Math 4400 Homework 0 (Solutions)

Due: Wednesday, May 17, 2017

1. **Linear algebra:** Let A be an $n \times n$-matrix with real entries, for some positive integer n. Suppose also that A is invertible. Show that $\det(A) \neq 0$. (Hint: if B is any other $n \times n$-matrix, then $\det(AB) = \det(A) \cdot \det(B)$.)

   Since A is invertible, it has an inverse, A^{-1}, such that $AA^{-1} = I$. Then $\det(AA^{-1}) = \det(I) = 1$. But $\det(AA^{-1}) = \det(A)\det(A^{-1})$. So we have $\det(A)\det(A^{-1}) = 1$. But this is impossible if $\det(A) = 0$, so we must have $\det(A) \neq 0$

2. Let $i = \sqrt{-1}$. What's i^{2017}? (Hint: start by simplifying $i, i^2, i^3, i^4, i^5, \ldots$ and see if you can find a pattern!)

   Since $i^4 = 1$, and $2017 = 2016 + 1 = 4 \cdot 504 + 1$, we see that

   $$i^{2017} = (i^4)^{504} \cdot i^1 = 1^{504} \cdot i^1 = i$$

3. **Some Set Theory:** Let A and B be two sets. Recall the following notation:

   - "$\in$" is read "in". So "$x \in A$" is read "x is in A" or "x is an element of A".
   - $A \cap B$ denotes the *intersection* of A and B, i.e. the set of things that are in A and also in B.
   - $A \cup B$ denotes the *union* of A and B, i.e. the set of things that are in A or in B (or both).
   - $A \setminus B$ denotes the *relative complement*, or *set difference* of B in A, i.e. the set of things that are in A but *not* in B.

   Now answer the following:

   (a) Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{4, 5, 6, 7, 8, 9, 10\}$. Find $A \cap B$, $A \cup B$, and $A \setminus B$.

   $A \cap B = \{4, 5, 6\}$, $A \cup B = \{1, 2, 3, \cdots, 10\}$, and $A \setminus B = \{1, 2, 3\}$.

   (b) Now let A, B, and C be any sets. Prove the following equality:

   $$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

   Let x be an arbitrary element of $(A \cup B) \cap C$. Then $x \in A \cup B$ and also $x \in C$. This gives us two possibilities: either $x \in A$ or $x \in B$. If $x \in A$, then we have $x \in A$ and also $x \in C$, so that would mean $x \in A \cap C$. Similarly, if $x \in B$, then $x \in B \cap C$. So we've shown that $x \in A \cap C$ or $x \in B \cap C$. In other words, $x \in (A \cap C) \cup (B \cap C)$. Thus, we have shown

   $$(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$$

   To show the other inclusion, let y be an arbitrary element of $(A \cap C) \cup (B \cap C)$. Then either $y \in A \cap C$ or $y \in B \cap C$. In other words, either:

   - y is in A and in C, or
   - y is in B and in C.

   Thus, $y \in A$ or $y \in B$. But no matter what, we have $y \in C$. In other words, $y \in (A \cup B) \cap C$.

4. Let $v \in \mathbb{R}^2$ and $w \in \mathbb{R}^2$ be two vectors. For any vector a, let $||a||$ denote the length of a. Prove the following inequality:

$$||v + w|| \leq ||v|| + ||w||$$

We can write $v = \langle v_1, v_2 \rangle$ and $w = \langle w_1, w_2 \rangle$ for some real numbers $v_1, v_2, w_1, w_2 \in \mathbb{R}$. By definition,

$$\begin{aligned} ||v + w|| &= \sqrt{(v_1 + w_1)^2 + (v_2 + w_2)^2} \\ &= \sqrt{v_1^2 + 2v_1w_1 + w_1^2 + v_2^2 + 2v_2w_2 + w_2^2}, \end{aligned}$$

and

$$||v|| + ||w|| = \sqrt{v_1^2 + v_2^2} + \sqrt{w_1^2 + w_2^2}.$$

Now, we know that $||v + w|| \geq 0$ and $||v|| + ||w|| \geq 0$, and furtner, the function $f(x) = x^2$ is increasing when $x \geq 0$. This means that

$$||v + w|| \leq ||v|| + ||w||$$

if and only if

$$||v + w||^2 \leq (||v|| + ||w||)^2 .$$

Thus, it's enough to show that

$$v_1^2 + 2v_1w_1 + w_1^2 + v_2^2 + 2v_2w_2 + w_2^2 \leq v_1^2 + v_2^2 + 2\sqrt{v_1^2 + v_2^2}\sqrt{w_1^2 + w_2^2} + w_1^2 + w_2^2.$$

By cancelling stuff out from either side, we see that it's enough to prove that

$$v \bullet w \leq ||v|| \cdot ||w||$$

where $v \bullet w$ denotes the dot product. This inequality follows from the formula

$$v \bullet w = ||v|| \cdot ||w|| \cos \theta$$

where θ is the angle between v and w, since $-1 \leq \cos \theta \leq 1$.