscr{C}_\gamma$ and not contained in IR_γ. It's clear that J is compatible with $\mathscr{C}_\gamma$, so we'll just show J is not contained in IR_γ. Let $\pi : R \to R/I$ be the natural surjection. As $(\mathscr{C}|_{R/I})_\gamma$ is F-regular and $\pi(d) \neq 0$, there exist e and $\overline{\varphi} \in (\mathscr{C}|_{R/I})_\gamma$ such that $\overline{\varphi}(F^e_* \pi(d)) = 1$. This means that $\varphi(F^e_* d) = 1 + x$ for some $x \in I$, where φ is any map in $\mathscr{C}_\gamma$ that induces $\overline{\varphi}$. But then $1 + x \in J$, so J is not contained in IR_γ. □

Lemma 3.1.10 (*Cf.* [Sch09, Lemma 6.13])**.** *Suppose $\varphi \in \mathscr{C}^R$, $c \in R$, and $b \in \varphi \cdot (cR)$. Then $b^2 \in \varphi^n \cdot (cR)$ for all $n > 0$.*

Proof. This proof is essentially the same as that of Lemma 6.13 of [Sch09]. We include it here for completeness.

We proceed by induction. The base case is given by the hypothesis. Suppose that $\varphi = \sum_i \varphi_i$, where $\varphi_i \in \mathscr{C}^R_{e_i}$. Then we compute:

$$\begin{aligned} b^2 \in b\,\varphi \cdot (cR) &= b \sum_i \varphi_i \left(F^{e_i}_* cR\right) = \sum_i \varphi_i \left(F^{e_i}_* b^{p^{e_i}} cR\right) \subseteq \sum_i \varphi_i \left(F^{e_i}_* b^2 cR\right) \\ &= \varphi \cdot (b^2 cR) \subseteq \varphi \cdot ((\varphi^n \cdot (cR))\, c) \subseteq \varphi^{n+1} \cdot (cR). \end{aligned}$$

□

Proposition 3.1.11 (*Cf.* [Sch09, Proposition 6.14])**.** *There is an element $b \in R \smallsetminus I$ such that for all $d \in R \smallsetminus I$, there exists $\Psi \in \mathscr{C}$ such that $b = \Psi \cdot d$.*

Proof. This proof is essentially the same as that of Proposition 6.14 of [Sch09]. We include it here for completeness.

Choose γ as in Proposition 3.1.9. Then there exists m and Ψ, of minimal degree $e_0 > 0$, such that $\gamma^m = \Psi \cdot 1$. By Lemma 3.1.10, $\gamma^{2m} \in \Psi^n \cdot R$ for all $n > 0$. We will show that $b = \gamma^{3m}$ works.

Let $d \in R \smallsetminus I$ be arbitrary. Then there exists Ψ_1 and m_1 such that $\gamma^{m_1} = \Psi_1 \cdot d$. If $m_1 < 3m$, then we're done. Otherwise, choose n such that $m_1 < mp^{ne_0}$ and write $\Psi^n = \sum_i \psi_i$ with $\psi_i \in \mathscr{C}_{e_i}$ for all i. Note that $e_i \geq ne_0$ for all i. Then we have:

$$\begin{aligned} \gamma^{3m} &= \gamma^m \gamma^{2m} \in \gamma^m \Psi^n \cdot (R) = \gamma^m \sum_i \psi_i(F^{e_i}_* R) = \sum_i \psi_i(F^{e_i}_* \gamma^{mp^{e_i}} R) \subseteq \sum_i \psi_i(F^{e_i}_* \gamma^{mp^{ne_0}} R) \\ &\subseteq \sum_i \psi_i(F^{e_i}_* \gamma^{m_1} R) = \Psi^n \cdot (\gamma^{m_1} R) \subseteq \Psi^n \Psi_1 \cdot (dR). \end{aligned}$$

□

Theorem 3.1.12 (*Cf.* [Sch09, Lemma 6.17 and Theorem 6.18])**.** *Let b be as in Proposition 3.1.11. Then*

$$\tau_I(R, \mathscr{C}) = \sum_{e \geq 0} \sum_{\varphi \in \mathscr{C}_e} \varphi(F^e_* b)$$

Proof. This proof is essentially the same as Lemma 6.17 and Theorem 6.18 of [Sch09]. We include it here for completeness.

Let $\tau_I(R, \mathscr{C}; b)$ denote the ideal

$$\sum_{e \geq 0} \sum_{\varphi \in \mathscr{C}_e} \varphi(F^e_* b),$$

and note that we have

$$\tau_I(R, \mathscr{C}; b) = \sum_{\varphi \in \mathscr{C}} \varphi \cdot b.$$

Then we need to show:

(a) $\tau_I(R, \mathscr{C}; b) \not\subseteq I$,

(b) $\tau_I(R, \mathscr{C}; b)$ is compatible with $\mathscr{C}$, and

(c) $\tau_I(R, \mathscr{C}; b)$ is contained in any other ideal satisfying (a) and (b).

For (a), it's enough to show that $b \in \tau_I(R, \mathscr{C}; b)$. This follows from Proposition 3.1.11, using $d = b$. Assertion (b) is clear from the construction of $\tau_I(R, \mathscr{C}; b)$.

For the final assertion, let J be some ideal satisfying (a) and (b). Choose some $d \in J \smallsetminus I$. Then

$$\sum_{\varphi \in \mathscr{C}} \varphi \cdot d \subseteq \sum_{\varphi \in \mathscr{C}} \varphi \cdot J \subseteq J$$

By Proposition 3.1.11, we have $b \in J$. But then

$$\tau_I(R, \underline{\mathfrak{a}}^t; b) = \sum_{\varphi \in \mathscr{C}} \varphi \cdot b \subseteq \sum_{\varphi \in \mathscr{C}} \varphi \cdot J \subseteq J$$

□

3.1.2 Basic Properties of $\tau_I(R, \mathscr{C})$

Here we explore the basic theory of $\tau_I(R, \mathscr{C})$. With the exception of the restriction theorem found in Proposition 3.1.14, the following are properties satisfied by all objects deserving of the name "test ideal." The restriction theorem suggests that $\tau_I(R, \mathscr{C})$ is a good candidate for a positive-characteristic analog to the adjoint ideals of birational geometry.

Lemma 3.1.13 (Monotonicity of test ideals)**.** *Suppose that $\mathscr{C} \subseteq \mathscr{D}$ are two Cartier algebras on R satisfying Setting 3.1.1. Then $\tau_I(R, \mathscr{C}) \subseteq \tau_I(R, \mathscr{D})$.*

Proof. The ideal $\tau_I(R, \mathscr{C})$ is the minimum element of the set

$$S_{\mathscr{C}} := \{\mathfrak{a} \subseteq R \mid \mathfrak{a} \not\subseteq I \text{ and } \mathfrak{a} \text{ is } \mathscr{C}\text{-compatible}\}$$

Similarly, $\tau_I(R, \mathscr{D})$ is the minimum element of the set

$$S_{\mathscr{D}} := \{\mathfrak{a} \subseteq R \mid \mathfrak{a} \not\subseteq I \text{ and } \mathfrak{a} \text{ is } \mathscr{D}\text{-compatible}\}$$

We see that $S_{\mathscr{C}} \supseteq S_{\mathscr{D}}$, whence the minimum of $S_{\mathscr{C}}$ is smaller than the minimum of $S_{\mathscr{D}}$. □

Proposition 3.1.14 (Restriction theorem)**.** *Let R, I, and $\mathscr{C}$ be as in Setting 3.1.1. Then we have* $\tau_I(R,\mathscr{C})R/I = \tau(R/I, \mathscr{C}|_{R/I})$.

Proof. The proof is very similar to that of Proposition 2.2.6. Let $\varphi \in \mathscr{C}_e|_{R/I}$. By definition, there exists some $\widehat{\varphi} \in \mathscr{C}_e$ such that the following diagram commutes:

$$\begin{array}{ccc} F^e_* R & \xrightarrow{\widehat{\varphi}} & R \\ {\scriptstyle F^e_*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ F^e_*(R/I) & \xrightarrow{\varphi} & R/I \end{array}$$

We see that

$$\varphi\left(F^e_* \tau_I(R,\mathscr{C})R/I\right) = \widehat{\varphi}(F^e_*\tau_I(R,\mathscr{C}))R/I \subseteq \tau_I(R,\mathscr{C})R/I.$$

By the minimality of $\tau(R/I,\mathscr{C}|_{R/I})$, it follows that $\tau_I(R,\mathscr{C})R/I \supseteq \tau(R/I,\mathscr{C}|_{R/I})$.

To get the reverse inclusion, it suffices to show that $\tau_I(R,\mathscr{C}) \subseteq \pi^{-1}\left(\tau(R/I,\mathscr{C}|_{R/I})\right)$. By definition, $\tau(R/I,\mathscr{C}|_{R/I}) \neq 0$, which means that $\pi^{-1}\left(\tau(R/I,\mathscr{C}|_{R/I})\right) \nsubseteq I$. Thus it suffices to check that $\pi^{-1}\left(\tau(R/I,\mathscr{C}|_{R/I})\right)$ is compatible with $\mathscr{C}$. To that end, let $\psi \in \mathscr{C}_e$ be arbitrary. As $\mathscr{C}$ is compatible with I, there exists some $\overline{\psi} \in \mathscr{C}|_{R/I}$ such that the following diagram commutes:

$$\begin{array}{ccc} F^e_* R & \xrightarrow{\psi} & R \\ {\scriptstyle F^e_*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ F^e_*(R/I) & \xrightarrow{\overline{\psi}} & R/I \end{array}$$

It follows from the above diagram and the $\mathscr{C}|_{R/I}$-compatibility of $\tau(R/I,\mathscr{C}|_{R/I})$ that:

$$\begin{aligned} \pi\circ\psi\left(F^e_*\pi^{-1}\left(\tau(R/I,\mathscr{C}|_{R/I})\right)\right) &= \overline{\psi}\circ F^e_*\pi\left(F^e_*\pi^{-1}\left(\tau(R/I,\mathscr{C}|_{R/I})\right)\right) \\ &= \overline{\psi}\left(F^e_*\tau(R/I,\mathscr{C}|_{R/I})\right) \\ &\subseteq \tau(R/I,\mathscr{C}|_{R/I}). \end{aligned}$$

In other words,

$$\psi\left(F^e_*\pi^{-1}\left(\tau(R/I,\mathscr{C}|_{R/I})\right)\right) \subseteq \pi^{-1}\left(\tau(R/I,\mathscr{C}|_{R/I})\right),$$

as desired. □

Definition 3.1.15. We say that an element b in $\tau_I(R,\mathscr{C}) \smallsetminus I$ is a *$\mathscr{C}$-test element along I*. If $\mathscr{C} = \mathscr{C}^{R,\prod_i \mathfrak{a}_i^{t_i}}$, then we say that b is an $\prod_i \mathfrak{a}_i^{t_i}$-test element along I.

Lemma 3.1.16. *Let c be a $\mathscr{C}$-test element along I. Then* $\tau_I(R,\mathscr{C}) = \sum_{e\geq 0}\sum_{\varphi\in\mathscr{C}_e}\varphi(F^e_* c)$.

Proof. For ease of notation, set

$$\tau_I(R, \mathscr{C}; c) = \sum_{e \geq 0} \sum_{\varphi \in \mathscr{C}_e} \varphi(F^e_* c).$$

As $c \notin I$, have that $\tau_I(R, \mathscr{C}; c) \nsubseteq I$ by Proposition 3.1.11. Now let $J \nsubseteq I$ be an ideal compatible with $\mathscr{C}$. By definition of $\tau_I(R, \mathscr{C})$, we know that $\tau_I(R, \mathscr{C}) \subseteq J$, so $c \in J$. It follows that $\varphi(F^e_* c) \subseteq \varphi(F^e_* J) \subseteq J$ for all $e \geq 0$ and all $\varphi \in \mathscr{C}_e$, since J is compatible with $\mathscr{C}$. Thus $\tau_I(R, \mathscr{C}; c) \subseteq J$. □

Lemma 3.1.17. *Let c be a $\mathscr{C}$-test element along I and let $e' \geq 0$. Then*

$$\tau_I(R, \mathscr{C}) = \sum_{e \geq e'} \sum_{\varphi \in \mathscr{C}_e} \varphi(F^e_* c).$$

Proof. Set $J = \sum_{e \geq e'} \sum_{\varphi \in \mathscr{C}_e} \varphi(F^e_* c)$. By Lemma 3.1.16, we see that $J \subseteq \tau_I(R, \mathscr{C})$. Thus it suffices to show that $J \nsubseteq I$ and that J is compatible with $\mathscr{C}$. The latter just follows from the definition of a Cartier algebra. To see that $J \nsubseteq I$, note that there exists some e'' and $\varphi \in \mathscr{C}_{e''}$ with $\varphi(F^{e''}_* c) \notin I$. This follows, for instance, from Lemma 3.1.16. Then by Lemma 3.1.10, we have $\varphi^n(F^{ne''}_* cR) \nsubseteq I$ for all n. In particular, we get the desired result by taking $n > e''/e'$. □

Lemma 3.1.18 (τ$_I$ localizes)**.** *Work in Setting 3.1.1 and let $W \subseteq R \smallsetminus I$ be a multiplicative set. Then*

$$W^{-1}\tau_I(R, \mathscr{C}) = \tau_{IW^{-1}R}\left(W^{-1}R, W^{-1}\mathscr{C}\right).$$

Proof. As R is a domain and $W \cap I = \emptyset$, the ideal $\tau_{IW^{-1}}(W^{-1}R, W^{-1}\mathscr{C})$ is well-defined. As R is F-finite and Noetherian, we have $W^{-1} \operatorname{Hom}_R(F^e_* R, R) = \operatorname{Hom}_{W^{-1}R}\left(F^e_* W^{-1}R, W^{-1}R\right)$ for all e. Then this lemma follows quickly from Theorem 3.1.12. □

In this section, we define the multiplier ideal of a pair (A, Z) *along a subscheme* $X \subseteq A$, which is a generalization of the adjoint ideal of a divisor D in A. For now, let A be a normal $\mathbb{Q}$-Gorenstein variety over a field of characteristic 0, $Z = \sum_{i=1}^m t_i Z_i$ a formal $\mathbb{R}$-linear combination of subschemes $Z_i \subseteq A$, and D a reduced Cartier divisor on A, such that no components of D are contained in any Z_i. Then we can define the adjoint ideal of the pair (A, Z) along the divisor D as follows.

Notation 3.1.19 (Exceptional locus)**.** Let $\pi\colon X' \to X$ be a birational map. By the *exceptional locus* of π, or $\operatorname{exc}(\pi)$, we mean the locus of points in X' at which π is not an isomorphism.

Definition 3.1.20 (Adjoint ideal along a divisor)**.** Let $\pi\colon \widetilde{A} \to A$ be a log-resolution of the pair (A, Z), meaning $\widetilde{A}$ is nonsingular, the inverse images $\pi^{-1}Z_i$ are divisors, $\operatorname{exc}(\pi)$ is a divisor, and

$\bigcup_i \pi^{-1}(Z_i) \cup \operatorname{exc}(\pi)$ has simple normal crossings support. By taking further blow-ups, we may assume that the strict transform $\pi_*^{-1}D$ is nonsingular. Then the *adjoint ideal of* (A, Z) *along* D is given by

$$\operatorname{adj}_D(A, Z) \coloneqq \pi_* \mathscr{O}_{\widetilde{A}} \left(\left\lceil K_{\widetilde{A}/A} - \pi^{-1}Z + \pi_*^{-1}D - \pi^*D \right\rceil \right),$$

where $\pi^{-1}Z \coloneqq \sum_i t_i \pi^{-1} Z_i$.

Compare the above with the multiplier ideal $\mathscr{J}(A, Z)$, which is defined as

$$\mathscr{J}(A, Z) \coloneqq \pi_* \mathscr{O}_{\widetilde{A}} \left(\left\lceil K_{\widetilde{A}/A} - \pi^{-1}Z \right\rceil \right)$$

Since $\pi_*^{-1}D - \pi^*D$ is always anti-effective, we have $\operatorname{adj}_D(A, Z) \subseteq \mathscr{J}(A, Z)$. According to Lazarsfeld [Laz04, 9.3.E], these adjoint ideals were first used by [Vaq94] and later rediscovered by [EL97]. Perhaps the most salient fact about these adjoint ideals is that they fit into the following exact sequence, whenever A and D are smooth:

$$0 \to \mathscr{J}(A, Z) \otimes \mathscr{O}_A(-D) \to \operatorname{adj}_D(A, Z) \to \mathscr{J}(D, Z|_D) \to 0, \tag{3.5}$$

Thus, adjoint ideals help us compare restrictions multiplier ideals with multiplier ideals of restrictions (*Cf.* [Laz04, Proposition 9.3.48]).

Takagi was the first to generalize the construction of $\operatorname{adj}_D(A, Z)$ to the case where D is a reduced closed subscheme of *arbitrary* codimension, by first performing a blow-up along said subscheme. His definition is given as follows:

Definition 3.1.21 ([Tak10, Definition 1.6])**.** Let A be a nonsingular variety over a field of characteristic 0 and let $X \subseteq A$ be a reduced closed subscheme such that each component of X has codimension c in A. Further, let $Z = \sum_{i=1}^m t_i Z_i$ be a formal sum of subschemes of A, where $t_i > 0$ for all i, and assume that no Z_i contains any component of X. Let $f\colon A' \to A$ be the blow-up of A along X and let $E_1, \ldots, E_s$ be the components of the exceptional divisor of f dominating the irreducible components of X. Let $g\colon \widetilde{A} \to A'$ be a log resolution of $(A', f^{-1}X + f^{-1}Z)$ such that $\sum_{j=1}^s g_*^{-1}E_j$ is nonsingular. Let $\pi = f \circ g$. Then we define the *adjoint ideal of* (A, Z) *along* X as:

$$\operatorname{adj}_X(A, Z) \coloneqq \pi_* \mathscr{O}_{\widetilde{A}} \left(\left\lceil K_{\widetilde{A}/A} - \pi^{-1}Z - c\pi^{-1}X + \sum_{j=1}^s g_*^{-1}E_j \right\rceil \right),$$

where $\pi^{-1}Z \coloneqq \sum_{i=1}^m t_i \pi^{-1} Z_i$.

Takagi shows in *op. cit.* that this adjoint ideal satisfies numerous desirable properties, such as an analog of (3.5) in the case where X is Gorenstein. Later, Eisenstein found another characterization of this adjoint ideal and used it to generalize (3.5) to the case where X is only $\mathbb{Q}$-Gorenstein (A is still required to be smooth) [Eis10]. Eisenstein further notes that Definition 3.1.21 makes sense even if A is not smooth. Indeed, we just need A to be smooth at the generic points of X, so that $\operatorname{exc}(f)$ will be a divisor.

The following is a slight generalization of Takagi's adjoint ideal. Namely, we allow ourselves greater flexibility in the formal sums Z we consider. We call this construction the *multiplier ideal of* (A, Z) *along* X.

Definition 3.1.22 (Multiplier ideal along a subscheme)**.** Let A be a $\mathbb{Q}$-Gorenstein scheme of finite type over a field of characteristic 0 and let X be a reduced subscheme of A of pure codimension c. Suppose also that A is smooth at the generic points of X. Let $Z = \sum_{i=1}^{m} t_i Z_i$ be a formal $\mathbb{Q}$-sum of subschemes of A such that Z equals cX at the generic points of X. We define $\mathscr{J}_X(A, Z)$ as follows: let $\pi_1 \colon A_1 \to A$ be a factorizing resolution of $X \subseteq A$ such that $\pi_1^{-1} Z \cup \operatorname{exc}(\pi_1)$ has simple normal crossings support[1] and the components of Z not vanishing along X lift to divisors. This is possible by [Eis10, Corollary 3.2]. Let $X_1 \subseteq A_1$ be the strict transform of X in A_1 and let $\pi_2 \colon A_2 \to A_1$ be the blow-up of A_1 along X_1. We get the following diagram:

$$\begin{array}{ccc} X_2 & \subseteq & A_2 \\ \downarrow & & \downarrow{\scriptstyle \pi_2} \\ X_1 & \subseteq & A_1 \\ \downarrow & & \downarrow{\scriptstyle \pi_1} \\ X & \subseteq & A\,. \end{array}$$

Then $X_2 := \operatorname{exc}(\pi_2)$ is a prime divisor dominating X_1. Let $\pi = \pi_1 \circ \pi_2$. We define *the multiplier ideal of* (A, Z) *along* X to be the ideal:

$$\mathscr{J}_X(A, Z) := \pi_* \mathscr{O}_{A_2}\left(\lceil K_{A_2/A} - \pi^{-1} Z \rceil + X_2\right).$$

The following lemma shows that $\mathscr{J}_X(A, Z)$ is indeed a generalization of Takagi's adjoint ideal $\operatorname{adj}_X(A, Z)$. This fact is implicit in the work of Eisenstein [Eis10].

Lemma 3.1.23 ([Eis10, Proof of proposition 3.5])**.** *Let A be a $\mathbb{Q}$-Gorenstein scheme of finite type over a field of characteristic 0 and let X be a subscheme of A with pure codimension c. Suppose A is smooth at the generic points of X. Let $Z = \sum_{i=1}^{m} t_i Z_i$ be a formal sum of subschemes of X such that none of the Z_i contain any component of X. Then*

$$\operatorname{adj}_X(A, Z) = \mathscr{J}_X(A, Z + cX).$$

[1] See [Eis10, Section 2] for the definition of a factorizing resolution and of "simple normal crossings support." Note that, by definition, $\operatorname{exc}(\pi_1)$ is a divisor on A_1.

Proof. Let $A_2 \xrightarrow{\pi_2} A_1 \xrightarrow{\pi_1} A$ be as in Definition 3.1.22. By the universal property of blow-ups, we have that π factors through $f\colon A' \to A$, i.e. the blow-up of X in A. Further, $g\colon A_2 \to A'$ is a log-resolution of $f^{-1}X \cup f^{-1}Z$. It follows:

$$\operatorname{adj}_X(A,Z) \coloneqq \pi_* \mathscr{O}_{A_2}\left(\left\lceil K_{A_2/A} - \pi^{-1}Z - c\pi^{-1}X + \sum_{j=1}^{s} g_*^{-1}E_j \right\rceil\right).$$

On the other hand,

$$\mathscr{J}_X(A, Z + cX) \coloneqq \pi_* \mathscr{O}_{A_2}\left(\lceil K_{A_2/A} - \pi^{-1}Z - c\pi^{-1}X\rceil + X_2\right).$$

So the result follows from the fact that $X_2 = \sum_{j=1}^{s} g_*^{-1}E_j$. □

3.2 Comparison with Eisenstein's Subadditivity Theorem

Our main technical result in this section is that the adjoint ideal equals the test ideal along a subscheme mod $p \gg 0$, even when A is singular. We will devote a subsection to each containment: in Section 3.2.1, we show the easier containment, which is that $\mathscr{J}_X$ reduces mod $p \gg 0$ to an ideal larger than τ_I. In Section 3.2.2 we show that $\mathscr{J}_X$ reduces mod $p \gg 0$ to an ideal contained in τ_I. This extends earlier results by Takagi in the setting where X is a divisor [Tak08] and in the setting where A is regular [Tak13]. We will mainly work in the following setting.

Setting 3.2.1. R is a $\mathbb{Q}$-Gorenstein ring essentially of finite type over a perfect field $\Bbbk$. $I \subseteq R$ is a prime ideal of height c and R_I is regular. For $1 \leq i \leq N$, $\mathfrak{a}_i \subseteq R$ are ideals and $t_i \geq 0$ are rational numbers. We further assume that $\mathfrak{a}_i R_I = I^{n_i} R_I$ for each i and also $\sum_i n_i t_i = c$. Set $\mathscr{C} = \mathscr{C}^{R, \prod_i \mathfrak{a}_i^{t_i}}$.

Further, set $A = \operatorname{Spec} R$, $X = \operatorname{Spec}(R/I) \subseteq A$, and $Z_i = \operatorname{Spec}(R/\mathfrak{a}_i) \subseteq A$. Let Z denote the formal sum $Z = \sum_i t_i Z_i$. If R has characteristic 0, let $S \subseteq k$ be a descent datum[2] and let $s \in \operatorname{MaxSpec} S$. Set $\kappa \coloneqq \kappa(s)$ and $p = \operatorname{char} \kappa$. Then we write R_κ to denote the mod-p reduction of R at s, and similarly for $I, \mathfrak{a}_i, A, X$, and Z_i. We further denote $\sum_i t_i (Z_i)_\kappa$ by Z_κ and we denote $\prod_i (\mathfrak{a}_i)_\kappa^{t_i}$ by $\underline{\mathfrak{a}}_\kappa^t$.

Note that this setting is agnostic to the characteristic of R.

Theorem 3.2.2. *Work in Setting 3.2.1 and assume that* $\operatorname{char} R = 0$. *Then for all* $s \in \operatorname{MaxSpec} S$ *sufficiently general we have* $\left(\mathscr{J}_X(A, Z)\right)_\kappa = \tau_{I_\kappa}(R_\kappa, \underline{\mathfrak{a}}_\kappa^t)$, *where* $\kappa = \kappa(s)$.

[2] See Section 1.2

Remark 3.2.3. Working in the setting of Definition 3.1.22, we can find a descent datum $B \subseteq k$ and reduce the maps $f\colon A' \xrightarrow{\psi} \overline{A} \xrightarrow{\pi} A$ modulo p. For $\mu \in \operatorname{MaxSpec} B$ sufficiently general, $\pi_\mu\colon \overline{A}_\mu \to A_\mu$ will still be a factorizing resolution of $X_\mu \subseteq A_\mu$. In this way, we can define the multiplier ideal $\mathscr{J}_{X_\mu}(A_\mu, Z_\mu)$ for μ sufficiently general. By generic freeness, we can choose our descent datum so that $\mathscr{O}_{A'_B}\left(\left\lceil K_{A'_B/A_B} - f_B^{-1} Z_B \right\rceil + X'_B\right)$, as well as all of its cohomology sheaves, are flat over B. It follows from [Har98, Lemma 4.1] that the mod p reduction of $\mathscr{J}_X(A,Z)$ at μ equals $\mathscr{J}_{X_{\kappa(\mu)}}\left(A_{\kappa(\mu)}, Z_{\kappa(\mu)}\right)$ for μ sufficiently general. Thus, to prove Theorem 3.2.2, it suffices to show that $\mathscr{J}_{X_\kappa}(A_\kappa, Z_\kappa) = \tau_{I_\kappa}\left(R_\kappa, \mathfrak{a}_\kappa^t\right)$ for all $s \in \operatorname{MaxSpec} S$ sufficiently general.

Before proving Theorem 3.2.2, we show how this theorem allows us to compare our subadditivity formula with the one obtained in [Eis10]. Consider the following setting:

Setting 3.2.4. R is a $\mathbb{Q}$-Gorenstein ring essentially of finite type over a field $\Bbbk$ of characteristic 0 and $\mathfrak{a}, \mathfrak{b} \subseteq R$ are ideals. Set $X = \operatorname{Spec} R$, $Z_1 = V(\mathfrak{a})$, and $Z_2 = V(\mathfrak{b})$. Let $t_1, t_2 \geq 0$ be rational numbers. Let $S \subseteq \Bbbk$ be a descent datum and $s \in \operatorname{MaxSpec} S$. Set $\kappa = \kappa(s)$.

In [Eis10], Eisenstein derives a new formula for restriction multiplier ideals to closed subschemes. By carefully studying the case $\Delta \subseteq X \times_{\Bbbk} X$, where $\Delta = X$ is the diagonal, he arrives at the containment:

$$\operatorname{adj}_\Delta\left(X \times_{\Bbbk} X, t_1 p_1^* Z_1 + t_2 p_2^* Z_2\right) \cdot \mathscr{O}_\Delta \supseteq \overline{\operatorname{Jac}_X} \mathscr{J}\left(X, t_1 Z_1 + t_2 Z_2\right) \tag{3.6}$$

where $p_i\colon X \times_{\Bbbk} X \to X$ are the projection maps [Eis10, Proof of Theorem 6.5]. The left-hand side of (3.6) is easily seen to be contained in the product of multiplier ideals, $\mathscr{J}(X, Z_1)\mathscr{J}(X, Z_2)$ [Eis10, Lemma 6.2]. Now,

$$\operatorname{adj}_\Delta\left(X \times_{\Bbbk} X, t_1 p_1^* Z_1 + t_2 p_2^* Z_2\right) = \mathscr{J}_\Delta\left(X \times_{\Bbbk} X, t_1 p_1^* Z_1 + t_2 p_2^* Z_2 + d\Delta\right)$$

where $d = \dim X$. By [HY03], we have

$$\left(\overline{\operatorname{Jac}_X} \mathscr{J}\left(X, t_1 Z_1 + t_2 Z_2\right)\right)_\kappa = \overline{\operatorname{Jac}(R_\kappa)}\tau\left(R_\kappa, (\mathfrak{a}_\kappa)^{t_1}(\mathfrak{b}_\kappa)^{t_2}\right)$$

for s sufficiently general. So we see, combining Theorem 3.2.2 with (3.6), that

$$\begin{aligned}&\tau_{(I_\Delta)_\kappa}\left((R \otimes_{\Bbbk} R)_\kappa, (\mathfrak{a} \otimes_{\Bbbk} R)_\kappa^{t_1} (R \otimes_{\Bbbk} \mathfrak{b})_\kappa^{t_2} (I_\Delta)_\kappa^d\right) \cdot (R \otimes_{\Bbbk} R)_\kappa / (I_\Delta)_\kappa \\ &\supseteq \overline{\operatorname{Jac}(R_\kappa)}\tau\left(R_\kappa, (\mathfrak{a}_\kappa)^{t_1}(\mathfrak{b}_\kappa)^{t_2}\right)\end{aligned}$$

for all s sufficiently general. As $\mathscr{C}^{(R\otimes_{\not{k}}R)_\kappa,(I_\Delta)^d_\kappa}$ is compatible with $(I_\Delta)_\kappa$ (*Cf.* Example 3.1.7), it follows from Lemma 3.1.13 that

$$\begin{aligned}&\tau_{(I_\Delta)_\kappa}\left((R\otimes_{\not{k}}R)_\kappa, (\mathfrak{a}\otimes_{\not{k}}R)^{t_1}_\kappa\cdot(R\otimes_{\not{k}}\mathfrak{b})^{t_2}_\kappa\cdot(I_\Delta)^d_\kappa\right)\\ \subseteq\,&\tau_{(I_\Delta)_\kappa}\left((R\otimes_{\not{k}}R)_\kappa, \mathscr{C}^{R_\kappa\otimes_\kappa R_\kappa,\,(I_\Delta)_\kappa\circlearrowleft}, (\mathfrak{a}\otimes_{\not{k}}R)^{t_1}_\kappa\cdot(R\otimes_{\not{k}}\mathfrak{b})^{t_2}_\kappa\right)\end{aligned}$$

Here we're using the fact that $R_\kappa\otimes_\kappa R_\kappa = (R\otimes_{\not{k}}R)_\kappa$. By Proposition 3.1.14, we have

$$\begin{aligned}&\tau_{(I_\Delta)_\kappa}\left((R\otimes_{\not{k}}R)_\kappa, \mathscr{C}^{R_\kappa\otimes_\kappa R_\kappa,\,(I_\Delta)_\kappa\circlearrowleft}, (\mathfrak{a}\otimes_{\not{k}}R)^{t_1}_\kappa\cdot(R\otimes_{\not{k}}\mathfrak{b})^{t_2}_\kappa\right)\cdot(R\otimes_{\not{k}}R)_\kappa/(I_\Delta)_\kappa\\ =&\tau(R_\kappa,\mathscr{D}^{(2)},\mathfrak{a}^{t_1}_\kappa\mathfrak{b}^{t_2}_\kappa).\end{aligned}$$

Thus we have shown, assuming Theorem 3.2.2:

Corollary 3.2.5. *Work in Setting 3.2.4. Then*

$$\overline{\mathrm{Jac}(R_\kappa)}\tau(R_\kappa,(\mathfrak{a}_\kappa)^{t_1}(\mathfrak{b}_\kappa)^{t_2})\subseteq\tau(R_\kappa,\mathscr{D}^{(2)},(\mathfrak{a}_\kappa)^{t_1}(\mathfrak{b}_\kappa)^{t_2})$$

for all s sufficiently general.

Namely, our subadditivity formula,

$$\tau(R_\kappa,\mathscr{D}^{(2)},(\mathfrak{a}_\kappa)^{t_1}(\mathfrak{b}_\kappa)^{t_2})\subseteq\tau(R_\kappa,(\mathfrak{a}_\kappa)^{t_1})\tau(R_\kappa,(\mathfrak{b}_\kappa)^{t_2})$$

is a sharper containment than the previously known formula,

$$\overline{\mathrm{Jac}(R_\kappa)}\tau(R_\kappa,(\mathfrak{a}_\kappa)^{t_1}(\mathfrak{b}_\kappa)^{t_2})\subseteq\tau(R_\kappa,(\mathfrak{a}_\kappa)^{t_1})\tau(R_\kappa,(\mathfrak{b}_\kappa)^{t_2}).$$

3.2.1 Proof that $(\mathscr{J}_X)_\kappa\supseteq\tau_{I_\kappa}$

We prove Theorem 3.2.2 in two parts. The first part is easier and just uses the minimality of τ_I. The second part follows an argument similar to [Tak08, Theorem 5.3] and requires a variant of Hara's surjectivity theorem [Har98].

Recall that for any normal F-finite scheme[3] X of characteristic p and any map $\varphi\colon F^e_*\mathscr{O}_X\to\mathscr{O}_X$ we can associate an effective $\mathbb{Q}$-divisor Δ_φ on X such that $K_X+\Delta_\varphi$ is $\mathbb{Q}$-Cartier with Cartier index not divisible by p (*Cf.* [BS13], [Sch09, Section 3]). If h is a global section of $\mathscr{O}_X$ and we set $\varphi_h=\varphi(F^e_*h\cdot -)$, then $\Delta_{\varphi_h}=\Delta_\varphi+1/(p^e-1)\operatorname{div}h$.

Now suppose that X is an F-finite integral scheme with fraction field sheaf $\mathscr{K}(X)$. As localization commutes with the F^e_* functor, any map $\varphi\colon F^e_*\mathscr{O}_X\to\mathscr{O}_X$ induces a map $\widehat{\varphi}\colon F^e_*\mathscr{K}(X)\to\mathscr{K}(X)$.

[3] Let X be a scheme of positive characteristic. Then the Frobenius map on each affine chart of X induces a morphism of schemes $F\colon X\to X$ called the *absolute Frobenius morphism*. We say X is F-finite if $F^e_*\mathscr{O}_X$ is a coherent $\mathscr{O}_X$-module for some (equivalently, all) $e>0$.

Lemma 3.2.6 (*Cf.* [Sch10, HW02])**.** *Let R be a normal F-finite domain and set $X = \operatorname{Spec} R$. Suppose $\pi\colon Y \to X$ is a log-resolution of the ideals $\mathfrak{a}_i \subseteq R$ and set $\mathfrak{a}_i\mathscr{O}_Y = \mathscr{O}_Y(-G_i)$. Let $t_i > 0$ be a collection of rational numbers. Then for any map $\varphi\colon F^e_*\mathscr{O}_X \to \mathscr{O}_X$ we have*

$$\widehat{\varphi}\left(F^e_*\underline{\mathfrak{a}}^{\lceil t(p^e-1)\rceil}\mathscr{O}_Y\left(\left\lceil K_Y - \pi^*(K_X+\Delta) - \sum_i t_iG_i\right\rceil + E\right)\right)$$
$$\subseteq \mathscr{O}_Y\left(\left\lceil K_Y - \pi^*(K_X+\Delta) - \sum_i t_iG_i\right\rceil + E\right),$$

where $\widehat{\varphi}$ is the induced map $F^e_*\mathscr{K}(X) \to \mathscr{K}(X)$, E is any effective divisor on Y, and Δ is any divisor on X such that $\Delta \le \Delta_\varphi$ and $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Proof. We work locally, so that $\pi_*K_Y = K_X$. Let $h \in \underline{\mathfrak{a}}^{\lceil t(p^e-1)\rceil}$. Then $\Delta_{\varphi_h} = \Delta_\varphi + 1/(p^e-1)\operatorname{div} h$. By the proof of [Sch10, Theorem 6.7], we have

$$\widehat{\varphi}_h\left(F^e_*\mathscr{O}_Y\left(\left\lceil K_Y - \pi^*\left(K_X+\Delta_\varphi+\frac{1}{p^e-1}\operatorname{div} h\right)\right\rceil + F\right)\right)$$
$$\subseteq \mathscr{O}_Y\left(\left\lceil K_Y - \pi^*\left(K_X+\Delta_\varphi+\frac{1}{p^e-1}\operatorname{div} h\right)\right\rceil + F\right)$$

for any integral effective divisor F. Set

$$F = \left\lceil K_Y - \pi^*(K_X+\Delta_\varphi) - \sum t_iG_i\right\rceil - \left\lceil K_Y - \pi^*\left(K_X+\Delta_\varphi+\frac{1}{p^e-1}\operatorname{div} h\right)\right\rceil$$

Then F is integral. Also F is effective, as $\operatorname{div} h \ge \sum\lceil t_i(p^e-1)\rceil G_i$, which means $1/(p^e-1)\operatorname{div} h \ge \sum t_iG_i$. Thus we have

$$\widehat{\varphi}_h\left(F^e_*\mathscr{O}_Y\left(\left\lceil K_Y - \pi^*(K_X+\Delta_\varphi) - \sum t_iG_i\right\rceil\right)\right) \subseteq \mathscr{O}_Y\left(\left\lceil K_Y - \pi^*(K_X+\Delta_\varphi) - \sum t_iG_i\right\rceil\right)$$

Then for any effective divisor E we have

$$\widehat{\varphi}_h\left(F^e_*\mathscr{O}_Y\left(\left\lceil K_Y - \pi^*(K_X+\Delta_\varphi) - \sum t_iG_i\right\rceil + E\right)\right)$$
$$\subseteq \mathscr{O}_Y\left(\left\lceil K_Y - \pi^*(K_X+\Delta_\varphi) - \sum t_iG_i\right\rceil + E\right)$$

using the projection formula, the fact that $(F^e)^*(\mathscr{O}_Y(E)) = \mathscr{O}_Y(p^eE)$, and the fact that $E \le p^eE$.

Similarly, for any $\Delta \le \Delta_\varphi$ with $K_X + \Delta$ $\mathbb{Q}$-Cartier, we have

$$\left\lceil K_Y - \pi^*(K_X+\Delta) - \sum t_iG_i\right\rceil - \left\lceil K_Y - \pi^*(K_X+\Delta_\varphi) - \sum t_iG_i\right\rceil \ge 0,$$

and so

$$\widehat{\varphi}_h\left(F^e_*\mathscr{O}_Y\left(\left\lceil K_Y - \pi^*(K_X+\Delta) - \sum t_iG_i\right\rceil + E'\right)\right)$$
$$\subseteq \mathscr{O}_Y\left(\left\lceil K_Y - \pi^*(K_X+\Delta) - \sum t_iG_i\right\rceil + E'\right),$$

for any effective divisor E', as desired. □

Theorem 3.2.7. *Work in Setting 3.2.1 and assume that* $\operatorname{char} R = 0$. *Then* $\mathscr{J}_{X_\kappa}(A_\kappa, Z_\kappa) \supseteq \tau_{I_\kappa}(R_\kappa, \underline{\mathfrak{a}}_\kappa^t)$ *for* s *sufficiently general.*

Proof. For s sufficiently general, we have that $p = \operatorname{char} \kappa$ does not divide the denominator of any t_i and thus $\tau_{I_\kappa}(R_\kappa, \underline{\mathfrak{a}}_\kappa^t)$ is well-defined. Fix such an s. We just need to prove two things:

- $\varphi\left(F_*^e \underline{\mathfrak{a}}_\kappa^{\lceil t(p^e-1)\rceil} \mathscr{J}_{X_\kappa}(A_\kappa, Z_\kappa)\right) \subseteq \mathscr{J}_{X_\kappa}(A_\kappa, Z_\kappa)$, for all $e > 0$ and $\varphi \in \mathscr{C}_e^R$, and
- $\mathscr{J}_{X_\kappa}(A_\kappa, Z_\kappa) \not\subseteq I_\kappa$.

Set $\mathfrak{a}_i \mathscr{O}_{A'} = \mathscr{O}_{A'}(-F_i)$. Then, by definition,

$$\mathscr{J}_{X_\kappa}(A_\kappa, Z_\kappa) = (f_\kappa)_* \mathscr{O}_{A'_\kappa}\left(\left\lceil K_{A'_\kappa} - f_\kappa^* K_{A_\kappa} - \sum_i t_i (F_i)_\kappa \right\rceil + X'_\kappa\right)$$

We see that the first assertion follows from Lemma 3.2.6, using $\Delta = 0$.

The second assertion is something we can check locally at I_κ, so now we assume that R_κ is a local ring with maximal ideal I_κ. But then, by assumption, $\mathscr{O}_{A'_\kappa}(-\sum_i t_i (F_i)_\kappa) = \mathscr{O}_{A'_\kappa}(-cX'_\kappa)$. So we see

$$\mathscr{J}_{X_\kappa}(A_\kappa, Z_\kappa) = (f_\kappa)_* \mathscr{O}_{A'_\kappa}\left(\lceil \psi^* K_{\overline{A}_\kappa} - f_\kappa^* K_{A_\kappa} \rceil\right)$$

and $\psi^* K_{\overline{A}_\kappa} - f_\kappa^* K_{A_\kappa}$ has no support along X'_κ. ☐

3.2.2 Proof that $(\mathscr{J}_X)_\kappa \subseteq \tau_{I_\kappa}$

For the other containment, we use a similar argument to the one in [Tak08]. First, we recall Hara's surjectivity theorem. The following statement is slightly stronger than the one found in [Har98]: in [Har98], the author assumes that X is the blow-up of an ideal sheaf on Y. However, the same proof actually shows the following statement, where X is just assumed to be projective over Y and smooth.

Theorem 3.2.8 ([Har98, Section 4.3])**.** *Let* $Y = \operatorname{Spec} R$, *where* R *is finitely generated over a field* $\Bbbk$ *of characteristic* 0, *and let* X *be a smooth Noetherian scheme projective over* Y. *Suppose* E *is a reduced simple normal crossings divisor on* X *and* D *an ample divisor with* $\operatorname{supp}(D - \lfloor D \rfloor) \subseteq \operatorname{supp} E$. *Choose some finitely generated* $\mathbb{Z}$*-subalgebra* B *of* $\Bbbk$, *over which we do our reduction mod* p. *For any closed point* $s \in S = \operatorname{Spec} B$ *with residue field* $\kappa = \kappa(s)$, *let* Y_κ, X_κ, E_κ, *and* D_κ *be the fibers of the corresponding objects over* s. *Then, for sufficiently general closed points* s,

$$\begin{array}{lll} (a) & H^j(X_\kappa, \Omega^i_{X_\kappa/\kappa}(\log E_\kappa)(-E_\kappa - \lfloor -p^e D_\kappa \rfloor)) = 0, & i+j > d, e \geq 0 \\ (b) & H^j(X_\kappa, \Omega^i_{X_\kappa/\kappa}(\log E_\kappa)(-E_\kappa - \lfloor -p^{e+1} D_\kappa \rfloor)) = 0, & i+j > 0, e \geq 0 \end{array}$$

where $d = \dim X$ *and* $p = \operatorname{char} \kappa(s)$.

Combined with [Har98, Proposition 3.6] (as stated), we obtain the following result:

Corollary 3.2.9 (*Cf.* [Har98, Section 4.4])**.** *Using notation as in Theorem 3.2.8, the map*

$$(F^e)^\vee\colon H^0(X_\kappa, F^e_*\omega_{X_\kappa}(\lceil p^e D_\kappa\rceil)) \to H^0(X_\kappa, \omega_{X_\kappa}(\lceil D_\kappa\rceil))$$

is surjective for $e > 0$ and for sufficiently general $s \in S$, where $\kappa = \kappa(s)$.

We will also need the following lemmas:

Lemma 3.2.10. *Work in Setting 3.2.1 and assume that R has characteristic 0. There exists $d \in R \smallsetminus I$ such that, for all sufficiently general s, a power of d_κ (depending on s) is an $\underline{\mathfrak{a}}^t_\kappa$-test element along I_κ in R_κ.*

Proof. If s is sufficiently general, then $p = \operatorname{char}\kappa(s)$ will not divide the denominator of any t_i and $\mathscr{C}_\kappa := \mathscr{C}^{R_\kappa, \underline{\mathfrak{a}}^t_\kappa}$ will satisfy the conditions of Setting 3.1.1. As R_I is regular, we can find a regular sequence $(x_1, \ldots, x_c)$ in R such that $(x_1, \ldots, x_c)R_I = IR_I$. Set $J = (x_1, \ldots x_c)$. Then there exists an element $d \in R \smallsetminus I$ such that R_d is regular, R_d/IR_d is regular, and $J^{n_i}R_d \subseteq \mathfrak{a}_i R_d$ for all i. Note that $(R_d)_\kappa$ is the same as R_κ localized at d_κ. We set $R_{d,\kappa} := (R_\kappa)_{d_\kappa}$ and $\mathscr{C}_{d,\kappa} := (\mathscr{C}_\kappa)_{d_\kappa}$. Then we have

$$\tau_{I_\kappa}(R_\kappa, \mathscr{C}_\kappa)R_{d,\kappa} = \tau_{I_\kappa R_{d,\kappa}}(R_{d,\kappa}, \mathscr{C}_{d,\kappa}) \supseteq \tau_{J_\kappa R_{d,\kappa}}(R_{d,\kappa}, J^c_\kappa R_{d,\kappa}),$$

where the containment follows quickly from the minimality of $\tau_{J_\kappa R_{d,\kappa}}(R_{d,\kappa}, J^c_\kappa R_{d,\kappa})$. Thanks to Lemma 3.2.11, we see that $\tau_{I_\kappa}(R_\kappa, \mathscr{C}_\kappa)R_{d,\kappa} = R_{d,\kappa}$. Thus, there exists some N such that $d^N_\kappa \in \tau_{I_\kappa}(R_\kappa, \mathscr{C}_\kappa)$. As $d^N_\kappa \in R_\kappa \smallsetminus I_\kappa$, we're done. □

Lemma 3.2.11. *Let k be a perfect field of characteristic p. Let R be a regular $\Bbbk$-algebra essentially of finite type and I a prime ideal generated by a regular sequence of length c. Suppose also that R/I is regular. Then $\tau_I(R, I^c) = R$.*

Proof. This fact is well-known to experts, and is essentially shown in [Tak13, Theorem 3.2]. □

Lemma 3.2.12. *Let $I \subseteq R$ be a prime ideal such that R_I is regular. Let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a}R_I = I^m R_I$ for some $m \geq 0$. Then there exists some $\xi \in R \smallsetminus I$ such that $\xi\overline{\mathfrak{a}^n} \subseteq \mathfrak{a}^n$ for all integers n.*

Proof. By [SH06, Proposition 5.3.4], there is some integer k, such that $\overline{\mathfrak{a}^n} = \mathfrak{a}^{n-k}\overline{\mathfrak{a}^k}$ for all $n \geq k$. As R_I is regular, we have IR_I is generated by a regular sequence and is therefore a *normal* ideal, meaning $I^n R_I$ is integrally closed for all n (see, for instance, [SH06, Exercise 5.7]). As integral closure of ideals commutes with localization, we have

$$\overline{\mathfrak{a}^n} R_I = \overline{\mathfrak{a}^n R_I} = \overline{I^{nm} R_I} = I^{nm} R_I = \mathfrak{a}^n R_I,$$

for all n. Thus, for $n = 1, \dots k$ there exist elements $\xi_n \in R \smallsetminus I$ satisfying $x_n \overline{\mathfrak{a}^n} \subseteq \mathfrak{a}^n$. Then we can set $\xi = \xi_1 \cdots \xi_k$. ☐

Lemma 3.2.13. *Work in Setting 3.2.1 and assume that* $\operatorname{char} R = 0$. *There exists some* $\xi \in R \smallsetminus I$ *such that* $\xi \overline{\underline{\mathfrak{a}}^{\lceil t(q-1) \rceil}} \subseteq \underline{\mathfrak{a}}^{\lceil t(q-1) \rceil}$ *for all* $p \gg 0$ *and all* $e > 0$ *sufficiently divisible, where* $q = p^e$.

Proof. For each i, write $t_i = a_i / b_i$. Set m to be the least common multiple of the b_i, so that for each i there exists an integer a_i' such that $t_i = a_i'/m$. For p sufficiently large, we have p does not divide b_i for each i. Then for e sufficiently divisible, we have $m \mid (p^e - 1)$. Thus

$$\prod_i \mathfrak{a}_i^{\lceil t_i (p^e - 1) \rceil} = \prod_i \mathfrak{a}_i^{a_i'(p^e-1)/m} = \left(\prod_i \mathfrak{a}_i^{a_i'} \right)^{\frac{p^e - 1}{m}}.$$

Then we can find the desired element ξ by applying Lemma 3.2.12 to the ideal $\mathfrak{a} = \prod_i \mathfrak{a}_i^{a_i'}$. ☐

Theorem 3.2.14. *Work in Setting 3.2.1 and assume that* $\operatorname{char} R = 0$. *Then* $\mathscr{J}_{X_\kappa}(A_\kappa, Z_\kappa) \subseteq \tau_{I_\kappa}(R_\kappa, \underline{\mathfrak{a}}_\kappa^t)$ *for all* s *sufficiently general.*

Proof. We start by finding a suitable resolution of A. In particular, we need a resolution $f \colon B \to A$ that can be used to compute $\mathscr{J}_X(A, Z)$ such that there exists a divisor V on B satisfying the following:

- $f^{-1} Z \cup \operatorname{exc}(f) \cup \operatorname{supp}(f^* K_A) \cup \operatorname{supp} V$ is a simple normal crossings scheme, [4]
- V does not dominate X, and
- $-\varepsilon V - F$ is relatively ample over A for any $\varepsilon > 0$ sufficiently small, where $\mathfrak{a}_i \mathscr{O}_B = \mathscr{O}_B(-F_i)$ and $F = \sum_i t_i F_i$.

As in Definition 3.1.22, let $\pi_1 \colon A_1 \to A$ be a factorizing resolution of $X \subseteq A$ such that the components of Z not containing X lift to divisors, and such that $\pi_1^{-1} Z \cup \operatorname{exc}(\pi_1) \cup \operatorname{supp}(\pi_1^* K_A)$ has simple normal crossings support. Let $X_1 \subseteq A_1$ be the strict transform of X in A_1 and let

[4] Recall from Notation 3.1.19 that by $\operatorname{exc}(f)$ we mean the locus of points at which f is not an isomorphism. See [Eis10, §2] for the definition of a simple normal crossings scheme.

$\pi_2 \colon A_2 \to A_1$ be the blow-up along X_1. Let X_2 be the reduced π_2-exceptional divisor dominating X_1. We have the following diagram:

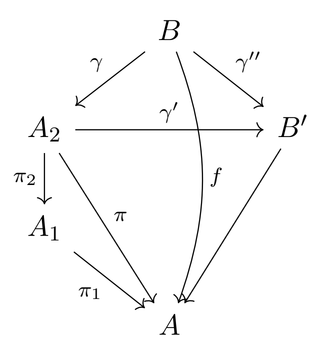
$$\begin{array}{ccc} X_2 & \subseteq & A_2 \\ \downarrow & & \downarrow{\scriptstyle \pi_2} \\ X_1 & \subseteq & A_1 \\ \downarrow & & \downarrow{\scriptstyle \pi_1} \\ X & \subseteq & A\,. \end{array}$$

Let $\mathfrak{a}_i \mathscr{O}_{A_2} = \mathscr{O}_{A_2}(-F_i')$ and set $F' = \sum t_i F_i'$. Setting $\pi = \pi_1 \circ \pi_2$, we get

$$\mathscr{J}_X(A, Z) = \pi_* \mathscr{O}_{A_2}\left(K_{A_2} - \lfloor \pi^* K_A + F' \rfloor + X_2\right),$$

by definition. However, we are not finished constructing the resolution we need, as we don't know whether a divisor V satisfying the properties above exists on A_2.

Note that $-F'$ is relatively big and relatively semi-ample over A. The latter implies that there exist natural numbers $a, N > 0$ and a map $\gamma' \colon X_2 \to \mathbb{P}^N_A$ such that $\mathscr{O}_{A_2}(-aF') \cong (\gamma')^* \mathscr{O}_{\mathbb{P}^N_A}(1)$. As $-F'$ is relatively big, γ' is birational onto its image, which we call B' (which is just the blow-up of Z in A). Note that γ' is an embedding at the generic point of X_2 and therefore an isomorphism over an open neighborhood of X in A. Thus the exceptional locus of γ' does not contain X_2, and we can perform a sequence of blow-ups of schemes not containing X_2, call it $\gamma \colon B \to A_2$, to get that $\operatorname{exc}(\gamma' \circ \gamma) \cup \gamma^{-1}\pi^{-1}Z \cup \operatorname{supp}(\gamma^*\pi^*K_A)$ has simple normal crossings support. Let $\gamma'' = \gamma' \circ \gamma''$ and $f = \pi \circ \gamma$. These definitions are illustrated by the following diagram:



We further define $\mathfrak{a}_i \mathscr{O}_B = \mathscr{O}_B(-F_i)$ for all i and $F = \sum t_i F_i$.

Note that the exceptional locus of γ'' does not dominate $\gamma'(X_2)$. Thus we can write γ'' as the blow-up of some ideal sheaf $\mathscr{I} \subseteq \mathscr{O}_{B'}$ that does not vanish along $\gamma'(X_2)$. Let W be the effective divisor on B such that $\mathscr{I}\mathscr{O}_B = \mathscr{O}_B(-W)$. Then we see that $\mathscr{O}_B(-W) \otimes (\gamma'')^* \mathscr{O}_{B'}(b)$ is very ample over A for all b sufficiently large. Set $V = \frac{1}{ab} W$. Then $\mathscr{O}_B(ab(-V - F)) \cong \mathscr{O}_B(-W) \otimes$

$(\gamma'')^*\mathscr{O}_{B'}(b)$, so $-V-F$ is ample over A. Because W is effective, we may assume that

$$\lfloor f^*K_A + F + V\rfloor = \lfloor f^*K_A + F\rfloor,$$

by choosing b sufficiently large. Fix canonical divisors K_{A_1}, K_{A_2}, and K_B so that $K_{A_1/A}$ is π_1-exceptional, $K_{A_2/A_1} = (c-1)X_2$, and K_{B/A_2} is γ-exceptional. Let $\widetilde{X} = \gamma_*^{-1}X_2$. Then

$$\gamma_*\mathscr{O}_B\left(K_B + \widetilde{X} - \lfloor f^*K_A + F\rfloor\right) = \mathscr{O}_{A_2}\left(K_{A_2} + X_2 - \lfloor \pi^*K_A + F'\rfloor\right),$$

just because adjoint ideals are well-defined. Indeed, it suffices to check the above equality after twisting each side by $\mathscr{O}_{A_2}(-\pi_2^*K_{A_1})$. As $K_{A_2} = (c-1)X_2 + \pi_2^*K_{A_1}$, we get

$$\begin{aligned}
&\gamma_*\mathscr{O}_B\left(K_B + \widetilde{X} - \lfloor f^*K_A + F\rfloor\right) \otimes \mathscr{O}_{A_2}\left(-\pi_2^*K_{A_1}\right)\\
=&\gamma_*\mathscr{O}_B\left(K_{B/A_2} + \gamma^*K_{A_2} - \gamma^*\pi_2^*K_{A_1} + \widetilde{X} - \lfloor f^*K_A + F\rfloor\right)\\
=&\gamma_*\mathscr{O}_B\left(K_{B/A_2} + (c-1)\gamma^*X_2 + \widetilde{X} - \lfloor f^*K_A + F\rfloor\right).
\end{aligned}$$

By our assumptions in Setting 3.2.1, we have $F' = cX_2 + G$ for some effective SNC $\mathbb{Q}$-divisor G not supported along X_2. Thus:

$$\begin{aligned}
&\gamma_*\mathscr{O}_B\left(K_{B/A_2} + (c-1)\gamma^*X_2 + \widetilde{X} - \lfloor f^*K_A + F\rfloor\right)\\
=&\gamma_*\mathscr{O}_B\left(K_{B/A_2} + (c-1)\gamma^*X_2 + \widetilde{X} - \lfloor f^*K_A + \gamma^*(cX_2 + G)\rfloor\right)\\
=&\gamma_*\mathscr{O}_B\left(K_{B/A_2} + \widetilde{X} - \lfloor f^*K_A + \gamma^*(X_2 + G)\rfloor\right).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\mathscr{O}_{A_2}\left(K_{A_2} + X_2 - \lfloor \pi^*K_A + F'\rfloor\right) \otimes \mathscr{O}_{A_2}\left(-\pi_2^*K_{A_1}\right)\\
=&\mathscr{O}_{A_2}\left((c-1)X_2 + X_2 - \lfloor \pi^*K_A + cX_2 + G\rfloor\right)\\
=&\mathscr{O}_{A_2}\left(X_2 - \lfloor \pi^*K_A + X_2 + G\rfloor\right).
\end{aligned}$$

By definition, we have

$$\begin{aligned}
\mathrm{adj}_X(A, \pi_2^*K_A + G) &= \gamma_*\mathscr{O}_B\left(K_{B/A_2} + \widetilde{X} - \lfloor f^*K_A + \gamma^*(X_2 + G)\rfloor\right)\\
&= \mathscr{O}_{A_2}\left(X_2 - \lfloor \pi^*K_A + X_2 + G\rfloor\right),
\end{aligned}$$

because both γ and the identity map are log-resolutions of $\pi_2^*K_A + G$. In particular, we obtain the equality $\mathscr{J}_X(A, Z) = f_*\mathscr{O}_B\left(K_B + \widetilde{X} - \lfloor f^*K_A + F\rfloor\right)$.

Employing Lemma 3.2.10, choose some $d \in R \smallsetminus I$ such that, for s sufficiently general, there is some N such that d_κ^N is an $\underline{\mathfrak{a}}_\kappa^t$-test element along I_κ in R_κ. Choose also an element $\xi \in R \smallsetminus I$ as in Lemma 3.2.13.

Next, we show there exists $\eta \in R \smallsetminus I$ satisfying

$$K_{B/A_2} + \gamma^*\pi_2^* K_{A_1} - \lfloor m f^* K_A \rfloor - \operatorname{div}_B \eta \le f^*\left((1-m)K_A\right) + \gamma^* X_2 - \widetilde{X}$$

for all m such that $(1-m)K_A$ is Cartier. As $f^*\left((1-m)K_A\right)$ is integral in this case, it suffices to find $\eta \in R \smallsetminus I$ satisfying

$$K_{B/A_2} + \gamma^*\pi_2^* K_{A_1} - \operatorname{div}_B \eta \le \lfloor f^* K_A \rfloor + \gamma^* X_2 - \widetilde{X}.$$

We compute:

$$\lfloor f^* K_A \rfloor + \gamma^* X_2 - \widetilde{X} - K_{B/A_2} - \gamma^*\pi_2^* K_{A_1} = \gamma^* X_2 - \widetilde{X} - K_{B/A_2} - \lceil \gamma^*\pi_2^* K_{A_1} - f^* K_A \rceil.$$

Recall that the exceptional locus of γ does not contain $\widetilde{X}$, so $\gamma^* X_2 - \widetilde{X} - K_{B/A_2}$ is not supported on $\widetilde{X}$. Since the support of $K_{A_1} - \pi_1^* K_A$ does not contain X_1, the support of

$$\gamma^*\pi_2^* K_{A_1} - f^* K_A = \gamma^*\pi_2^*\left(K_{A_1} - \pi_1^* K_A\right)$$

does not contain $\widetilde{X}$. Further, this divisor is f-exceptional. Thus we have

$$H^0\left(A, f_*\mathscr{O}_B\big(\lfloor f^* K_A \rfloor + \gamma^* X_2 - \widetilde{X} - K_{B/A_2} - \gamma^*\pi_2^* K_{A_1}\big)\right) \subseteq R$$

and also

$$H^0\left(A, f_*\mathscr{O}_B\big(\lfloor f^* K_A \rfloor + \gamma^* X_2 - \widetilde{X} - K_{B/A_2} - \gamma^*\pi_2^* K_{A_1}\big)\right) \not\subseteq I,$$

so we can find the desired element η by taking

$$\eta \in H^0\left(A, f_*\mathscr{O}_B\big(\lfloor f^* K_A \rfloor + \gamma^* X_2 - \widetilde{X} - K_{B/A_2} - \gamma^*\pi_2^* K_{A_1}\big)\right) \smallsetminus I.$$

The utility of this η will be made apparent in Claim 3.2.15.

Next, we define

$$D = f^* K_A + F + V + \varepsilon \operatorname{div}_B(d\xi\eta),$$

where $\varepsilon > 0$ is chosen to be small enough such that

$$\lfloor D \rfloor = \lfloor f^* K_A + F \rfloor.$$

Note that D is f-anti-ample, as $F+V$ is f-anti-ample, f^*K_A is f-numerically trivial, and $\operatorname{div}_B(d\xi\eta)$ is f-anti-nef. For all $m \in \mathbb{N}_{>0}$, we have a short exact sequence of sheaves on A',

$$0 \to \mathscr{O}_B\left(K_B - \lfloor mD \rfloor\right) \to \mathscr{O}_B(K_B - \lfloor mD \rfloor + \widetilde{X}) \to \mathscr{A}_m \to 0.$$

for some $\mathscr{A}_m$ supported on $\widetilde{X}$.

Then we get an exact sequence

$$\begin{aligned} 0 &\to H^0(B, \mathscr{O}_B(K_B - \lfloor mD \rfloor)) \to H^0(B, \mathscr{O}_B(K_B - \lfloor mD \rfloor + \widetilde{X})) \\ &\to H^0(\widetilde{X}, \mathscr{A}_m) \to H^1(B, \mathscr{O}_B(K_B - \lfloor mD \rfloor)), \end{aligned}$$

Now, $-mD$ is an f-big and f-nef divisor whose fractional part has simple normal crossings support. By Kawamata-Viehweg vanishing [Laz04, Corollary 9.1.20], we have

$$H^1(B, \mathscr{O}_B(K_B - \lfloor mD \rfloor)) = H^1(B, \mathscr{O}_B(K_B + \lceil -mD \rceil)) = 0.$$

Thus, we have the short exact sequence,

$$\begin{aligned} 0 &\to H^0(B, \mathscr{O}_B(K_B - \lfloor mD \rfloor)) \to H^0(B, \mathscr{O}_B(K_B - \lfloor mD \rfloor + \widetilde{X})) \\ &\to H^0(\widetilde{X}, \mathscr{A}_m) \to 0. \end{aligned}$$

Next, we compute $\mathscr{A}_m$. By our assumptions in Setting 3.2.1, we can express F as $F = c\widetilde{X} + G$, where G is a $\mathbb{Q}$-divisor whose support does not contain $\widetilde{X}$, and c is the codimension of X in A. Thus, we get

$$\begin{aligned} &\mathscr{O}_B\left(K_B - \lfloor mf^*K_A + mF + mV + m\varepsilon \operatorname{div}_B(d\xi\eta) \rfloor + \widetilde{X}\right) \\ =&\mathscr{O}_B\left(K_B + \widetilde{X} - \lfloor mf^*K_A + mG + mV + m\varepsilon \operatorname{div}_B(d\xi\eta) \rfloor\right) \otimes \mathscr{O}_B(-mc\widetilde{X}) \end{aligned}$$

Now, $\mathscr{O}_B(c\widetilde{X}) \otimes \mathscr{O}_{\widetilde{X}}$ is a line bundle on $\widetilde{X}$, so we can fix some integral divisor Σ on $\widetilde{X}$ so that $\mathscr{O}_{\widetilde{X}}(\Sigma) \cong \mathscr{O}_B(c\widetilde{X}) \otimes \mathscr{O}_{\widetilde{X}}$. As $(K_B + \widetilde{X})|_{\widetilde{X}} \sim K_{\widetilde{X}}$, it follows that

$$\mathscr{A}_m \cong \mathscr{O}_{\widetilde{X}}\left(K_{\widetilde{X}} - \lfloor m\widetilde{D} \rfloor\right)$$

where we set $\widetilde{D} :- (f^*K_A + G + V + \varepsilon \operatorname{div}_B(d\xi\eta))\,|_{\widetilde{X}} + \Sigma$. Note that restricting to $\widetilde{X}$ here commutes with rounding down because D has simple normal crossings support and D is supported on $\widetilde{X}$. Also, note that $\widetilde{D}$ is anti-ample over A. As Σ is integral, the fractional part of $\widetilde{D}$ has simple normal crossings support.

Set $p = \operatorname{char} \kappa(s)$. For s sufficiently general, we have:

- The map

$$(F^e_{B_\kappa})^\vee \colon H^0\left(B_\kappa, F^e_*\omega_{B_\kappa}\left(-\lfloor p^e D_\kappa\rfloor\right)\right) \to H^0\left(B_\kappa, \omega_{B_\kappa}\left(-\lfloor D_\kappa\rfloor\right)\right)$$

  is a surjection for all $e > 0$. This is possible by Corollary 3.2.9.

- The map

$$(F^e_{\widetilde{X}_\kappa})^\vee \colon H^0\left(\widetilde{X}_\kappa, F^e_*\omega_{\widetilde{X}_\kappa}\left(-\lfloor p^e \widetilde{D}_\kappa\rfloor\right)\right) \to H^0\left(\widetilde{X}_\kappa, \omega_{\widetilde{X}_\kappa}\left(-\lfloor \widetilde{D}_\kappa\rfloor\right)\right)$$

  is a surjection for all e. This is also possible by Corollary 3.2.9. Note that $\widetilde{X}$ is projective over X, so the original statement of Hara's surjectivity theorem would not apply to $\widetilde{X} \to X$.

- p does not divide the Cartier index of K_A.

- p does not divide the denominator of any t_i.

- Some power of d is an $\underline{\mathfrak{a}}^t_\kappa$-test element along I_κ in R_κ.

- $H^0(B, \omega_B(-\lfloor mD\rfloor))_\kappa = H^0(B_\kappa, \omega_{B_\kappa}(-\lfloor mD_\kappa\rfloor))$ for all $m \in \mathbb{N}_{>0}$. This is possible because D is a $\mathbb{Q}$-divisor. Indeed, let v be the least common multiple of the denominators of the coefficients appearing in D. Then

$$\{\lfloor mD\rfloor \mid m \geq 0\} = \{uvD + \lfloor mD\rfloor \mid 0 \leq m < v, u \geq 0\}.$$

  By generic freeness, we can choose our descent datum S to ensure that the coherent sheaves $\omega_B(-\lfloor mD\rfloor)$ and $\mathscr{O}_B(-vD)$, as well as their cohomologies, are flat for $0 \leq m < v$. Then the result follows from [Har98, Lemma 4.1].

- Similarly, we can ensure that $H^0(B, \omega_B(\widetilde{X} - \lfloor mD\rfloor))_\kappa = H^0(B_\kappa, \omega_{B_\kappa}(\widetilde{X}_\kappa - \lfloor mD_\kappa\rfloor))$ for all $m \in \mathbb{N}_{>0}$, and

- $H^0(\widetilde{X}, \omega_{\widetilde{X}}(-\lfloor m\widetilde{D}\rfloor))_\kappa = H^0(\widetilde{X}_\kappa, \omega_{\widetilde{X}_\kappa}(-\lfloor m\widetilde{D}_\kappa\rfloor))$ for all $m \in \mathbb{N}_{>0}$.

Fix such an s. Fix also a number N so that d^N_κ is an $\underline{\mathfrak{a}}^t_\kappa$-test element along I_κ in R_κ. Then for all $e \in \mathbb{N}$ sufficiently divisible we have:

- $\xi\overline{\underline{\mathfrak{a}}^{\lceil t(p^e-1)\rceil}} \subseteq \underline{\mathfrak{a}}^{\lceil t(p^e-1)\rceil}$,

- $p^e\varepsilon > N$,

- $(1-p^e)K_A$ is Cartier, and

- $t_i(p^e-1) \in \mathbb{Z}$ for all i.

Fix such an e. With that taken care of, reduce the whole setup modulo p at κ and set $q = p^e$. We get the following diagram:

$$\begin{array}{ccccccccc} 0 \to & H^0(B_\kappa, F^e_*\omega_{B_\kappa}(-\lfloor qD_\kappa \rfloor)) & \to & H^0(B_\kappa, F^e_*\omega_{B_\kappa}(\widetilde{X}_\kappa - \lfloor qD_\kappa \rfloor)) & \to & H^0(\widetilde{X}_\kappa, F^e_*\omega_{\widetilde{X}_\kappa}(-\lfloor q\widetilde{D}_\kappa \rfloor)) & \to 0 \\ & \downarrow (F^e_{B_\kappa})^\vee & & \downarrow (F^e_{B_\kappa})^\vee & & \downarrow (F^e_{\widetilde{X}_\kappa})^\vee & \\ 0 \longrightarrow & H^0(B_\kappa, \omega_{B_\kappa}(-\lfloor D_\kappa \rfloor)) & \longrightarrow & H^0(B_\kappa, \omega_{B_\kappa}(\widetilde{X}_\kappa - \lfloor D_\kappa \rfloor)) & \longrightarrow & H^0(\widetilde{X}_\kappa, \omega_{\widetilde{X}_\kappa}(-\lfloor \widetilde{D}_\kappa \rfloor)) & \longrightarrow 0 \end{array}$$

By the five lemma, as well as our assumptions on p, we see that

$$(F^e_{B_\kappa})^\vee \colon H^0(B_\kappa, F^e_*\omega_{B_\kappa}(\widetilde{X}_\kappa - \lfloor qD_\kappa \rfloor)) \to H^0(B_\kappa, \omega_{B_\kappa}(\widetilde{X}_\kappa - \lfloor D_\kappa \rfloor))$$

is a surjection. But the target of the above map is exactly (the global sections of) $\mathscr{J}_{X_\kappa}(A_\kappa, Z_\kappa)$ by Lemma 3.1.23. Thus it's enough to show that the image of this map is contained in $\tau_{I_\kappa}(R_\kappa, \underline{\mathfrak{a}}^t_\kappa)$. The following claim follows from a straightforward computation:

Claim 3.2.15. *In the above set-up, we have*

$$H^0(B_\kappa, F^e_*\omega_{B_\kappa}(\widetilde{X}_\kappa - \lfloor qD_\kappa \rfloor)) \subseteq R_\kappa\left((1-q)K_{A_\kappa}\right) \overline{\prod_i (a_i)_\kappa^{t_i(q-1)}} d_\kappa^N \xi_\kappa^N.$$

Proof of claim. In this proof, we work exclusively modulo p at κ and we abuse notation by omitting the subscripts κ. We compute:

$$\begin{aligned} & H^0(B, F^e_*\omega_B(\widetilde{X} - \lfloor qD \rfloor)) \\ = \; & H^0\left(A, f_*F^e_*\mathscr{O}_B\left(K_B + \widetilde{X} - \lfloor qf^*K_A + qF + qV + q\varepsilon \operatorname{div}_B(d\xi\eta) \rfloor\right)\right) \\ \subseteq \; & H^0\left(A, f_*F^e_*\mathscr{O}_B\left(K_B + \widetilde{X} - \lfloor qf^*K_A \rfloor - \lfloor qF \rfloor - \lfloor qV \rfloor - \lfloor q\varepsilon \rfloor \operatorname{div}_B(d\xi\eta)\right)\right) \\ \subseteq \; & H^0\left(A, f_*F^e_*\mathscr{O}_B\left(K_B + \widetilde{X} - \lfloor qf^*K_A \rfloor - \lfloor qF \rfloor - N \operatorname{div}_B(d\xi\eta)\right)\right) \end{aligned}$$

Here, we're using the fact that H is anti-effective. Note that $K_{A_2} = \pi_2^*K_{A_1} + (c-1)X_2$ by [Har77, Exercise II.8.5], and so $K_B = K_{B/A_2} + \gamma^*(\pi_2^*K_{A_1} + (c-1)X_2)$. Then it follows from our choice of q and the construction of η that

$$\begin{aligned} & H^0\left(A, f_*F^e_*\mathscr{O}_B\left(K_B + \widetilde{X} - \lfloor qf^*K_A \rfloor - \lfloor qF \rfloor - N \operatorname{div}_B(d\xi\eta)\right)\right) \\ = & H^0\Big(A, f_*F^e_*\mathscr{O}_B\Big(K_{B/A_2} + \gamma^*\pi_2^*K_{A_1} + (c-1)\gamma^*X_2 \\ & \qquad\qquad + \widetilde{X} - \lfloor qf^*K_A \rfloor - \lfloor qF \rfloor - N \operatorname{div}_B(d\xi\eta)\Big)\Big) \\ \subseteq & H^0\left(A, f_*F^e_*\mathscr{O}_B\left(f^*\left((1-q)K_A\right) + c\gamma^*X_2 - \lfloor qf^*K_A \rfloor - \lfloor qF \rfloor - N \operatorname{div}_B(d\xi)\right)\right) \end{aligned}$$

Next, we examine the term $-\lfloor qF \rfloor$. For each i, we can write $F_i = \widetilde{F}_i + a_i\gamma^*X$ for some $a_i \in \mathbb{N}$, where $\widetilde{F}_i$ is not supported along $\widetilde{X}$. Since we assumed $Z = cX$ at the generic point of X, we have $\sum_i t_i a_i = c$. Then we see

$$-\lfloor qF \rfloor = -\lfloor \sum_i qt_i\widetilde{F}_i + qt_ia_i\gamma^*X \rfloor = -\lfloor \sum_i qt_i\widetilde{F}_i \rfloor - qc\gamma^*X.$$

Thus we have:

$$\begin{aligned}
&H^0\left(A, f_*F^e_*\mathscr{O}_B\left(f^*\left((1-q)K_A\right) + c\gamma^*X_2 - \lfloor qf^*K_A \rfloor - \lfloor qF \rfloor - N\operatorname{div}_B(d\xi)\right)\right)\\
\subseteq &H^0\left(A, f_*F^e_*\mathscr{O}_B\left(f^*\left((1-q)K_A\right) - \lfloor qf^*K_A \rfloor - \sum_i \lfloor qt_i \rfloor \widetilde{F}_i + (c-qc)\gamma^*X_2\right.\right.\\
&\qquad\left.\left. - N\operatorname{div}_B(d\xi)\right)\right).
\end{aligned}$$

Note that for all i, $\lfloor qt_i \rfloor \geq \lfloor (q-1)t_i \rfloor = (q-1)t_i$. As $\widetilde{F}_i$ is effective for each i, we get

$$\begin{aligned}
-\sum_i \lfloor qt_i \rfloor \widetilde{F}_i + (c-qc)\gamma^*X_2 &\leq -\sum_i (q-1)t_i\widetilde{F}_i - (q-1)c\gamma^*X_2\\
&\leq -\sum_i \left((q-1)t_i\widetilde{F}_i + (q-1)t_ia_i\gamma^*X_2\right)\\
&\leq -\sum_i (q-1)t_iF_i.
\end{aligned}$$

By the construction of η, it follows that

$$\begin{aligned}
&H^0\left(A, f_*F^e_*\mathscr{O}_B\left(f^*\left((1-q)K_A\right) - \lfloor qf^*K_A \rfloor - \sum_i \lfloor qt_i \rfloor \widetilde{F}_i + (c-qc)\gamma^*X_2\right.\right.\\
&\qquad\left.\left. - N\operatorname{div}_B(d\xi)\right)\right)\\
\subseteq &H^0\left(A, f_*F^e_*\mathscr{O}_B\left(f^*((1-q)K_A) - \sum_i (q-1)t_iF_i - N\operatorname{div}_B(d\xi)\right)\right)\\
\subseteq &R\left((1-q)K_A\right)\overline{\prod_i \mathfrak{a}_i^{t_i(q-1)}}d^N\xi^N,
\end{aligned}$$

as desired. □

By the construction of ξ, we have $\xi_\kappa^N\overline{\prod_i (a_i)_\kappa^{t_i(q-1)}} \subseteq \prod_i (a_i)_\kappa^{t_i(q-1)}$. Thus we're done, by Lemma 3.1.16. Indeed, the map $(F^e_{A'_\kappa})^\vee$ may be identified with the "evaluate at 1" map from $\operatorname{Hom}_{R_\kappa}(F^e_*R_\kappa, R_\kappa)$ to R_κ via the isomorphism

$$\operatorname{Hom}_{R_\kappa}(F^e_*R_\kappa, R_\kappa) \cong F^e_*R_\kappa\left((1-q)K_{A_\kappa}\right),$$

see [Sch09]. So we have shown that

$$\mathscr{J}_{X_\kappa}(A_\kappa, Z_\kappa) \subseteq \varphi\left(F^e_* \underline{\mathfrak{a}}_\kappa^{\lceil t(p^e-1)\rceil} d_\kappa^N\right)$$

for some $\varphi \in \mathscr{C}_e^{R_\kappa}$. □

It's worth noting the following analog to Lemma 1.1.15(b). This proposition follows from Lemma 3.2.12, and it provides further evidence that $\tau_I(R, \mathscr{C})$ is well-behaved in Setting 3.2.1.

Proposition 3.2.16. *Work in Setting 3.2.1 and assume that* char $R = p$. *For each* i, *let* $\mathfrak{b}_i = \overline{\mathfrak{a}_i}$ *be the integral closure of* $\mathfrak{a}_i$. *Then*

$$\tau_I\left(R, \prod_i \mathfrak{a}_i^{t_i}\right) = \tau_I\left(R, \prod_i \mathfrak{b}_i^{t_i}\right).$$

Proof. The $\subseteq$ inclusion follows from Lemma 3.1.13, so we prove the reverse inclusion. As $\mathfrak{b}_i^n \subseteq \overline{\mathfrak{a}_i^n}$ for all i and n, it follows from Lemma 3.2.12 that there exists some $\xi \in R \smallsetminus I$ such that

$$\xi \prod_i \mathfrak{b}_i^{t_i(p^e-1)} \subseteq \prod_i \mathfrak{b}_i^{t_i(p^e-1)}.$$

Let $d \in \tau_I(R, \prod_i \mathfrak{a}_i) \smallsetminus I$ be arbitrary. Then Lemma 3.1.16 tells us:

$$\begin{aligned}\tau_I\left(R, \prod_i \mathfrak{b}_i\right) &= \sum_{e\geq 0}\sum_{\varphi\in\mathscr{C}^R} \varphi\left(F^e_* d\xi \prod_i \mathfrak{b}_i^{t_i(p^e-1)}\right) \\ &\subseteq \sum_{e\geq 0}\sum_{\varphi\in\mathscr{C}^R} \varphi\left(F^e_* d \prod_i \mathfrak{a}_i^{t_i(p^e-1)}\right) = \tau_I\left(R, \prod_i \mathfrak{b}_i\right).\end{aligned}$$

□

Chapter 4

Application: Diagonal F-Regularity and Symbolic Powers of Ideals

> It's invaluable to have a friend who shares your interests and helps you stay motivated.
>
> Maryam Mirzakhani

In this chapter, we use the subadditivity formulas from Chapter 2 to make progress on the containment problem of ordinary and symbolic powers of ideals. Namely, we define the notion of *Diagonally F-regular* singularities, which are milder than strongly F-regular singularities. A nontrivial result is that there exist diagonally F-regular singularities which are not regular, *Cf.* Theorem 4.4.1. This in turn yields a new family of rings known to satisfy the *Uniform Symbolic Topology Property*, or USTP. Not only that, but we shall see that diagonally F-regular rings satisfy USTP in an effective way. The work in this chapter was done jointly with Javier Carvajal-Rojas.

4.1 Introduction

Let $I \subseteq R$ be a radical ideal. Then the n-th *symbolic power* of I is given by

$$I^{(n)} := \bigcap_{\mathfrak{p} \in \operatorname{Ass}(I)} I^n R_{\mathfrak{p}} \cap R.$$

See [DDG$^+$17] for a survey on symbolic powers of ideals. If I is not radical, then one can instead take the intersection over the minimal primes of I instead of the associated primes; it's unclear which notion is preferable. Observe that if $I = \mathfrak{p}$ is a prime ideal, then $\mathfrak{p}^{(n)} = \mathfrak{p}^n R_{\mathfrak{p}} \cap R$. In this case, $\mathfrak{p}^{(n)}$ is the unique $\mathfrak{p}$-primary component of $\mathfrak{p}^n$. Further, it's evident that symbolic powers are larger than ordinary powers, meaning $I^n \subseteq I^{(n)}$ for all n, and that the symbolic powers of an ideal form a descending chain $I = I^{(1)} \supseteq I^{(2)} \supseteq I^{(3)} \supseteq \cdots$.

Recall that any ideal $I \subseteq R$ gives a topology on R, called the *I-adic topology* on R. A basis of open sets in this topology is given by the cosets $x + I^n$ where n is an integer. See [AM69, Chapter 10] for more details. If I is radical, then we can also define the *I-symbolic* topology on R by using $x + I^{(n)}$ as our basis of open sets. The symbolic and adic topologies of an ideal will be equivalent provided that

$$\forall b \, \exists a : I^a \subseteq I^{(b)}, \text{ and } \forall d \, \exists c : I^{(c)} \subseteq I^d. \tag{4.1}$$

The first condition is always satisfied, as we can take $a = b$, so the second condition is the pertinent one.

It appears Hartshorne was the first to ask about when the symbolic and ordinary topologies of a prime ideal were equivalent [Har70, §7]. He showed that, if R is a complete local Noetherian ring and $\mathfrak{p} \subseteq R$ is a prime ideal with $\dim(R/\mathfrak{p}) = 1$, then the $\mathfrak{p}$-adic and $\mathfrak{p}$-symbolic topologies are equivalent if and only if all the associated primes of R are contained $\mathfrak{p}$. Later, Schenzel extended these results to arbitrary primes $\mathfrak{p} \subseteq R$.

These sorts of questions are related to determining the Noetherianity of the symbolic Rees algebra $S(\mathfrak{p}) := \bigoplus_n \mathfrak{p}^{(n)}$, which is an important question in its own right. Note that $S(\mathfrak{p})$ is a graded ring. A quick computation shows that, if $S(\mathfrak{p})$ is generated in elements of degree $\leq k$, then we must have $\mathfrak{p}^{(kn)} \subseteq \mathfrak{p}^n$ for all n. Thus, the finite generation of $S(\mathfrak{p})$ implies a strong form of equivalence of the $\mathfrak{p}$-adic and $\mathfrak{p}$-symbolic topologies on R: we can bound c by a linear function of d in (4.1). In this case we say the $\mathfrak{p}$-adic and $\mathfrak{p}$-symbolic topologies are *linearly equivalent*, though this terminology was once reserved for the case when c was bounded by a linear function of d of *slope 1*.

One might hope that the converse of this statement holds, that is, that $S(\mathfrak{p})$ is Noetherian whenever the $\mathfrak{p}$-adic and $\mathfrak{p}$-symbolic topologies are linearly equivalent. Swanson showed this is far from true: in fact, she showed that if the $\mathfrak{p}$-adic and $\mathfrak{p}$-symbolic topologies are equivalent, *then they must be linearly equivalent* [Swa00].

Shortly thereafter, a breakthrough result of Ein, Lazarsfeld, and Smith established a uniform bound on the constant c of (4.1) independent of the ideal I. Namely, they showed that in a d-dimensional regular ring of equal characteristic 0, we have $I^{(dn)} \subseteq I^n$ for all radical ideals I and all n [ELS01]. This result has since been extended to the positive characteristic setting [HH02] and the mixed characteristic setting [MS17]. Huneke–Katz–Validashti coined the term *Uniform Symbolic Topology Property*, or USTP, to describe this situation. Namely, a ring R has USTP if there exists some h such that for all $\mathfrak{p} \subseteq R$ and all $n \geq 0$, we have $\mathfrak{p}^{(hn)} \subseteq \mathfrak{p}^n$ [HKV15]. They show in *loc.cit.* that USTP is descends under finite extensions (assuming some mild technical hypotheses—for instance, if both rings are F-finite) and pose the following question:

Question 4.1.1. Let $(R, \mathfrak{m})$ be a complete local domain. Does R have USTP?

In this chapter, we introduce a new class of F-singularity, called *diagonal F-regularity*, which are designed to have the USTP property. We show that, for rings essentially of finite type over a perfect field, regularity is strictly stronger than diagonal F-regularity and diagonal F-regularity is strictly stronger than strong F-regularity. This yields a new class of rings with USTP.

4.1.1 A Discussion on Diagonal F-Regularity

The concept of diagonal F-regularity came about naturally from our efforts to extend to the method introduced in [ELS01] to the non-regular setting. To see how, we summarize Ein–Lazarsfeld–Smith's argument here, following the presentation of [ST12]. In positive characteristic[1], the crux of Ein–Lazarsfeld–Smith's argument is the following chain of containments:

$$\mathfrak{p}^{(hn)} \overset{①}{\subseteq} \tau\Big(\mathfrak{p}^{(hn)}\Big) = \tau\left(\left(\mathfrak{p}^{(hn)}\right)^{n/n}\right) \overset{②}{\subseteq} \tau\left(\left(\mathfrak{p}^{(hn)}\right)^{1/n}\right)^{n} \overset{③}{\subseteq} \mathfrak{p}^{n} \tag{4.2}$$

We ask ourselves: which of these containments break down in the non-regular setting? Containment ① holds in any strongly F-regular ring. Containment ② holds by the subadditivity theorem for test ideals—this theorem requires the ambient ring R to be regular. Containment ③ holds quite generally (for $h = \operatorname{ht}\mathfrak{p}$), as we shall discuss in the proof of Theorem 4.2.4.

So, in order to apply this technique to the non-regular case, we must deal with containment ②. Luckily, we have the handy new subadditivity formula of Theorem 2.2.11. Namely, we have

$$\tau\left(\mathscr{D}^{(n)}(R), \mathfrak{p}^{(hn)}\right) \subseteq \tau\left(\left(\mathfrak{p}^{(hn)}\right)^{1/n}\right)^{n}.$$

Thus, we recover the result of Ein–Lazarsfeld–Smith whenever we have

$$\mathfrak{p}^{(hn)} \subseteq \tau\left(\mathscr{D}^{(n)}(R), \mathfrak{p}^{(hn)}\right)$$

for all $\mathfrak{p}$ and all n. A simple calculation shows that, in fact, we have $\mathfrak{a} \subseteq \tau\left(\mathscr{D}^{(n)}(R), \mathfrak{a}\right)$ for all ideals $\mathfrak{a} \subseteq R$ as long as $\tau\left(\mathscr{D}^{(n)}(R)\right) = R$; *Cf.* Proposition 4.2.3. We call the latter condition *n-diagonal F-regularity*. We say that a ring is *diagonally F-regular* if it is n-diagonally F-regular for all integers $n > 0$.

The following theorem summarizes the above discussion.

[1]Ein–Lazarsfeld–Smith's original argument uses *multiplier ideals*, which are only known to exist in characteristic 0. Their argument was adapted to positive characteristic rings by Nobuo Hara [Har05] and to mixed-characteristic rings by Linquan Ma and Karl Schwede [MS17]. Hara and Ma–Schwede achieved this by using positive characteristic and mixed characteristic analogs of multiplier ideals, respectively.

Theorem (Theorem 4.2.4)**.** *If R is a diagonally F-regular $\mathscr{k}$-algebra essentially of finite type, then R has USTP with $h = \dim R$.*

As we shall see, every essentially smooth $\mathscr{k}$-algebra is diagonally F-regular. What's more significant, and much harder to show, is that the converse does not hold. Indeed, we will show the following:

Theorem (Theorem 4.4.1)**.** *Let $\mathscr{k}$ be a perfect field of positive characteristic, and let $r, s \geq 1$ be integers. Then the affine cone over the Segre embedding of $\mathbb{P}^r_{\mathscr{k}} \times \mathbb{P}^s_{\mathscr{k}}$ is diagonally F-regular.*

Of course, the affine cone over any embedding of $\mathbb{P}^r_{\mathscr{k}} \times \mathbb{P}^s_{\mathscr{k}}$ is an isolated singularity, and USTP is known to hold for isolated singularities by [HKV09]. Nonetheless, our result has the virtue of being effective in the sense that we determine the number h explicitly, and show h is as small as we might expect it to be. We also observe that the class of diagonally F-regular F-finite $\mathscr{k}$-algebras is closed under tensor products over $\mathscr{k}$:

Theorem. *Let R and S be $\mathscr{k}$-algebras essentially of finite type, where $\mathscr{k}$ is a perfect field of characteristic p. If R and S are diagonally F-regular, then so is $R \otimes_{\mathscr{k}} S$.*

This implies that the class of diagonally F-regular singularities includes some non-isolated singularities. To our knowledge, this gives a new class of examples where USTP is known to hold. We note that Robert Walker obtains orthogonal results to Theorem 4.2.4 and Theorem 4.4.1 using complementary techniques; see [Wal17a, Wal17b] for precise statements.

4.2 Diagonal F-Regularity and USTP

In this section we define diagonal F-regularity and show that the approach of [ELS01] works in the diagonally F-regular setting.

Definition 4.2.1 (Diagonal F-regularity)**.** Let $\mathscr{k}$ be a field of positive characteristic. We say that a $\mathscr{k}$-algebra R is *n-diagonally F-regular* if $\mathscr{D}^{(n)}(R)$ is F-regular. We say that R is *diagonally F-regular* if $\mathscr{D}^{(n)}(R)$ is F-regular for all $n \in \mathbb{N}$.

Remark 4.2.2 (Diagonal F-regularity and diagonal F-splitting)*.* Note that R is diagonally F-split if and only if $\mathscr{D}^{(2)}(R)$ is F-split, and so 2-diagonal F-regularity can be seen as a strengthening of diagonal F-splitting[2]. Indeed, a ring R is defined to be diagonally F-split whenever there is a splitting $\varphi \in \mathscr{C}^{R^{\otimes 2}}$ compatible with I_Δ. It is clear that $\mathscr{D}^{(2)}(R)$ is F-split whenever R is diagonally

[2]See [Pay09, Ram85, RR85] for more on diagonal F-splittings and why they are important.

F-split. On the other hand, suppose that $\varphi \in \mathscr{D}_e^{(2)}(R)$ is a splitting. Then φ admits a lifting $\widehat{\varphi}$ in $\mathscr{C}_e^{R^{\otimes 2}}$, with $\varphi(1 \otimes 1) = 1 \otimes 1 + f$, for some $f \in I_\Delta$. Further, we have that $\varphi \otimes \varphi$ is an F-splitting of $R^{\otimes 2}$. It follows that $\widehat{\varphi} - f \cdot \varphi \otimes \varphi$ is an F-splitting of $R^{\otimes 2}$ compatible with I_Δ.

Proposition 4.2.3 (Properties of diagonal test ideals for USTP)**.** *Let R be a reduced $\mathscr{k}$-algebra, $\mathfrak{a} \subset R$ an ideal not contained in a minimal prime of R, $t \in \mathbb{R}_{\geq 0}$, and $n, m \in \mathbb{N}$. Then the following properties hold:*

(a) (Nonambiguity) $\tau\big(R, \mathscr{D}^{(n)}, \mathfrak{a}^{mt}\big) = \tau\Big(R, \mathscr{D}^{(n)}, \big(\mathfrak{a}^m\big)^t\Big)$,

(b) (Fundamental lower-bound) $\mathfrak{a} \cdot \tau\big(R, \mathscr{D}^{(n)}\big) \subset \tau\big(R, \mathscr{D}^{(n)}, \mathfrak{a}\big)$, so that $\mathfrak{a} \subset \tau\big(R, \mathscr{D}^{(n)}, \mathfrak{a}\big)$ if R is diagonally F-regular,

(c) (Subadditivity) $\tau\big(R, \mathscr{D}^{(n)}, \mathfrak{a}^{tn}\big) \subset \tau\big(R, \mathfrak{a}^t\big)^n$.

Proof. The nonambiguity property (a) follows from part (b) of Lemma 1.1.15. The subadditivity formula (c) follows from Theorem 2.2.11. For part (b), we will show the following much more general claim: if $\mathscr{C}$ is any Cartier algebra on R, then $\mathfrak{a}\tau(R, \mathscr{C}) \subseteq \tau(R, \mathscr{C}, \mathfrak{a})$. It follows that if $\tau(R, \mathscr{C}) = R$, then $\mathfrak{a} \subseteq \tau(R, \mathscr{C}, \mathfrak{a})$.

To prove the claim, let c be a test element for $\mathscr{C}^{\mathfrak{a}}$. Then c is also a test element for $\mathscr{C}$. We compute:

$$\begin{aligned}
\mathfrak{a}\tau(R, \mathscr{C}) &= \mathfrak{a} \sum_{e \geq 0} \sum_{\varphi \in \mathscr{C}_e} \varphi\left(F_*^e c\right) \\
&= \sum_{e \geq 0} \sum_{\varphi \in \mathscr{C}_e} \varphi\left(F_*^e c \mathfrak{a}^{[p^e]}\right) \\
&\subseteq \sum_{e \geq 0} \sum_{\varphi \in \mathscr{C}_e} \varphi\left(F_*^e c \mathfrak{a}^{p^e - 1}\right) \\
&= \tau(R, \mathscr{C}, \mathfrak{a}).
\end{aligned}$$

□

With the above properties in hand, we show that USTP is satisfied by diagonally F-regular rings. We do this by making our discussion in the preceding section rigorous.

Theorem 4.2.4. *Let R be a diagonally F-regular $\mathscr{k}$-algebra, and let $\mathfrak{p} \in$ Spec$R$ be an ideal of height h. Then $\mathfrak{p}^{(hn)} \subset \mathfrak{p}^n$ for all $n \in \mathbb{N}$.*

Proof. We can also assume that $\mathfrak{p}$ is not the maximal ideal of R, because in that case $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ for all n. This implies that the residue field of R at $\mathfrak{p}$ is transcendental over $\mathscr{k}$, and so $\kappa(\mathfrak{p})$ is infinite.[3]

[3]Recall that $\kappa(\mathfrak{p})/\mathscr{k}$ is algebraic if and only if $\mathfrak{p}$ is maximal in R.

As mentioned in Section 4.1.1, our strategy for proving this theorem is to enlarge the scope of the proof in [ST12, §6.3] and [SZ15, §4.3]. We just need to verify that the upper-bound

$$\tau\left(R, \left(\mathfrak{p}^{(hn)}\right)^{1/n}\right) \subset \mathfrak{p} \tag{4.3}$$

holds for all $n \in \mathbb{N}$, all prime ideals $\mathfrak{p} \subset R$, and all R under our consideration. This inclusion can be checked after localizing at $\mathfrak{p}$, which means that we may assume R is local with maximal ideal $\mathfrak{p}$ and infinite residue field. But then in that case $\mathfrak{p}^{(hn)} = \mathfrak{p}^{hn}$. Therefore, the left-hand side in (4.3) simply becomes

$$\tau\left(\left(\mathfrak{p}^{(hn)}\right)^{1/n}\right) = \tau\left(\left(\mathfrak{p}^{hn}\right)^{1/n}\right) = \tau\left(\mathfrak{p}^{hn/n}\right) = \tau\left(\mathfrak{p}^{h}\right).$$

Using [SH06, Theorems 8.3.7 and 8.3.9], just as in [ST12, Proof of Theorem 6.23], we have that $\mathfrak{p}$ admits a reduction,[4] say $\mathfrak{q} \subset \mathfrak{p}$, generated by $h = \dim R_{\mathfrak{p}}$ elements or fewer.[5] Hence,

$$\tau\left(\mathfrak{p}^{h}\right) = \tau\left(\mathfrak{q}^{h}\right) \subset \mathfrak{q} \subset \mathfrak{p},$$

where the penultimate inclusion is nothing but a consequence of the Briançon–Skoda theorem for test ideals [HH90], [BS19, Proposition 4.2]. The equality simply follows from unambiguity and the invariance of test ideals under integral closure, see [ST12, Theorem 6.9].

Thus, for all $\mathfrak{p} \in \operatorname{Spec} R$ and $n \in \mathbb{N}$ we have the following:

$$\mathfrak{p}^{(hn)} \overset{(1)}{\subset} \tau\left(R, \mathscr{D}^{(n)}, \mathfrak{p}^{(hn)}\right) \overset{(\text{🚲})}{=} \tau\left(R, \mathscr{D}^{(n)}, \left(\mathfrak{p}^{(hn)}\right)^{n/n}\right) \overset{(2)}{\subset} \tau\left(R, \left(\mathfrak{p}^{(hn)}\right)^{1/n}\right)^{n} \overset{(3)}{\subset} \mathfrak{p}^{n}$$

Here, (1) follows from R being diagonally F-regular and Proposition 4.2.3. The equality (🚲) is simply unambiguity, whereas (2) follows from subadditivity and (3) is just (4.3) raised to the n-th power. □

Remark 4.2.5. Thus, if R is diagonally F-regular, we have $\mathfrak{p}^{(dn)} \subset \mathfrak{p}^{n}$, where $d = \dim R$, for *all* $\mathfrak{p} \in \operatorname{Spec} R$. In fact $\mathfrak{p}^{((d-1)n)} \subset \mathfrak{p}^{n}$ holds because symbolic and ordinary powers of maximal ideals are the same. Indeed, suppose $\mathfrak{m} \in \operatorname{Spec} R$ is maximal. We know that the n-th symbolic power of any prime ideal $\mathfrak{p}$ is the unique $\mathfrak{p}$-primary component of $\mathfrak{p}^{n}$, so $\mathfrak{m}^{(n)}$ is the $\mathfrak{m}$-primary component of $\mathfrak{m}^{n}$. But powers of maximal ideals are always primary, so $\mathfrak{m}^{(n)} = \mathfrak{m}^{n}$.

4.3 On the Class of Diagonally F-Regular Rings

In this section, we show how torsion elements in the class group of a ring R give an obstruction to diagonal F-regularity. We also show how to make new diagonally F-regular $\mathscr{k}$-algebras from old ones. We start with the following simple observation about the class of diagonally F-regular rings.

[4] That is, a subideal with the same integral closure.

[5] Here it is where we need the residue field to be infinite.

Proposition 4.3.1. *Essentially smooth $\Bbbk$-algebras are diagonally F-regular. Further, n-diagonally F-regular $\Bbbk$-algebras are strongly F-regular, in particular normal and Cohen–Macaulay.*

Proof. The second statement is obvious, whereas the former is a consequence of Kunz's theorem [Kun69] just as in . Indeed, if R is smooth over $\Bbbk$, then $R^{\otimes n}$ is smooth and therefore regular for all n. Thus Kunz's theorem tells us that $F^e_* R^{\otimes n}$ is a projective $R^{\otimes n}$-module, which implies that $\mathscr{D}^{(n)}(R) = \mathscr{C}^R$ for all n. Similarly, if R is a localization of S, where S is a smooth $\Bbbk$ algebra, then $R^{\otimes n}$ is a localization of $S^{\otimes n}$ and so $R^{\otimes n}$ is still regular. The result follows. □

It follows from the following proposition that the class of diagonally F-regular $\Bbbk$-algebras is properly contained in the class of strongly F-regular ones. In the next subsection, we will show that the class of diagonally F-regular $\Bbbk$-algebras properly contains the class of essentially smooth algebras.

We thank Linquan Ma for sharing the following observation:

Proposition 4.3.2. *Let $(R, \mathfrak{m})$ be a local normal domain essentially of finite type over $\Bbbk$, with $R/\mathfrak{m}$ infinite. If R is diagonally F-regular, then the divisor class group $\mathrm{Cl}(R)$ is torsion-free. In fact, if $\mathrm{Cl}(R)$ has r-torsion[6], then $\mathscr{D}^{(nr)}(R)$ is not F-regular for $n \gg 0$.*

Proof. Suppose $\mathscr{D}^{(nr)}$ is F-regular for all n. Then for all prime ideals $\mathfrak{p}$, we have

$$\mathfrak{p}^{(hnr)} \subset \mathfrak{p}^{nr}$$

where h is the height of $\mathfrak{p}$. By assumption, there exists some non-principal prime ideal $\mathfrak{q}$ in R of height 1 such that $\mathfrak{q}^{(r)}$ is principal. Thus $\mathfrak{q}^{(rn)} = \mathfrak{q}^{rn}$ is principal for all n.

However, this cannot happen. Indeed, since R is a normal domain, we know that principal ideals of R are integrally closed, and so the analytic spread of $\mathfrak{q}$ is at least 2. This tells us that the fiber cone of $\mathfrak{q}$,

$$F_{\mathfrak{q}} = \frac{R}{\mathfrak{m}} \oplus \frac{\mathfrak{q}}{\mathfrak{m}\mathfrak{q}} \oplus \frac{\mathfrak{q}^2}{\mathfrak{m}\mathfrak{q}^2} \oplus \cdots$$

has dimension at least 2, so the Hilbert function of $F_{\mathfrak{q}}$, $h(F_{\mathfrak{q}}, n)$, agrees with a non-constant polynomial for $n \gg 0$. But we know that $h(F_{\mathfrak{q}}, n) = \mu(\mathfrak{q}^n)$ by Nakayama's lemma, so $\mathfrak{q}^{rn}$ is not principal for $n \gg 0$. □

Example 4.3.3. By the above proposition, we see that Veronese subrings of polynomial rings are never diagonally F-regular, *Cf.* Example 5.0.9.

[6]That is, some element of $\mathrm{Cl}(R)$ is annihilated by r.

By [Car17, Theorem G], if $s(R) > 1/2$ then $\mathrm{Cl}(R)$ is torsion-free. In light of Proposition 4.3.2, we suspect there is an interesting connection between diagonally F-regular rings and rings with F-signature greater than $1/2$. For example, we pose the following question:

Question 4.3.4. If $s(R) > 1/2$, is R diagonally F-regular? In particular, is

$$\Bbbk[x_1, \dots, x_d] \,\big/\, \big(x_1^2 + \cdots + x_d^2\big)$$

diagonally F-regular for all $d \geq 4$? We note that there exist diagonally F-regular rings with F-signature less than $1/2$, by Theorem 4.4.1 and work of Anurag Singh [Sin05, Example 6].

The following proposition shows that the class of diagonally F-regular $\Bbbk$-algebras is closed under tensor product.

Proposition 4.3.5. *Let $\Bbbk$ be a perfect field and let R and S be n-diagonally F-regular $\Bbbk$-algebras. Then $R \otimes_{\Bbbk} S$ is a n-diagonally F-regular $\Bbbk$-algebra.*

Proof. We prove this via global F-signatures [DSPY16]. For simplicity, write $a = a_e\big(R, \mathscr{D}^{(n)}\big)$ and $b = a_e\big(S, \mathscr{D}^{(n)}\big)$. Suppose

$$\varphi : F^e_* R \twoheadrightarrow R^{\oplus a}, \quad \psi : F^e_* S \twoheadrightarrow S^{\oplus b}$$

are surjections, such that each composition

$$F^e_* R \xrightarrow{\varphi} R^{\oplus a} \xrightarrow{\pi_i} R$$

is in $\mathscr{D}^{(n)}(R)$. Similarly, each composition

$$F^e_* S \xrightarrow{\psi} S^{\oplus b} \xrightarrow{\sigma_j} S$$

is in $\mathscr{D}^{(n)}(S)$. Then we get a surjection of $R \otimes S$-modules

$$F^e_*(R \otimes S) \cong F^e_* R \otimes F^e_* S \xrightarrow{\varphi \otimes \psi} R^{\oplus a} \otimes S^{\oplus b} \cong (R \otimes S)^{\oplus ab}$$

Then each composition

$$(\pi_i \circ \varphi) \otimes (\sigma_j \circ \psi) \colon F^e_*(R \otimes S) \xrightarrow{\varphi \otimes \psi} (R \otimes S)^{\oplus ab} \xrightarrow{\pi_i \otimes \sigma_j} R \otimes S$$

is in the Cartier algebra $\mathscr{D}^{(n)}(R \otimes S)$. Indeed, given any maps $\theta \in \mathscr{D}^{(n)}_e(R)$ and $\eta \in \mathscr{D}^{(n)}_e(S)$, with liftings $\widehat{\theta} \in \mathscr{C}_e^{R^{\otimes n}}$ and $\widehat{\eta} \in \mathscr{C}_e^{S^{\otimes n}}$, one checks that $\widehat{\theta} \otimes \widehat{\eta}$ is a lifting of $\theta \otimes \eta$ by a diagram chase. Thus, $a_e\big(R \otimes S, \mathscr{D}^{(n)}(R \otimes S)\big) \geq ab$. It follows that

$$s\big(R \otimes S, \mathscr{D}^{(n)}(R \otimes S)\big) \geq s\big(R, \mathscr{D}^{(n)}(R)\big) \cdot s\big(S, \mathscr{D}^{(n)}(S)\big) > 0. \qquad \square$$

4.4 Segre Products of Polynomial Rings are Diagonally F-Regular

This section will be spent proving the following theorem:

Theorem 4.4.1. *Let R be the Segre product $\Bbbk[x_0,\dots,x_r]\#\Bbbk[y_0,\dots,y_s]$, with $r,s>0$, and $\Bbbk$ perfect. Then R is diagonally F-regular.*

Combined with Theorem 4.2.4, we get the following corollary:

Corollary 4.4.2 (USTP holds for Segre varieties)**.** *Let $R=\Bbbk[x_0,\dots,x_r]\#\Bbbk[y_0,\dots,y_s]$, and let $\mathfrak{p}\subset R$ be a prime ideal. Then $\mathfrak{p}^{(hn)}\subset\mathfrak{p}^n$ for all n, where $h=\dim R-1=r+s$.*

Remark 4.4.3. Let $\ell/\Bbbk$ be a finitely generated field extension over a perfect field. Then in view of Proposition 4.3.5 and Theorem 4.4.1 we have that $R_\ell=\ell[x_0,\dots,x_r]\#\ell[y_0,\dots,y_s]$ is a diagonally F-regular $\Bbbk$-algebra. In particular, USTP holds for R_ℓ as well.

Combining Theorem 4.4.1 and Proposition 4.3.5, we obtain the following observation:

Corollary 4.4.4. *The class of diagonally F-regular $\Bbbk$-algebras includes some non-isolated singularities.*

We now prove Theorem 4.4.1. We begin with some lemmas which inform our approach. The first is a criterion for verifying the F-regularity of a Cartier algebra $\mathscr{C}\subset\mathscr{C}^R$. We presume it is well-known among experts, but we give a proof for sake of completeness. We are thankful to Karl Schwede for bringing it to our attention, thus significantly simplifying part of our argument.

Lemma 4.4.5 (*Cf.* [BMRS15, Proposition 4.5], [HH89, Theorem 3.3], [BS19, Lemma 2.3])**.** *Let R be a ring, $\mathscr{C}\subset\mathscr{C}^R$ a Cartier R-algebra, and $f\in R^\circ$. Suppose that $\mathscr{C}_f$ is an F-regular Cartier algebra on R_f and moreover that there is $\varphi\in\mathscr{C}_e$, for some e, such that $\varphi(F^e_*f)=1$. Then $\mathscr{C}$ is F-regular.*

Proof. We must prove that $1\in\tau(R,\mathscr{C})$. Then our first hypothesis is $\tau(R_f,\mathscr{C})\ni 1$. However, $\tau(R_f,\mathscr{C}_f)=\tau(R,\mathscr{C})_f$. Putting these two statements together we get that $f^n\in\tau(R,\mathscr{C})$ for some $n\in\mathbb{N}$.

If there is $e\in\mathbb{N}$ and $\psi\in\mathscr{C}_e$ such that $\psi(F^e_*f^n)=1$, then we would be done, for

$$1=\psi(F^e_*f^n)\in\psi\big(F^e_*\tau(R,\mathscr{C})\big)\subset\tau(R,\mathscr{C}).$$

Our second hypothesis is that there exists $e\in\mathbb{N}$ and $\varphi\in\mathscr{C}_e$ such that $\varphi(F^e_*f)=1$. We prove inductively that the same holds for all powers of f. Indeed, say $\psi\in\mathscr{C}_d$ is such that $\psi(F^d_*f^{m-1})=1$.

Then it follows that $\vartheta \coloneqq \varphi \cdot \psi \cdot f^{p^d-1}$ is such that $\vartheta(F_*^{e+d} f^m) = 1$, for

$$\begin{aligned}\vartheta(F_*^{e+d} f^m) = \varphi\Big(F_*^e \psi\big(F_*^d f^{p^d-1} f^m\big)\Big) = \varphi\Big(F_*^e \psi\big(F_*^d f^{p^d} f^{m-1}\big)\Big) &= \varphi\Big(F_*^e f \psi\big(F_*^d f^{m-1}\big)\Big) \\ &= \varphi\Big(F_*^e f \cdot 1\Big) = 1. \qquad \square\end{aligned}$$

Next, we observe that diagonal Cartier algebras localize:

Lemma 4.4.6. *Let R be a ring essentially of finite type over a perfect field $\Bbbk$ of positive characteristic. Let $f \in R$. Then $\mathscr{D}^{(n)}(R)_f = \mathscr{D}^{(n)}(R_f)$ for all n. Here $\mathscr{D}^{(n)}(R)_f$ the Cartier algebra on R_f induced by $\mathscr{D}^{(n)}(R)$, as described in Notation 1.1.13.*

Proof. Let $\varphi \in \mathscr{D}^{(n)}(R)$. By assumption, there exists some $\widehat{\varphi}$ fitting in the following diagram:

$$\begin{array}{ccc} F_*^e\left(R^{\otimes_\Bbbk n}\right) & \xrightarrow{\ \widehat{\varphi}\ } & R^{\otimes_\Bbbk n} \\ {\scriptstyle F_*^e \mu_n}\downarrow & & \downarrow{\scriptstyle \mu_n} \\ F_*^e R & \xrightarrow{\ \varphi\ } & R \end{array}$$

As applying F_*^e commutes with localization, applying the functor $- \otimes_{R^{\otimes n}} (R_f)^{\otimes n}$ to this diagram yields the following diagram of $R^{\otimes n}$-linear maps,

$$\begin{array}{ccc} F_*^e\left(R_f^{\otimes_\Bbbk n}\right) & \xrightarrow{\ \widehat{\varphi}\ } & R_f^{\otimes_\Bbbk n} \\ {\scriptstyle F_*^e \mu_n}\downarrow & & \downarrow{\scriptstyle \mu_n} \\ F_*^e R_f & \xrightarrow{\ \varphi\ } & R_f \end{array},$$

where R is a $R^{\otimes n}$-module via μ. This shows that φ induces a map in $\mathscr{D}^{(n)}(R_f)$. In other words, $\mathscr{D}^{(n)}(R)_f \subseteq \mathscr{D}^{(n)}(R_f)$.

To see the other direction, let $\psi \in \mathscr{D}^{(n)}(R_f)$. As R is F-finite, we may write $\psi = \frac{1}{f^a}\psi^\circ$ for some $\psi^\circ \in \operatorname{Hom}_R(F_*^e R, R)$ and some a. Here we're again using the fact that $F_*^e(R_f) = (F_*^e R)_f$. We wish to show that $f^N \psi^\circ \in \mathscr{D}^{(n)}(R)$ for some N. By assumption, there exists some $\widehat{\psi}$ fitting into the following diagram:

$$\begin{array}{ccc} F_*^e\left(R_f^{\otimes_\Bbbk n}\right) & \xrightarrow{\ \widehat{\psi}\ } & R_f^{\otimes_\Bbbk n} \\ {\scriptstyle F_*^e \mu_n}\downarrow & & \downarrow{\scriptstyle \mu_n} \\ F_*^e R_f & \xrightarrow{\ \psi\ } & R_f \end{array}.$$

As $R^{\otimes_\Bbbk n}$ is F-finite, and $R_f^{\otimes_\Bbbk n} = (R^{\otimes_\Bbbk n})_{f\otimes\cdots\otimes f}$, we may write $\widehat{\psi} = \frac{1}{(f\otimes\cdots\otimes f)^b}\widehat{\psi}^\circ$ for some $b > 0$ and some $\widehat{\psi}^\circ \in \operatorname{Hom}_{R^{\otimes n}}(F_*^e R^{\otimes n}, R^{\otimes n})$. Let $c = \max\{\lceil a/n \rceil, b\}$. Then $(f \otimes \cdots \otimes f)^{c-b}\widehat{\psi}^\circ$ is a lifting of $f^{c-a}\psi^\circ$, and $(f \otimes \cdots \otimes f)^{c-b}\widehat{\psi}^\circ(F_*^e R^{\otimes_\Bbbk n}) \subseteq R^{\otimes_\Bbbk n}$. Thus $f^{c-a}\psi^\circ \in \mathscr{D}^{(n)}(R)$, as desired. $\square$

Combining the above two facts, we get the following

Proposition 4.4.7. *Let R be a ring essentially of finite type over a perfect field $\Bbbk$ of positive characteristic. Let $f \in R$ be an element so that R_f is n-diagonally F-regular, and suppose there exist $e > 0$ and $\varphi \in \mathscr{D}_e^{(n)}(R)$ such that $\varphi(F_*^e f) = 1$. Then R is n-diagonally F-regular.*

Observe that R can be realized as the following subring of $S := \Bbbk[x_0, \ldots, x_r, y_0, \ldots, y_s]$:

$$R = \Bbbk[x_0y_0, \ldots, x_iy_j, \ldots, x_ry_s] \subset \Bbbk[x_0, \ldots, x_r, y_0, \ldots, y_s] = S.$$

Fix an integer $n > 1$. Note that $R_{x_0y_0}$ is smooth, and thus diagonally F-regular. By Proposition 4.4.7, it suffices to find an integer e and a map $\varphi \in \mathscr{D}_e^{(n)}(R)$ with $\varphi(F_*^e x_0y_0) = 1$, in order to prove Theorem 4.4.1. It turns out that finding the correct map φ is easy; the hard part is checking that $\varphi \in \mathscr{D}^{(n)}(R)$. Our strategy will be to work mostly in the polynomial ring S. This is possible thanks to the following lemma:

Lemma 4.4.8. *The Frobenius trace $\Phi^e \in \mathscr{C}_e^S$ restricts to a map in $\mathscr{C}_e^R$, i.e. $\Phi^e(F_*^e R) \subset R$, so that there is a commutative diagram*

$$\begin{array}{ccc} F_*^e R & \xrightarrow{\Phi^e} & R \\ \big\downarrow & & \big\downarrow \\ F_*^e S & \xrightarrow{\Phi^e} & S \end{array}$$

Proof. Let $x_0^{a_0} \cdots x_r^{a_r} \cdot y_0^{b_0} \cdots y_s^{b_s}$ be a monomial in R, meaning

$$\sum_{i=0}^{r} a_i = \sum_{i=0}^{s} b_i. \tag{4.4}$$

For convenience, we will use the notation

$$\boldsymbol{x}^{a_\bullet} := x_0^{a_0} \cdots x_r^{a_r}, \quad \boldsymbol{y}^{b_\bullet} := y_0^{b_0} \cdots y_s^{b_s}.$$

Write using the Euclidean algorithm,

$$a_i =: \mu_i q + \alpha_i, \quad 0 \le \alpha_i \le q - 1. \tag{4.5}$$

Similarly,

$$b_i =: \nu_i q + \beta_i, \quad 0 \le \beta_i \le q - 1. \tag{4.6}$$

In such a way that,

$$F_*^e \boldsymbol{x}^{a_\bullet} \boldsymbol{y}^{b_\bullet} = \boldsymbol{x}^{\mu_\bullet} \boldsymbol{y}^{\nu_\bullet} \cdot F_*^e \boldsymbol{x}^{\alpha_\bullet} \boldsymbol{y}^{\beta_\bullet}.$$

Therefore,

$$\Phi^e\big(F^e_* \boldsymbol{x}^{a_\bullet} \boldsymbol{y}^{b_\bullet}\big) = \begin{cases} \boldsymbol{x}^{\mu_\bullet} \boldsymbol{y}^{\nu_\bullet} & \text{if } \alpha_i, \beta_i = q-1, \\ 0 & \text{otherwise.} \end{cases}$$

Now, combining (4.4), (4.5) and (4.6) we get

$$\Big(\sum \mu_i\Big) q + \sum \alpha_i = \Big(\sum \nu_i\Big) q + \sum \beta_i, \tag{4.7}$$

Introducing the notation $\mu \coloneqq \sum_i \mu_i$ etcetera, we conclude that $\mu = \nu$ if and only if $\alpha = \beta$. In particular, if $\alpha_i, \beta_i = q-1$ then $\mu = \nu$, meaning that

$$\boldsymbol{x}^{\mu_\bullet} \boldsymbol{y}^{\nu_\bullet} \in R,$$

as desired. This proves the lemma. □

Continuing with the proof of Theorem 4.4.1, consider the map

$$\varphi_e \coloneqq \Phi^e \cdot x_0^{q-2} x_1^{q-1} \cdots x_r^{q-1} y_0^{q-2} y_1^{q-1} \cdots y_s^{q-1} \in \mathscr{C}_e^S.$$

Since

$$x_0^{q-2} x_1^{q-1} \cdots x_r^{q-1} y_0^{q-2} y_1^{q-1} \cdots y_s^{q-1} \in R,$$

we have that φ_e also restricts to a map in $\mathscr{C}_e^R$. Moreover,

$$\varphi_e(F^e_* x_0 y_0) = \Phi^e\Big(F^e_* x_0^{q-1} \cdots x_r^{q-1} y_0^{q-1} \cdots y_s^{q-1}\Big) = 1.$$

Hence, it suffices to prove that $\varphi_e \in \mathscr{D}^{(n)}(R)$ for e large enough. Our strategy will be to show the following.

Claim 4.4.9. *There exists a lifting of $\varphi_e \in \mathscr{C}_e^S$ to $\mathscr{C}_e^{S^{\otimes n}}$, say*

$$\begin{array}{ccc} F^e_* S^{\otimes n} & \xrightarrow{\widehat{\varphi}_e} & S^{\otimes n} \\ {\scriptstyle F^e_*\Delta_n}\big\downarrow & & \big\downarrow{\scriptstyle \Delta_n} \\ F^e_* S & \xrightarrow{\varphi_e} & S \end{array}$$

such that $\widehat{\varphi}_e$ restricts to $R^{\otimes n}$, i.e. $\widehat{\varphi}_e\big(R^{\otimes n}\big) \subset R^{\otimes n}$, for $e \gg 0$.

It suffices to show this claim, for then the restriction of $\widehat{\varphi}_e$ to $F^e_* R^{\otimes n}$ will be a lifting of the map $\varphi_e \colon F^e_* R \to R$. We are going to spend the rest of the section proving Claim 4.4.9. For this, we use the following notation,

$$S^{\otimes n} = \mathbb{k}[\boldsymbol{x}_1, \boldsymbol{y}_1, \ldots, \boldsymbol{x}_n, \boldsymbol{y}_n],$$

where

$$\boldsymbol{x}_k \coloneqq x_{0,k}, x_{1,k}, \dots, x_{r,k}$$

and similarly for $\boldsymbol{y}_k$, where the second subscript of $x_{i,k}$ (resp. $y_{j,k}$) denotes which copy of the n-fold tensor product it corresponds to. We also write

$$R^{\otimes n} = \Bbbk \left[x_{i,k} y_{j,k} \;\middle|\; \begin{array}{l} 1 \leq i \leq r, \\ 0 \leq j \leq s, \\ 0 \leq k \leq n \end{array} \right]$$

so that a monomial

$$\prod_{k=1}^{n} \boldsymbol{x}_k^{a_{\bullet,k}} \boldsymbol{y}_k^{b_{\bullet,k}} \in S^{\otimes n}$$

belongs to $R^{\otimes n}$ if and only if

$$a_k = b_k$$

for all k, where we use the notation

$$a_k \coloneqq \sum_{i=0}^{r} a_{i,k} \quad \text{and} \quad \boldsymbol{x}_k^{a_{\bullet,k}} \coloneqq \prod_{i=0}^{r} x_{i,k}^{a_{i,k}}$$

and similarly for b_k and $\boldsymbol{y}_k^{b_{\bullet,k}}$. To be clear, the second subscript always denotes which factor of the n-fold tensor product we are working in.

Recall that

$$F_*^e \prod_{k=1}^{n} \boldsymbol{x}_k^{a_{\bullet,k}} \boldsymbol{y}_k^{b_{\bullet,k}}, \quad 0 \leq a_{i,k}, b_{j,k} \leq q-1$$

is a (free) basis of $F_*^e S^{\otimes n}$ as an $S^{\otimes n}$-module. We will construct the map $\widehat{\varphi}_e$ from Claim 4.4.9 explicitly by assigning values for $\widehat{\varphi}_e$ at each of these basis elements, pursuant to the two conditions:

(a) $\Delta_n \circ \widehat{\varphi}_e = \varphi_e \circ F_*^e \Delta_n$, and

(b) $\widehat{\varphi}_e\big(F_*^e R^{\otimes n}\big) \subset R^{\otimes n}$.

For some basis elements, it is easy to figure out where we can send them. For others, it is a more delicate question. We begin by taking care of the easy ones.

As we will see, one case when it is easy is when our basis element is in the kernel of $\varphi_e \circ F_*^e \Delta_n$. Let $\psi := \varphi_e \circ F_*^e \Delta_n$. Then we have

$$\begin{aligned}
&\psi\left(F^e_* \prod_{k=1}^n \boldsymbol{x}_k^{a_{\bullet,k}} \boldsymbol{y}_k^{b_{\bullet,k}}\right) \\
=&\varphi_e\left(F^e_* \boldsymbol{x}^{\sum_k a_{\bullet,k}} \boldsymbol{y}^{\sum_k b_{\bullet,k}}\right) \\
=&\Phi^e\Big(F^e_* x_0^{q-2+\sum_k a_{0,k}} x_1^{q-1+\sum_k a_{1,k}} \cdots x_r^{q-1+\sum_k a_{r,k}} \cdot \\
&\qquad y_0^{q-2+\sum_k b_{0,k}} y_1^{q-1+\sum_k b_{1,k}} \cdots y_s^{q-1+\sum_k b_{s,k}}\Big).
\end{aligned}$$

This will be nonzero precisely when

$$\begin{aligned}
&\sum_k a_{0,k}, \sum_k b_{0,k} \equiv 1 \mod q, \\
&\sum_k a_{i,k}, \sum_k b_{j,k} \equiv 0 \mod q, \text{ where } 1 \le i \le r, 1 \le j \le s.
\end{aligned} \tag{😸}$$

Let $\upsilon(x) = \lfloor x/q \rfloor$. Hence, in case (😸) we have

$$\begin{aligned}
&\Phi^e\Big(F^e_* x_0^{q-2+\sum_k a_{0,k}} x_1^{q-1+\sum_k a_{1,k}} \cdots x_r^{q-1+\sum_k a_{r,k}} \cdot y_0^{q-2+\sum_k b_{0,k}} y_1^{q-1+\sum_k b_{1,k}} \cdots y_s^{q-1+\sum_k b_{s,k}}\Big) \\
=&\boldsymbol{x}^{\upsilon(\sum_k a_{\bullet,k})} \boldsymbol{y}^{\upsilon(\sum_k b_{\bullet,k})}.
\end{aligned}$$

In summary,

$$\psi\left(F^e_* \prod_{k=1}^n \boldsymbol{x}_k^{a_{\bullet,k}} \boldsymbol{y}_k^{b_{\bullet,k}}\right) = \begin{cases} \boldsymbol{x}^{\upsilon(\sum_k a_{\bullet,k})} \boldsymbol{y}^{\upsilon(\sum_k b_{\bullet,k})} & \text{if condition (😸) holds,} \\ 0 & \text{otherwise.} \end{cases}$$

If condition (😸) does not hold, we set

$$\widehat{\varphi}_e\left(F^e_* \prod_{k=1}^n \boldsymbol{x}_k^{a_{\bullet,k}} \boldsymbol{y}_k^{b_{\bullet,k}}\right) = 0 \in R^{\otimes n}.$$

The next case that is easy to deal with is the case where our generator of $F^e_* S^{\otimes n}$ has nothing to do with $F^e_* R^{\otimes n}$. More precisely, if we have

$$\left(S^{\otimes n} F^e_* \prod_{k=1}^n \boldsymbol{x}_k^{a_{\bullet,k}} \boldsymbol{y}_k^{b_{\bullet,k}}\right) \cap F^e_* R^{\otimes n} = 0$$

then the value we assign to

$$\widehat{\varphi}_e\left(F^e_* \prod_{k=1}^n \boldsymbol{x}_k^{a_{\bullet,k}} \boldsymbol{y}_k^{b_{\bullet,k}}\right)$$

has no bearing on whether $\widehat{\varphi}_e(F^e_* R^{\otimes n}) \subset R^{\otimes n}$. So for these generators we only need to worry about the requirement that $\Delta_n \circ \widehat{\varphi}_e = \varphi_e \circ F^e_* \Delta_n$. We deduce which generators satisfy this condition in the following lemma.

Lemma 4.4.10. *$F^e_* R$ is generated as an R-submodule of $F^e_* S$ by the elements*

$$\boldsymbol{x}^{\mu_\bullet} \cdot F^e_* \boldsymbol{x}^{\alpha_\bullet} \boldsymbol{y}^{\beta_\bullet}, \quad 0 \leq \alpha_i, \beta_j \leq q-1$$

such that $sq \geq \mu q = \beta - \alpha \geq 0$, along with the elements

$$\boldsymbol{y}^{\nu_\bullet} \cdot F^e_* \boldsymbol{x}^{\alpha_\bullet} \boldsymbol{y}^{\beta_\bullet}, \quad 0 \leq \alpha_i, \beta_j \leq q-1$$

such that $rq \geq \nu q = \alpha - \beta \geq 0$. Moreover, $F^e_* R^{\otimes n}$ is generated as an $R^{\otimes n}$-module by tensor products of these generators. Here, we are still using the notation $\mu = \sum_{i=0}^r \mu_i$ and $\nu = \sum_{j=0}^s \nu_j$, and similarly for α and β.

In particular, the ring $F^e_* R^{\otimes n}$ is contained in the direct summand of $F^e_* S^{\otimes n}$ generated as a (free) $S^{\otimes n}$-module by monomials of the form

$$F^e_* \prod_k \boldsymbol{x}_k^{a_{\bullet,k}} \boldsymbol{y}_k^{b_{\bullet,k}}$$

such that $b_k - a_k \equiv 0 \mod q$ for all k, $1 \leq k \leq n$. Here, we are still using the notation $b_k := \sum_{j=0}^s b_{j,k}$ and $a_k := \sum_{i=0}^r a_{i,k}$.

Proof. We observed in the proof of Lemma 4.4.8 that elements in $F^e_* R$ are $\Bbbk$-linear combinations of elements of the form

$$\boldsymbol{x}^{\mu_\bullet} \boldsymbol{y}^{\nu_\bullet} \cdot F^e_* \boldsymbol{x}^{\alpha_\bullet} \boldsymbol{y}^{\beta_\bullet}, \quad 0 \leq \alpha_i, \beta_j \leq q-1$$

such that

$$\mu q + \alpha = \nu q + \beta,$$

equivalently,

$$(\mu - \nu) q = \beta - \alpha. \tag{4.8}$$

In particular, $\mu - \nu$ and $\beta - \alpha$ have both the same sign (including zero). Note that

$$\beta - \alpha \in \{-(r+1)(q-1), -(r+1)(q-1)+1, \dots, -1, 0, 1, \dots, (s+1)(q-1)\}$$

and $(\mu - \nu) q \in q\mathbb{Z}$. We see that the intersection of these two sets is $\{-rq, -rq+1, \dots, sq\}$, assuming $q > \max\{r+1, s+1\}$. Therefore for (4.8) to hold there are three possibilities: if $\mu - \nu = 0$, then both monomials $\boldsymbol{x}^{\mu_\bullet} \boldsymbol{y}^{\nu_\bullet}$ and $\boldsymbol{x}^{\alpha_\bullet} \boldsymbol{y}^{\beta_\bullet}$ are in R. Otherwise, if $\mu - \nu > 0$ (respectively, $\mu - \nu < 0$), then the monomial $\boldsymbol{x}^{\mu_\bullet} \boldsymbol{y}^{\nu_\bullet}$ can be factored as a product of a monomial in R times a monomial $\boldsymbol{x}^{\mu'_\bullet}$ (respectively, $\boldsymbol{y}^{\nu'_\bullet}$) with $\mu' = \mu - \nu$ (respectively, $\nu' = \nu - \mu$). This proves the lemma. □

The above being said, we proceed as follows. If we have $b_k - a_k \not\equiv 0 \mod q$ for some $1 \le k \le n$, we set

$$\widehat{\varphi}_e\left(F_*^e \prod_k \boldsymbol{x}_k^{a_{\bullet,k}} \boldsymbol{y}_k^{b_{\bullet,k}}\right) = \psi\Big(F_*^e \prod_k \boldsymbol{x}_k^{a_{\bullet,k}} \boldsymbol{y}_k^{b_{\bullet,k}}\Big) \otimes 1 \otimes \cdots \otimes 1.$$

Note that this is consistent with our earlier assignment, even if condition (🐱) does not hold.

Now we come to the hard part of this proof. We are given a monomial that satisfies condition (🐱) and also satisfies $b_k - a_k \equiv 0 \mod q$ for all k and we need to figure out where $\widehat{\varphi}_e$ should send it to. Our idea is quite simple, though it might be lost in the cumbersome notation. Thus it makes sense to do an example first.

Example 4.4.11. Say $p = 5, e = 1, n = 2$, and $r = s = 1$. Let $F_*g := F_*x_{0,1}x_{1,1}y_{0,1}^3y_{1,1}^4 \otimes x_{1,2}^4y_{0,2}^3y_{1,2}$ be the generator in question. To figure out where we should send this generator, we first compute $\varphi_1 \circ F_*\Delta_2(F_*g)$:

$$\varphi_1 \circ F_*\Delta_2\left(F_*x_{0,1}x_{1,1}y_{0,1}^3y_{1,1}^4 \otimes x_{1,2}^4y_{0,2}^3y_{1,2}\right) = \varphi_e\left(x_0x_1^5y_0^6y_1^5\right) = x_1y_0y_1$$

Now, $F_*g \notin F_*R^{\otimes 2}$, as $x_0x_1y_0^3y_1^4 \notin R$, but there are certainly many S-multiples of F_*g that land in $F_*R^{\otimes 2}$. Wherever we send F_*g, we need to make sure that these S-multiples get sent to $R^{\otimes 2}$.

Luckily, as described in Lemma 4.4.10, the multiples of F_*g that appear in $F_*R^{\otimes 2}$ have a very precise form. The point is that the monomial $F_*^e x_{0,1}x_{1,1}y_{0,1}^3y_{1,1}^4$ has a surplus of 5 more y's than x's. To multiply this monomial into F_*^eR, we must balance this out by multiplying by one more x relative to the number of y's (which becomes a surplus of 5 more x's than y's once we move them across the F_*).

So for instance, $x_{0,1} \otimes 1 \cdot F_*x_{0,1}x_{1,1}y_{0,1}^3y_{1,1}^4 \otimes x_{1,2}^4y_{0,2}^3y_{1,2} \in F_*R^{\otimes 2}$. This means that, if we set

$$\widehat{\varphi}_e\left(F_*x_{0,1}x_{1,1}y_{0,1}^3y_{1,1}^4 \otimes x_{1,2}^4y_{0,2}^3y_{1,2}\right) = x_{0,1}^{c_{0,1}}x_{1,1}^{c_{1,1}}y_{0,1}^{d_{0,1}}y_{1,1}^{d_{1,1}} \otimes x_{0,2}^{c_{0,2}}x_{1,2}^{c_{1,2}}y_{0,2}^{d_{0,2}}y_{1,2}^{d_{1,2}}$$

we must have

$$1 + c_{0,1} + c_{1,1} = d_{0,1} + d_{1,1}, \text{ and } c_{0,2} + c_{1,2} = d_{0,2} + d_{1,2}.$$

In other words, $\widehat{\varphi}_1(F_*g)$ needs to have one more y than it does x's in the first tensor factor and the same number of x's and y's in the second tensor factor. We do this by "taking" one of the y's from the product $x_1y_0y_1$ (it does not matter which) and "giving" it to the first tensor factor of $\widehat{\varphi}_1(F_*g)$. For instance, we can set the first tensor factor of $\widehat{\varphi}_1(F_*g)$ to be y_0. Then we give the rest of the product $x_1y_0y_1$ to the second tensor factor. At the end of the day we have

$$\widehat{\varphi}_1(F_*g) = y_{0,1} \otimes x_{1,2}y_{1,2}$$

and we see that $x_{0,1}\widehat{\varphi}_1(F_*g) \in R^{\otimes 2}$ and $\Delta_2 \circ \widehat{\varphi}_1(F_*g) = x_1y_0y_1$, as desired. □

In what follows, we use the same technique as in the above example, but in a more general setting. We go through each tensor factor of the generator $F^e_* g$ and we ask: does it have more y's than x's? If so, we take the correct number of y's from $\varphi_e \circ F^e_* \Delta_n(F^e_* g)$ and give them to the corresponding tensor factor of $\widehat{\varphi}_e(F^e_* g)$. Similarly, if that tensor factor of $F^e_* g$ has more x's, we take the correct number of x's from $\varphi_e \circ F^e_* \Delta_n(F^e_* g)$ and give them to the corresponding tensor factor of $\widehat{\varphi}_e(F^e_* g)$. The fact that $\varphi_e \circ F^e_* \Delta_n(F^e_* g)$ will always have enough x's and y's to do this process is expressed by (4.10). The fact that, after removing these x's and y's, whatever is left of $\varphi_e \circ F^e_* \Delta_n(F^e_* g)$ will be an element of R is expressed by (4.9). We can then tack on these left-overs to any tensor factor of $\widehat{\varphi}_e(F^e_* g)$ to ensure that we have $\Delta_n \circ \widehat{\varphi}_e(F^e_* g) = \varphi_e \circ F^e_* \Delta_n(F^e_* g)$.

Recall that $\upsilon(x) = \lfloor x/q \rfloor$.

Lemma 4.4.12. *Let $F^e_* \prod_k \boldsymbol{x}_k^{a_{\bullet,k}} \boldsymbol{y}_k^{b_{\bullet,k}}$ be an $S^{\otimes n}$-module generator of $F^e_* S^{\otimes n}$ satisfying condition (😸), and suppose $b_k - a_k \equiv 0 \mod q$ for all k with $1 \leq k \leq n$. Then*

$$\sum_{k=1}^{n} b_k - a_k = q\left(\sum_{j=0}^{s} \upsilon\left(\sum_{k=1}^{n} b_{j,k}\right) - \sum_{i=0}^{r} \upsilon\left(\sum_{k=1}^{n} a_{i,k}\right)\right). \tag{4.9}$$

Moreover, setting

$$(\mu_{+,k}, \nu_{+,k}) = \begin{cases} \big((b_k - a_k)/q, 0\big), & b_k - a_k \geq 0, \\ \big(0, (a_k - b_k)/q\big), & b_k - a_k < 0 \end{cases}$$

we have

$$\sum_{j=0}^{s} \upsilon\left(\sum_{k=1}^{n} b_{j,k}\right) \geq \sum_{k=1}^{n} \mu_{+,k} =: \mu_+, \qquad \sum_{i=0}^{r} \upsilon\left(\sum_{k=1}^{n} a_{i,k}\right) \geq \sum_{k=1}^{n} \nu_{+,k} =: \nu_+. \tag{4.10}$$

Assuming this lemma, we define

$$\widehat{\varphi}_e\left(F^e_* \prod_k \boldsymbol{x}_k^{a_{\bullet,k}} \boldsymbol{y}_k^{b_{\bullet,k}}\right) = \vartheta \cdot \prod_{k=1}^{n} \vartheta_k$$

where $\vartheta_k \in \Bbbk[\boldsymbol{x}_k, \boldsymbol{y}_k] \subset S^{\otimes n}$ is defined inductively as follows.

For ϑ_1, if $b_1 - a_1 \geq 0$ then $b_1 - a_1 = \mu_{+,1} q$. Let $f_1 = 1$ and let g_1 be some factor of $\boldsymbol{y}^{\upsilon(\sum_k b_{\bullet,k})}$ of degree $\mu_{+,1}$. This is possible by (4.10), as $\sum_{j=0}^{s} \upsilon\big(\sum_{k=1}^{n} b_{j,k}\big) \geq \mu_{+,1}$. For all k, let $\varpi_k : S \to S^{\otimes n}$ be the canonical homomorphism that sends S to the k-th factor of the tensor product. Then $\vartheta_1 = \varpi_1(g_1)$.

Similarly, if $b_1 - a_1 < 0$, we know that $a_1 - b_1 = \nu_{+,1} q$. We let f_1 be some factor of $\boldsymbol{x}^{\upsilon(\sum_k a_{\bullet,k})}$ of degree $\nu_{+,1}$ and let $g_1 = 1$. This is again possible by (4.10). Then we define $\vartheta_1 = \varpi_1(f_1)$.

Having defined ϑ_k, f_k, and g_k for $i = 1, \ldots, m$. We define ϑ_{m+1} as follows: if $b_{m+1} - a_{m+1} \geq 0$ then let $f_{m+1} = 1$ and let g_{m+1} be some factor of

$$\boldsymbol{y}^{\upsilon(\sum_k b_{\bullet,k})} \Big/ g_1 \cdots g_m$$

of degree $\mu_{+,m+1}$. We know that this is always possible by (4.10). Then $\vartheta_{m+1} = \varpi_{m+1}(g_{m+1})$. Similarly, if $b_{m+1} - a_{m+1} < 0$, we let f_{m+1} be some factor of

$$\boldsymbol{x}^{\upsilon(\sum_k a_{\bullet,k})} \Big/ f_1 \cdots f_m$$

of degree $\nu_{+,m+1}$ and let $g_{m+1} = 1$. Then $\vartheta_{m+1} = \varpi_{m+1}(f_{m+1})$.

Having defined ϑ_k for $k = 1, \ldots, n$, we simply let

$$\vartheta = \varpi_1 \left(\boldsymbol{x}^{\upsilon(\sum_k a_{\bullet,k})} \boldsymbol{y}^{\upsilon(\sum_k b_{\bullet,k})} \Big/ f_1 \cdots f_n g_1 \cdots g_n \right)$$

it is clear from the definition of ψ that $\widehat{\varphi}_e$ satisfies

$$\Delta_n \circ \widehat{\varphi}_e = \psi.$$

It remains to check that $\widehat{\varphi}_e \left(F_*^e R^{\otimes n} \right) \subset R^{\otimes n}$. It is enough to check that $\widehat{\varphi}_e$ sends each of the $R^{\otimes n}$-module generators from Lemma 4.4.10 to $R^{\otimes n}$. Recall any such generator can be written as

$$\prod_{k=1}^{n} \boldsymbol{z}_k^{\rho_{\bullet,k}} \cdot F_*^e \prod_{k=1}^{n} \boldsymbol{x}_k^{a_{\bullet,k}} \boldsymbol{y}_k^{b_{\bullet,k}}$$

where, for each k, $\boldsymbol{z}_k^{\rho_{\bullet,k}} = \boldsymbol{x}_k^{\mu_{\bullet,k}}$ if $b_k - a_k \geq 0$, and $\boldsymbol{z}_k^{\rho_{\bullet,k}} = \boldsymbol{y}_k^{\nu_{\bullet,k}}$ if $b_k - a_k < 0$. Here, as in Lemma 4.4.10, $\sum_i \mu_{i,k} = (b_k - a_k)/q$ and $\sum_j \nu_{j,k} = (a_k - b_k)/q$. Note that $\sum_i \mu_{i,k}$ and $\sum_j \nu_{j,k}$ are respectively the quantities $\mu_{+.k}$ and $\nu_{+,k}$ defined in Lemma 4.4.12. Then

$$\widehat{\varphi}_e \left(\prod_{k=1}^{n} \boldsymbol{z}_k^{\rho_{\bullet,k}} \cdot F_*^e \prod_{k=1}^{n} \boldsymbol{x}_k^{a_{\bullet,k}} \boldsymbol{y}_k^{b_{\bullet,k}} \right) = \prod_{k=1}^{n} \boldsymbol{z}_k^{\rho_{\bullet,k}} \cdot \widehat{\varphi}_e \left(F_*^e \prod_{k=1}^{n} \boldsymbol{x}_k^{a_{\bullet,k}} \boldsymbol{y}_k^{b_{\bullet,k}} \right) = 0$$

if condition (😺) is not satisfied by $\prod_{k=1}^{n} \boldsymbol{x}_k^{a_{\bullet,k}} \boldsymbol{y}_k^{b_{\bullet,k}}$. Otherwise,

$$\begin{aligned} \widehat{\varphi}_e \left(\prod_{k=1}^{n} \boldsymbol{z}_k^{\rho_{\bullet,k}} \cdot F_*^e \prod_{k=1}^{n} \boldsymbol{x}_k^{a_{\bullet,k}} \boldsymbol{y}_k^{b_{\bullet,k}} \right) &= \prod_{k=1}^{n} \boldsymbol{z}_k^{\rho_{\bullet,k}} \cdot \widehat{\varphi}_e \left(F_*^e \prod_{k=1}^{n} \boldsymbol{x}_k^{a_{\bullet,k}} \boldsymbol{y}_k^{b_{\bullet,k}} \right) \\ &= \prod_{k=1}^{n} \boldsymbol{z}_k^{\rho_{\bullet,k}} \cdot \vartheta \cdot \prod_{k=1}^{n} \vartheta_k \\ &= \vartheta \cdot \prod_{k=1}^{n} \boldsymbol{z}_k^{\rho_{\bullet,k}} \vartheta_k. \end{aligned}$$

For each k, if $b_k - a_k \geq 0$, then $\boldsymbol{z}_k^{\rho_{\bullet,k}} = \boldsymbol{x}_k^{\mu_{\bullet,k}}$ and ϑ_k is a monomial in $\{y_{0,k}, \dots, y_{s,k}\}$ of degree $\mu_{+,k}$. Similarly, if $b_k - a_k < 0$, then by construction $\boldsymbol{z}_k^{\rho_{\bullet,k}} = \boldsymbol{y}_k^{\nu_{\bullet,k}}$ and ϑ_k is a monomial in $\{x_{0,k}, \dots, x_{r,k}\}$ of degree $\nu_{+,k}$. In either case, we see that

$$\prod_{k=1}^{n} \boldsymbol{z}_k^{\rho_{\bullet,k}} \vartheta_k \in R^{\otimes n}.$$

So it just remains to show that $\vartheta \in R^{\otimes n}$. To see this, it suffices to show that

$$\boldsymbol{x}^{\upsilon\left(\sum_k a_{\bullet,k}\right)} \boldsymbol{y}^{\upsilon\left(\sum_k b_{\bullet,k}\right)} \Big/ f_1 \cdots f_n g_1 \cdots g_n \in R.$$

That is what (4.9) is all about, for the degrees in terms of y's and x's in this monomial are, respectively,

$$\sum_{j=0}^{s} \upsilon\left(\sum_k b_{j,k}\right) - \sum_k \nu_{+,k}, \quad \sum_{i=0}^{r} \upsilon\left(\sum_k a_{i,k}\right) - \sum_k \mu_{+,k}.$$

In order to prove these two numbers are equal, it suffices to show that

$$\sum_{j=0}^{s} \upsilon\left(\sum_k b_{j,k}\right) - \sum_{i=0}^{r} \upsilon\left(\sum_k a_{i,k}\right) = \sum_k \nu_{+,k} - \sum_k \mu_{+,k}.$$

However, the right-hand side is nothing but $\sum_k (b_k - a_k)/q$, so this follows from (4.9). This shows that $\widehat{\varphi}_e\left(F_*^e R^{\otimes n}\right) \subset R^{\otimes n}$.

All that remains now is to prove Lemma 4.4.12.

Proof of Lemma 4.4.12. To prove (4.9), we just switch the order of summation:

$$\sum_{k=1}^{n} b_k - a_k = \sum_{k=1}^{n}\left(\sum_{j=0}^{s} b_{j,k} - \sum_{i=0}^{r} a_{i,k}\right) = \sum_{j=0}^{s}\sum_{k=1}^{n} b_{j,k} - \sum_{i=0}^{r}\sum_{k=1}^{n} a_{i,k}.$$

As (😸) holds, we know that, for $i, j \geq 1$, we have $\sum_{k=1}^{n} b_{j,k} = q\upsilon\left(\sum_{k=1}^{n} b_{j,k}\right)$ and $\sum_{k=1}^{n} a_{i,k} = q\upsilon\left(\sum_{k=1}^{n} a_{i,k}\right)$. On the other hand, for $i = j = 0$, we rather have $\sum_{k=1}^{n} b_{j,k} = q\upsilon\left(\sum_{k=1}^{n} b_{j,k}\right) + 1$ and $\sum_{k=1}^{n} a_{i,k} = q\upsilon\left(\sum_{k=1}^{n} a_{i,k}\right) + 1$. In particular, for all i and j we have

$$\sum_{k=1}^{n} b_{j,k} - \sum_{k=1}^{n} a_{i,k} = q\left(\upsilon\left(\sum_{k=1}^{n} b_{j,k}\right) - \upsilon\left(\sum_{k=1}^{n} a_{i,k}\right)\right)$$

which finishes the proof of equation (4.9).

To prove (4.10), it is enough to show

$$q\sum_{j=0}^{s} \upsilon\left(\sum_{k=1}^{n} b_{j,k}\right) \geq q\sum_{k=1}^{n} \mu_{+,k}$$

(by symmetry, we will not have to check the other inequality). To see this, note that, by condition (🐱) we have

$$q \sum_{j=0}^{s} \upsilon \left(\sum_{k=1}^{n} b_{j,k} \right) = \sum_{k=1}^{n} \sum_{j=0}^{s} b_{j,k} - 1.$$

Further, we have

$$q \sum_{k=1}^{n} \mu_{+,k} \leq \sum_{k=1}^{n} |b_k - a_k| \leq \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} \sum_{j=0}^{s} b_{j,k}.$$

However, by condition (🐱), we see that

$$\sum_{k=1}^{n} \sum_{j=0}^{s} b_{j,k} \equiv 1 \mod q$$

so we have

$$q \sum_{k=1}^{n} \mu_{+,k} \neq \sum_{k=1}^{n} \sum_{j=0}^{s} b_{j,k}$$

Thus,

$$q \sum_{k=1}^{n} \mu_{+,k} \leq \sum_{k=1}^{n} \sum_{j=0}^{s} b_{j,k} - 1 = q \sum_{j=0}^{s} \upsilon \left(\sum_{k=1}^{n} b_{j,k} \right).$$

□

This proves Claim 4.4.9 and therefore Theorem 4.4.1.

Chapter 5

Computing $\mathscr{D}^{(2)}(R)$ for Affine Toric Varieties

> It eluded us then, but that's no matter—tomorrow we will run faster, stretch out our arms further. . .
>
> F. Scott Fitzgerald

Our next goal is to find a nice description of the diagonal Cartier algebras $\mathscr{D}^{(2)}(R)$. The case where R is a normal affine semigroup ring over a field k (equivalently, $\operatorname{Spec} R$ is an affine toric variety over k) turns out to be quite tractable.

Setting 5.0.1. We let Σ be a fan in $\mathbb{R}^n$ and $X = X(\Sigma)$ the associated toric variety over a perfect field k of characteristic $p > 0$. By $\Sigma(1)$ we mean the set of rays (i.e. 1-dimensional cones) in Σ. We assume, for simplicity, that Σ has a unique n-dimensional cone σ. We let R be the coordinate ring of X. In particular, R is a *toric ring*, that is, a normal affine semigroup ring. For all rays $\rho \in \Sigma(1)$, we let v_ρ denote the primitive generator of ρ. That is, v_ρ is the shortest nonzero vector in $\mathbb{Z}^n \cap \rho$.

Choose $e > 0$ and set $q = p^e$. In this chapter, it will be more convenient to use the notation $R^{1/q}$ rather than $F^e_* R$. Working in Setting 5.0.1, we have a nice k-basis for $\operatorname{Hom}_R(R^{1/q}, R)$. In particular, we let

$$k[T] = k\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]$$

be the coordinate ring of the n-dimensional torus T. Similarly, we let

$$k[T \times T] = k[T] \otimes_k k[T]$$

be the coordinate ring of the $2n$-dimensional torus, $T \times T$. We let $q = p^e$, and we adopt the notation,

$$\frac{1}{q}\mathbb{Z}^n := \left\{ \left(\frac{a_1}{q}, \cdots, \frac{a_n}{q} \right) \,\middle|\, a_1, \cdots, a_n \in \mathbb{Z} \right\}.$$

For any vector $u \in \frac{1}{q}\mathbb{Z}^n$, we adopt the shorthand notation

$$x^u := \prod_{i=1}^{n} x_i^{u_i}.$$

Then for all $a \in \frac{1}{q}\mathbb{Z}^n$ we define a map π_a in terms of its action on monomials: for all $u \in \frac{1}{q}\mathbb{Z}^n$, set

$$\pi_a(x^u) = \begin{cases} x^{a+u}, & a+u \in \mathbb{Z}^n \\ 0, & \text{otherwise} \end{cases}$$

It's not hard to see that these maps π_a generate all the maps $k[T]^{1/q} \to k[T]$:

Lemma 5.0.2 ([Pay09])**.** *In the notation above, the set $\{\pi_a \mid a \in \frac{1}{q}\mathbb{Z}^n\}$ is a k-vector space basis of* $\operatorname{Hom}_{k[T]}\left(k[T]^{1/q}, k[T]\right)$.

Further, every map $R^{1/q} \to R$ extends to a unique map $k[T]^{1/q} \to k[T]$. Thus, each map in $\operatorname{Hom}(R^{1/q}, R)$ is just a map $k[T]^{1/q} \to k[T]$ that happens to send $R^{1/q}$ into R. Payne has characterized such maps: we define the *anticanonical polytope* of R,

$$P_{-K_X} = \{u \in \mathbb{R}^n \mid \langle u, v_\rho\rangle \geq -1 \text{ for all rays } \rho \in \Sigma(1)\}$$

Then we have:

Proposition 5.0.3 ([Pay09])**.** *Work in Setting 5.0.1. Then the set of maps π_a, where a is any element of* $\operatorname{int}(P_{-K_R}) \cap \frac{1}{q}\mathbb{Z}^n$*, forms a k-vector space basis for* $\operatorname{Hom}_R(R^{1/q}, R)$.

Our characterization of $\mathscr{D}^{(2)}$ is as follows:

Theorem 5.0.4. *Work in Setting 5.0.1. Then $\mathscr{D}_e^{(2)}(R)$ is generated as a k-vector space by the maps π_a such that $a \in \frac{1}{p^e}\mathbb{Z}^n \cap \operatorname{int}(P_{-K_X})$ and the interior of $P_{-K_X} \cap (a - P_{-K_X})$ contains a representative of each equivalence class in $\frac{1}{p^e}\mathbb{Z}^n/\mathbb{Z}^n$.*

First, we must prove a lemma. This lemma is similar to [CHP+18, Theorem 7.3]. Note however that we do not assume that φ is a splitting.

Lemma 5.0.5. *Let $\varphi = \sum c_{a,a'}\pi_a \otimes \pi_{a'}$ be a map in $\operatorname{Hom}_{k[T\times T]}\left(k[T\times T]^{1/q}, k[T\times T]\right)$. Then φ is compatible with I_Δ if and only if for all equivalence classes $[u_1], [u_2] \in \frac{1}{q}\mathbb{Z}^n/\mathbb{Z}^n$, we have, for all $d \in \frac{1}{q}\mathbb{Z}^n$,*

$$\sum_{a\in[u_1]} c_{a,d-a} = \sum_{b\in[u_2]} c_{b,d-b}$$

Proof. Note that the ideal $I_\Delta^{1/q} \subseteq k[T \times T]^{1/q}$ is generated by the elements

$$\left\{ x^u \otimes x^{-u} - 1 \;\middle|\; u \in \frac{1}{q}\mathbb{Z}^n \right\}.$$

Since $k[T \times T]$ is a smaller ring than $k[T \times T]^{1/q}$, we need more elements to generate $I_\Delta^{1/q}$ as a $k[T \times T]$-module. However, elements of the form $x^v \otimes x^{v'}$, where v and v' are vectors in $\frac{1}{q}\mathbb{Z}^n$, generate $k[T \times T]^{1/q}$ as a $k[T \times T]$-module (indeed, as a k-vector space). Thus, the set

$$\left\{ x^v \otimes x^{v'} \left(x^u \otimes x^{-u} - 1 \right) \;\middle|\; u, v, v' \in \frac{1}{q}\mathbb{Z}^n \right\}$$

generates $I_\Delta^{1/q}$ as a module over $k[T \times T]$.

Suppose $\varphi = \sum_{a,a'} c_{a,a'} \pi_a \otimes \pi_{a'}$ is compatible with the diagonal. This is equivalent to asserting that

$$\varphi \left(x^v \otimes x^{v'} \left(x^u \otimes x^{-u} - 1 \right) \right) \equiv 0 \bmod I_\Delta \tag{5.1}$$

for all $u, v, v' \in \frac{1}{q}\mathbb{Z}^n$. Set $\varphi_{v,v'} := \varphi(x^v \otimes x^{v'} \cdot \underline{\quad})$. Then the condition in (5.1) is equivalent to saying

$$\varphi_{v,v'} \left(x^u \otimes x^{-u} - 1 \right) \equiv 0 \bmod I_\Delta,$$

for all $u, v, v' \in \frac{1}{q}\mathbb{Z}^n$, or in other words,

$$\varphi_{v,v'} \left(x^u \otimes x^{-u} \right) \equiv \varphi_{v,v'}(1) \bmod I_\Delta. \tag{5.2}$$

Now, it's easy to see that $\pi_a \otimes \pi_{a'} \left(x^v \otimes x^{v'} \cdot \underline{\quad} \right) = \pi_{a+v} \otimes \pi_{a'+v'}$. Thus

$$\varphi_{v,v'} = \sum_{a,a' \in \frac{1}{q}\mathbb{Z}^n} c_{a,a'} \pi_{a+v} \otimes \pi_{a'+v'} = \sum_{a,a' \in \frac{1}{q}\mathbb{Z}^n} c_{a-v,a'-v'} \pi_a \otimes \pi_{a'}$$

This means that (5.2) is equivalent to saying

$$\sum_{\substack{a \in -u+\mathbb{Z}^n, \\ a' \in u+\mathbb{Z}^n}} c_{a-v,a'-v'} x^{a+u} \otimes x^{a'-u} \equiv \sum_{b,b' \in \mathbb{Z}^n} c_{b-v,b'-v'} x^b \otimes x^{b'} \bmod I_\Delta,$$

and this is the case if and only if

$$\sum_{\substack{a \in -u+\mathbb{Z}^n, \\ a' \in u+\mathbb{Z}^n}} c_{a-v,a'-v'} x^{a+a'} = \sum_{b,b' \in \mathbb{Z}^n} c_{b-v,b'-v'} x^{b+b'}. \tag{5.3}$$

Now, the above is an equality of Laurent polynomials, so it holds if and only if the corresponding coefficients for each exponent of x are the same. So our initial assertion (5.1) holds if and only if, for all $d \in \mathbb{Z}^n$ and all $u, v, v' \in \frac{1}{q}\mathbb{Z}^n$, we have

$$\sum_{a \in -u+\mathbb{Z}^n} c_{a-v,d-a-v'} = \sum_{b \in \mathbb{Z}^n} c_{b-v,d-b-v'}$$

(In other words, we're setting $d = a+a' = b+b'$ in (5.3).) By setting $U = -u-v$ and $D = d-v'-v$, the above is equivalent to

$$\sum_{a\in U+\mathbb{Z}^n} c_{a,D-a} = \sum_{b\in -v+\mathbb{Z}^n} c_{b,D-b}$$

where U, v, and D independently range over $\frac{1}{q}\mathbb{Z}^n$. This completes the proof. □

Corollary 5.0.6. *Let R be a toric ring. The Cartier algebra $\mathscr{C}^{R\otimes_k R, I_\Delta \circlearrowleft}$ is "graded", in the sense that the map $\sum_{a,a'} c_{a,a'}\pi_a\otimes\pi_{a'}$ is compatible with the diagonal if and only if, for each $d\in\frac{1}{q}\mathbb{Z}^n$, we have $\sum_{a+a'=d} c_{a,a'}\pi_a\otimes\pi_{a'}$ is compatible with the diagonal. It follows that $\mathscr{D}^{(2)}(R)$ is generated over k by the maps π_d in $\mathscr{D}^{(2)}(R)$.*

Proof. First, we focus on the case that $R = k[T]$. The first part follows immediately from the above lemma: a map $\sum_{a,a'} c_{a,a'}\pi_a\otimes\pi_{a'}$ satisfies the condition in Lemma 5.0.5 if and only if the maps $\sum_{a+a'=d} c_{a,a'}\pi_a\otimes\pi_{a'}$ satisfy the condition in Lemma 5.0.5 for all d.

For the second assertion, let $\psi\in\mathscr{C}^{R\otimes_k R, I_\Delta \circlearrowleft}$ be arbitrary. Then $\overline{\psi} := \psi|_{R\otimes_k R/I_\Delta}$ is an arbitrary element of $\mathscr{D}^{(2)}(R)$. If

$$\psi = \sum_{a,a'} b_{a,a'}\pi_a\otimes\pi_{a'},$$

then by the first assertion, we have $\psi' := \sum_{a+a'=u} b_{a,a'}\pi_a\otimes\pi_{a'}$ is also in $\mathscr{C}^{R\otimes_k R, I_\Delta \circlearrowleft}$ for all $u\in\frac{1}{q}\mathbb{Z}^n$. Now let $v\in\frac{1}{q}\mathbb{Z}^n$ be arbitrary. We compute:

$$\psi'|_\Delta(x^v) = \left.\left(\sum_a b_{a,u-a}\pi_a(x^v)\otimes\pi_{u-a}(1)\right)\right|_{R\otimes_k R/I_\Delta} \tag{5.4}$$

$$= \left.\left(\sum_{\substack{a\in -v+\mathbb{Z}^n\\ \text{where } u-a\in\mathbb{Z}^n}} b_{a,u-a}x^{a+v}\otimes x^{u-a}\right)\right|_{R\otimes_k R/I_\Delta} \tag{5.5}$$

$$= \begin{cases}\left(\sum_{a\in -v+\mathbb{Z}^n} b_{a,u-a}\right)x^{u+v}, & u+v\in\mathbb{Z}^n\\ 0 & \text{otherwise}\end{cases} \tag{5.6}$$

$$= \left(\sum_{a\in -v+\mathbb{Z}^n} b_{a,u-a}\right)\pi_u(x^v) \tag{5.7}$$

So if we write $\overline{\psi} = \sum_a c_a\pi_a$, we see that $\psi'|_{R\otimes_k R/I_\Delta} = c_u\pi_u$. Thus either $c_u = 0$ or $\pi_u\in\mathscr{D}^{(2)}(R)$, as desired.

For the general case, we just note that each map $(R\otimes_k R)^{1/q}\to R\otimes_k R$ extends to a unique map $k[T\times T]^{1/q}\to k[T\times T]$, and one of these maps is compatible with the diagonal whenever the other map is (*Cf.* [BK07, Lemma 1.1.7]). □

Note that, a priori, it looks like the coefficient of $\pi_u(x^v)$ in (5.7) depends on v (or even worse, on our choice of lifting of x^v to $R \otimes_k R$), but by Lemma 5.0.5, this is not the case.

Proof of Theorem 5.0.4. Any map in $\psi \in \operatorname{Hom}_{R\otimes_k R}((R \otimes_k R)^{1/q}, R \otimes_k R)$ extends to a map $\widehat{\psi} \in \operatorname{Hom}_{k[T\times T]}(k[T \times T]^{1/q}, k[T \times T])$, and two maps agree if and only if their extensions to the torus agree. By [BK07, Lemma 1.1.7], ψ is compatible with the diagonal if and only if $\widehat{\psi}$ is compatible with the diagonal. This means the map π_d is in $\mathscr{D}^{(2)}(R)$ if and only if there exists some map

$$\varphi = \sum_{a,a'} c_{a,a'} \pi_a \otimes \pi_{a'} \colon k[T \times T]^{1/q} \to k[T \times T]$$

compatible with I_Δ that restricts to a map $(R \otimes_k R)^{1/q} \to R \otimes_k R$, such that the sum $\sum_{a+a'=d} c_{a,a'}$ is non-zero. By Lemma 5.0.5, this means for all $[u_1], [u_2] \in \frac{1}{q}\mathbb{Z}^n/\mathbb{Z}^n$, we have

$$\sum_{a\in[u_1]} c_{a,d-a} = \sum_{b\in[u_2]} c_{b,d-b}.$$

Further, if $\sum_{a+a'=d} c_{a,a'} \neq 0$, then there exists some $[u] \in \frac{1}{q}\mathbb{Z}^n/\mathbb{Z}^n$ such that $\sum_{a\in[u]} c_{a,d-a} \neq 0$. This is just because

$$\sum_{a+a'=d} c_{a,a'} = \sum_{[u]\in\frac{1}{q}\mathbb{Z}^n/\mathbb{Z}^n} \sum_{a\in[u]} c_{a,d-a}$$

Using Lemma 5.0.5 again, this means that for *all* $[u] \in \frac{1}{q}\mathbb{Z}^n/\mathbb{Z}^n$, the sum $\sum_{a\in[u]} c_{a,d-a}$ is nonzero. In particular, for all $[u] \in \frac{1}{q}\mathbb{Z}^n/\mathbb{Z}^n$, there is some $a \in [u]$ such that $c_{a,d-a} \neq 0$. Since φ restricts to a map $F^e_*(R \otimes_k R) \to R \otimes_k R$, this means $a, d - a \in \operatorname{int}(P_{-K_X})$, by Proposition 5.0.3. In other words, a is in the interior of $P_{-K_X} \cap (d - P_{-K_X})$.

Conversely, given some d, suppose that each equivalence class $[u]$ has a representative in the interior of $P_{-K_X} \cap (d - P_{-K_X})$. Then we can label these representatives $a_1, \ldots, a_N$. Then $\sum_i \pi_{a_i} \otimes \pi_{d-a_i}$ is a map compatible with the diagonal, and its restriction to the diagonal is π_d. This is just an application of equation (5.7); in this case, there is only one nonzero coefficient $b_{a,d-a}$ where a is in any particular equivalence class of $\frac{1}{q}\mathbb{Z}^n/\mathbb{Z}^n$. □

Remark 5.0.7. Theorem 5.0.4 can be seen as a generalization of [Pay09, Theorem 1.2]. Indeed, the following lemma shows that an affine toric variety is diagonally F-split if and only if $\pi_0 \in \mathscr{D}^{(2)}$. Thus we recover [Pay09, Theorem 1.2] by setting $a = 0$ in the statement of Theorem 5.0.4.

Lemma 5.0.8. *Let R be a toric ring. The following are equivalent:*

(i) R is diagonally F-split

(ii) $\pi_0 \in \mathscr{D}_e^{(2)}(R)$ *for some* $e > 0$

(iii) $\pi_0 \in \mathscr{D}_1^{(2)}(R)$.

Proof. To see that (i) implies (ii), suppose that R is diagonally F-split. By [Pay09, Proposition 4.5], there exists some $e > 0$ and some map

$$\varphi = \sum_{a \in \frac{1}{q}\mathbb{Z}^n} c_a \pi_a \in \mathscr{D}_e^{(2)}.$$

with $c_0 \neq 0$. By Corollary 5.0.6, we have $\pi_0 \in \mathscr{D}_e^{(2)}(R)$.

To see that (ii) implies (iii), suppose that $\pi_0 \in \mathscr{D}_e^{(2)}(R)$ and set $q = p^e$. As $R^{1/p} \subseteq R^{1/q}$, the map π_0 restricts to a map $R^{1/p} \to R$. One checks that this restriction is in $\mathscr{D}_1^{(2)}(R)$, for instance by using Theorem 5.0.4.

Finally, (iii) implies (i) by definition. □

Example 5.0.9. Consider the case $R = k[x, y, z]/(xy - z^2)$, and assume that $\operatorname{char} k > 2$. To use the techniques in this chapter, we use the presentation $R = k[y, xy, xy^2]$. Then Figure 5.1 shows the polytope P_{-K_X}. Using Theorem 5.0.4, one can compute $\mathscr{D}^{(2)}(R)$:

$$\mathscr{D}^{(2)}(R) = \bigoplus_{e \geq 0} F_*^e \left\langle x^{p^e+1} y^{\frac{p^e+1}{2}}, x^{p^e} y^{\frac{p^e+1}{2}}, xy^{\frac{p^e+1}{2}}, y^{\frac{p^e+1}{2}}, x^{p^e-1} y^{p^e-1} \right\rangle \operatorname{Hom}_R(F_*^e R, R) \quad (5.8)$$

In terms of the more familiar presentation, $R \cong k[x^2, xy, y^2]$, this formula becomes

$$\mathscr{D}^{(2)}(R) = \bigoplus_{e \geq 0} F_*^e \left\langle x^{p^e+1}, x^{p^e} y, x^{p^e-1} y^{p^e-1}, xy^{p^e}, y^{p^e+1} \right\rangle \operatorname{Hom}_R(F_*^e R, R)$$

To see this, we will prove the following:

(i) If $a > -1$ and $b > a/2$, then $\pi_{(a,b)} \in \mathscr{D}^{(2)}(R)$

(ii) If $a > 0$ and $b > (a-1)/2$, then $\pi_{(a,b)} \in \mathscr{D}^{(2)}(R)$

(iii) $\pi_{(0,0)} \in \mathscr{D}^{(2)}(R)$

(iv) The maps $\pi_{(0,-1/q)}, \pi_{(-1/q,-1/q)}$, and $\pi_{(-2/q,-2/q)}$ are not in $\mathscr{D}^{(2)}(R)$.

Because $\mathscr{D}^{(2)}(R)$ is a Cartier algebra and $\pi_a(x^b \cdot -) = \pi_{a+b}$ for all $a, b \in \frac{1}{q}\mathbb{Z}^n$, it follows from (iv) that the maps described in (i)–(iii) are the only maps in $\mathscr{D}^{(2)}(R)$. Another way to see this is to notice that, for any map π_v not among those described in (i)–(iii), the corresponding polytope

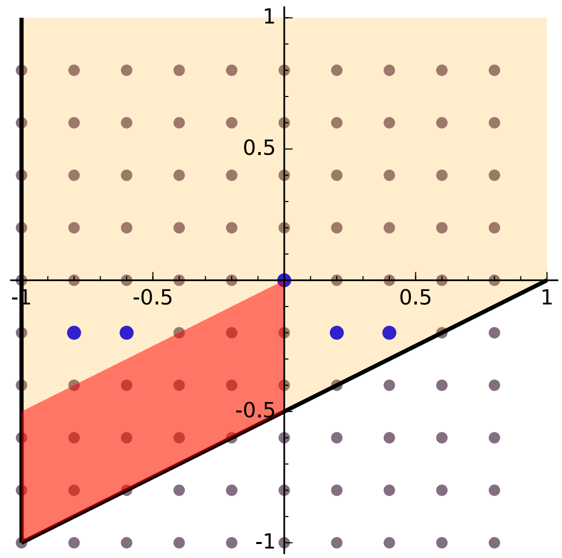


Figure 5.1: The polytope P_{-K_X} for the quadric cone $R = k[y, xy, x^2y]$, along with the fractional lattice $\frac{1}{5}\mathbb{Z}^2$. The area in red denotes the set of maps not in $\mathscr{D}_1^{(2)}(R)$. The points in blue denote the generators of $\mathscr{D}_1^{(2)}(R)$ over $F_*^1 R$.

$P_{-K_X} \cap (v - P_{-K_X})$ is contained in the polytope corresponding to one of the maps described in (iv). Consequently we see that $\mathscr{D}_e^{(2)}$ is generated over $F_*^e R$ by the maps π_v, where

$$v \in \left\{ \left(\frac{1-q}{q}, \frac{(q+1)/2}{q}\right), \left(\frac{2-q}{q}, \frac{(q+1)/2}{q}\right), (0,0), \left(\frac{1}{q}, \frac{(q+1)/2}{q}\right), \left(\frac{2}{q}, \frac{(q+1)/2}{q}\right) \right\}$$

As $\operatorname{Hom}_R(F_*^e R, R)$ is generated as an $F_*^e R$-module by $\pi_{\left(\frac{1-q}{q}, \frac{1-q}{q}\right)}$, we get that $\mathscr{D}^{(2)}(R)$ has the description given in equation (5.8).

So, let $(a, b), (\alpha, \beta) \in \mathbb{R}^2$. Then

$$\begin{aligned} P_{(a,b)} &:= \operatorname{int}\left(P_{-K_X} \cap ((a,b) - P_{-K_X})\right) \\ &= \{(x,y) \mid -1 < x < a+1, -1 < 2y - x < 1 - a + 2b\} \end{aligned}$$

We wish to find an integer translation of (α, β) in $P_{(a,b)}$, where (a, b) is as in (i) or (ii). Let $\overline{\alpha} = \alpha - \lceil \alpha \rceil$. If (a, b) is as in (i), then we have

$$\begin{cases} (\overline{\alpha}, \beta - \lfloor \beta \rfloor) \in P_{(a,b)}, & 2(\beta - \lfloor \beta \rfloor) - \overline{\alpha} \leq 1 \\ (\overline{\alpha}, \beta - \lfloor \beta \rfloor - 1) \in P_{(a,b)}, & 2(\beta - \lfloor \beta \rfloor) - \overline{\alpha} > 1 \end{cases}$$

If (a, b) is as in (ii), then we have

$$\begin{cases} (\overline{\alpha} + 1, \beta - \lfloor \beta \rfloor - 1) \in P_{(a,b)}, & \frac{1}{2}\overline{\alpha} + \frac{1}{2} < \beta - \lfloor \beta \rfloor \\ (\overline{\alpha}, \beta - \lfloor \beta \rfloor - 1) \in P_{(a,b)}, & \frac{1}{2}\overline{\alpha} - \frac{1}{2} < \beta - \lfloor \beta \rfloor \leq \frac{1}{2}\overline{\alpha} + \frac{1}{2} \\ (\overline{\alpha} + 1, \beta - \lfloor \beta \rfloor) \in P_{(a,b)}, & 0 \leq \beta - \lfloor \beta \rfloor \leq \frac{1}{2}\overline{\alpha} - \frac{1}{2} \end{cases}$$

To check (iii), we note that

$$\begin{cases} (\overline{\alpha}, \beta - \lfloor \beta \rfloor) \in P_{(0,0)}, & 2(\beta - \lfloor \beta \rfloor) - \overline{\alpha} < 1 \\ (\overline{\alpha}, \beta - \lfloor \beta \rfloor - 1) \in P_{(0,0)}, & 2(\beta - \lfloor \beta \rfloor) - \overline{\alpha} > 1 \\ (\overline{\alpha} + 1, \beta - \lfloor \beta \rfloor) \in P_{(0,0)}, & 2(\beta - \lfloor \beta \rfloor) - \overline{\alpha} = 1 \end{cases}$$

Here we're using the fact that $\operatorname{char} k \neq 2$ to see that $\overline{\alpha} < 0$ if $2(\beta - \lfloor \beta \rfloor) - \overline{\alpha} = 1$. Indeed, the point $(0, \frac{1}{2})$ has no integer translation in $P_{(0,0)}$, so $\pi_0 \notin \mathscr{D}^{(2)}(R)$ if $\operatorname{char} k = 2$.

Finally, to check (iv), we note that $(0, \frac{(q-1)/2}{q})$ has no integer translations in $P_{(0,-1/q)}$. The polytope $P_{(-1/q,-1/q)}$ has no integer translations of $(0, \frac{(q-1)/2}{q})$ nor, for that matter, of $(-\frac{1}{q}, \frac{(q-1)/2}{q})$. The polytope $P_{(-2/q,-1/q)}$ has no integer translations of $(-\frac{1}{q}, \frac{(q-1)/2}{q})$.

We can use this calculation to compute the F-signature of $\mathscr{D}^{(2)}(R)$ in the sense of [BST12]. Indeed, we see that the only splitting of $R \to R^{1/q}$ contained in $\mathscr{D}^{(2)}(R)$ is $\pi_{(0,0)}$. Thus $s(\mathscr{D}^{(2)}(R)) = 0$.

Example 5.0.10. Let $R = k[x, y, z, xyz^{-1}] \cong k[s, t, u, v]/(st - uv)$. The cone σ of R is given by the extremal rays:

$$(1, 0, 0)\,, \quad (0, 1, 0)\,, \quad (1, 0, 1)\,, \quad (0, 1, 1)$$

We will show the following:

Claim 5.0.11. *Let $e > 0$. If $(a, b, c) \in P_{-K} \cap \frac{1}{q}\mathbb{Z}^3$ and $a + b + c > -1$, then $\pi_{(a,b,c)}$ is in $\mathscr{D}_e^{(2)}$.*

To see this claim, it's enough to consider the case $-1 < a, b \leq 0$, since $\varphi(F_*^e x \cdot -) \in \mathscr{D}^{(2)}$ whenever $\varphi \in \mathscr{D}^{(2)}$ and $x \in R$. The key point is that then

$$2 + a + c > 1 - b \geq 1, \text{ and} \tag{5.9}$$

$$2 + b + c > 1 - a \geq 1. \tag{5.10}$$

Set $\vec{d} = (a, b, c)$. Let $(\alpha, \beta, \gamma) \in \mathbb{R}^3$. We wish to find an integer translation of (α, β, γ) is in $P \cap (\vec{d} - P)$. We start by translating the polytope $P \cap (\vec{d} - P)$ by $(1, 1, 0)$: the resulting polytope is described as the set of (x, y, z) satisfying the inequalities

$$0 < x < 2 + a$$

$$0 < y < 2 + b$$

$$0 < x + z < 2 + a + c$$

$$0 < y + z < 2 + b + c$$

We may assume, without loss of generality, that $0 < \alpha, \beta \leq 1$ and $0 \leq \gamma < 1$. Note that we automatically have

$$0 < \alpha < 2 + a, \quad 0 < \beta < 2 + b, \quad 0 < \alpha + \gamma, \quad 0 < \beta + \gamma$$

If we happen to have $\alpha + \gamma < 2 + a + c$ and $\beta + \gamma < 2 + b + c$, then we're done. So suppose otherwise. Without loss of generality, we may assume that $\alpha + \gamma \geq 2 + a + c$. If we also have $\beta + \gamma \geq 2 + b + c$, then the point $(\alpha, \beta, \gamma - 1)$ is in $P \cap (\vec{d} - P)$: since $\alpha + \gamma < 2$, we have $\alpha + \gamma - 1 < 1 < 2 + a + c$ by equation (5.9). On the other hand, $\alpha + \gamma \geq 2 + b + c > 1$, so $\alpha + \gamma - 1 > 0$. Similarly, we have $0 < \beta + \gamma - 1 < 2 + b + c$.

Now suppose that $\alpha + \gamma \geq 2 + a + c$ but $\beta + \gamma < 2 + b + c$. If $\beta + \gamma > 1$, then the point $(\alpha, \beta, \gamma - 1)$ is again in $P \cap (\vec{d} - P)$, as clearly we have $0 < \beta + \gamma - 1 < 2 + b + c$. On the other hand, if $\beta + \gamma \leq 1$, then $(\alpha, \beta + 1, \gamma - 1)$ is in $P \cap (\vec{d} - P)$. Indeed, we just have to check that $\beta + 1 < 2 + b$, or in other words, that $b > \beta - 1$. We know, by assumption, that $b > -1 - a - c$, so it suffices to check that $-a - c \geq \beta$. As $\gamma \geq 2 + a + c - \alpha$, we have

$$\beta \leq 1 - \gamma \leq 1 - (2 + a + c - \alpha) = -a - c + \alpha - 1 < -a - c$$

The last inequality comes from the assumption that $0 \leq \alpha < 1$. This proves the claim.

By [WY04, Von11], the splittings of $R^{1/q}$ correspond to the points in $\frac{1}{q}\mathbb{Z}^3 \cap P_{\text{sig}}$, where P_{sig} is the polytope given by

$$P_{\text{sig}} := \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} -1 < \langle 1, 0, 0 \rangle \cdot x \leq 0 \\ -1 < \langle 0, 1, 0 \rangle \cdot x \leq 0 \\ -1 < \langle 1, 0, 1 \rangle \cdot x \leq 0 \\ -1 < \langle 0, 1, 1 \rangle \cdot x \leq 0 \end{array} \right\}.$$

This polytope is depicted in Figure 5.2. As seen in the figure, the plane $x + y + z = -1$ cuts this polytope in half. This shows that $s(\mathscr{D}^{(2)}(R)) \geq s(R)/2 = 1/3$. Calculations in Macaulay2 [GS] suggest that there are no further maps in $\mathscr{D}^{(2)}(R)$ and that $s(\mathscr{D}^{(2)}(R)) = 1/3$.

5.1 Measuring singularities with $\mathscr{D}^{(2)}$

Theorem 2.3.1 suggests that rings with milder singularities have larger Cartier algebras $\mathscr{D}^{(2)}$. We wonder whether the singularities of a ring can be well understood just by considering, in some sense, the size of $\mathscr{D}^{(2)}$. The following conjecture would be a natural place to start in order to develop such a theory:

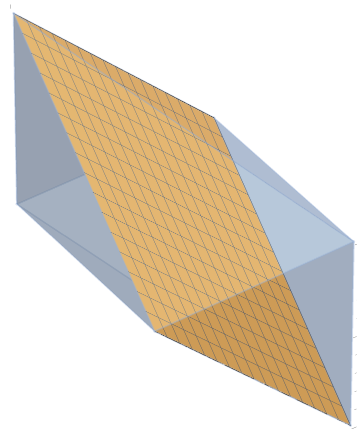


Figure 5.2: Comparison of maps in $\mathscr{D}^{(2)}(R)$ and a polytope whose volume is the F-signature of R, according to [WY04] (*Cf.* [Von11]). The plane is given by $x + y + z = -1$. All fractional lattice points lying above the plane correspond to maps in $\mathscr{D}^{(2)}(R)$.

Conjecture 5.1.1. *Let R be a finitely-generated algebra over a perfect field of positive characteristic. Then $\mathscr{D}^{(2)}(R) = \mathscr{C}^R$ if and only if R is regular.*

One direction is clear: if R is regular then so is $R \otimes_k R$, meaning that $F^e_*(R \otimes_k R)$ is a projective $R \otimes_k R$-module for all $e > 0$. As $R \otimes_k R \to R$ is a surjective map of $R \otimes_k R$ modules, the universal property of projective modules tells us that any map $F^e_* R \to R$ will lift:

$$\begin{array}{ccc} F^e_*(R \otimes_k R) & \overset{\widehat{\varphi}}{\dashrightarrow} & R \otimes_k R \\ \Big\downarrow{\scriptstyle F^e_*\mu} & & \Big\downarrow{\scriptstyle \mu} \\ F^e_* R & \xrightarrow{\ \varphi\ } & R \end{array}$$

Here, we're thinking of R and $F^e_* R$ as $R \otimes_k R$-modules via μ. We have the following partial converses.

Proposition 5.1.2. *Suppose k is a perfect field of positive characteristic and R is a reduced k-algebra essentially of finite type. If $\mathscr{D}^{(2)}(R) = \mathscr{C}^R$, then R is strongly F-regular.*

Proof. Apply Corollary 3.2.5 to the case where $\mathfrak{a} = \mathfrak{b} = R$. We see that $0 \neq \tau(R) \subseteq \tau(R)^2$, so $\tau(R) = \tau(R)^2$. As $\tau(R)$ contains a regular element of R, it follows from Nakayama's lemma that $\tau(R) = R$. □

We also know that the converse of Conjecture 5.1.1 holds in the $\mathbb{Q}$-Gorenstein toric case, using Watanabe and Yoshida's characterisation of the F-signature in that setting. The point of the $\mathbb{Q}$-Gorenstein condition is just so that we know P_{-K_X} is a translation of the dual cone $\sigma^\vee$ of R. Note that, by [CLS11, Proposition 4.2.7], this includes the case when the cone σ of R is simplicial, and in particular all toric surfaces.

Proposition 5.1.3. *Work in Setting 5.0.1. Suppose also that R is $\mathbb{Q}$-Gorenstein and $\mathscr{D}^{(2)}(R) = \mathscr{C}^R$. Then R is regular.*

Proof. It follows from Lemma 5.1.4 that $P_{-K_X} = \sigma^\vee + v$ for some $v \in \mathbb{R}^n$ such that $\langle v, v_\rho \rangle = -1$ for all $\rho \in \Sigma(1)$. Let $Q = \{x \mid \forall \rho \in \Sigma(1) : 0 < \langle x, v_\rho \rangle < 1\}$. By [Von11], we know that $s(R) = \mathrm{vol}(Q)$. The key point is to notice that

$$P_{-K_X} \cap (v - P_{-K_X}) = (\sigma^\vee + v) \cap (-\sigma^\vee) = \{x \mid \forall \rho \in \Sigma(1) : -1 < \langle x, v_\rho \rangle < 0\} = -Q.$$

Thus $s(R) = \mathrm{vol}(P_{-K_X} \cap (v - P_{-K_X}))$.

Now, the function $\mathbb{R}^n \to \mathbb{R}$ given by $d \mapsto \mathrm{vol}(P_{-K_X} \cap (d - P_{-K_X}))$ is continuous, as this intersection is always compact. By taking e sufficiently large, we can find a lattice point $d \in \frac{1}{p^e}\mathbb{Z}^n \cap P_{-K_X}$ arbitrarily close to v. Thus we can make $\mathrm{vol}(P_{-K_X} \cap (d - P_{-K_X}))$ arbitrarily close to $\mathrm{vol}(P_{-K_X} \cap (v - P_{-K_X}))$. Since $\mathscr{D}^{(2)}(R) = \mathscr{C}^R$, we have that $\pi_d \in \mathscr{D}^{(2)}(R)$ and also that R is diagonally F-split. Then $\mathrm{vol}(P_{-K_X} \cap (v - P_{-K_X})) \geq 1$ by Lemma 5.1.5. Thus $s(R) = 1$ and R is regular. □

Lemma 5.1.4. *Work in Setting 5.0.1 and suppose that R is $\mathbb{Q}$-Gorenstein. Then $P_{-K_X} = \sigma^\vee + v$ where v is a vector satisfying $\langle v, v_\rho \rangle = -1$ for all $\rho \in \sigma(1)$.*

Proof. This can be seen in a few different ways, but here's one. Let r be the Cartier index of R, so that rK_X is Cartier. As $-rK_X = \sum_{\rho \in \sigma(1)} rD_\rho$, we have by [CLS11, Theorem 4.2.8] that there exists w such that $\langle w, v_\rho \rangle = -r$ for all $\rho \in \sigma(1)$. Then we certainly have $\frac{1}{r}w + \sigma^\vee \subseteq P_{-K}$. On the other hand, for any $x \in P_{-K}$ we have $\left\langle x - \frac{1}{r}w, v_\rho \right\rangle > 0$ for all ρ, meaning $\frac{1}{r}w + \sigma^\vee = P_{-K}$. So we set $v = \frac{1}{r}w$. □

Lemma 5.1.5. *Let R be a diagonally split n-dimensional affine toric variety. For all e and all $d \in \frac{1}{p^e}\mathbb{Z}^n$, if $\pi_d \colon F^e_* R \to R$ is in $\mathscr{D}^{(2)}_e(R)$ then $\mathrm{vol}(P_{-K_X} \cap (d - P_{-K_X})) \geq 1$.*

Proof. For all $e' > e$, let $\pi^{e'}_d = \pi_d \cdot (\pi_0)^{e'-e} \in \mathscr{C}^R_{e'}$. This is the map $F^{e'}_* R \to R$ corresponding to the lattice point $d \in \frac{1}{p^{e'}}\mathbb{Z}^n$. The map π_0 is in $\mathscr{D}^{(2)}_1(R)$ by (5.0.8), so we have $\pi^{e'}_d \in \mathscr{D}^{(2)}(R)$ since $\mathscr{D}^{(2)}(R)$ is a Cartier algebra. By Theorem 5.0.4 the polytope $P_{-K_X} \cap (d - P_{-K_X})$ contains at least $p^{e'n}$ fractional lattice points in $\frac{1}{p^{e'}}\mathbb{Z}^n$. Then we're done, as for any polytope $P \subseteq \mathbb{R}^n$ we have

$$\mathrm{vol}(P) = \lim_{m \to \infty} \frac{\#\left\{\frac{1}{m}\mathbb{Z}^n \cap P\right\}}{m^n}$$

This is a well-known fact; see for instance [MS06, Theorem 2.2]. □

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