Math 4400 Homework 1

Due: Monday, May 22nd, 2017

Feel free to work with your classmates, but everyone must turn in their own assignment. Please make a note of who you worked with on each problem. Also, please give me an estimate of how long this assignment took to complete.

Let me know if you find a typo, or you're stuck on any of the problems.

1. Prove the following statements:

   (a) $\displaystyle\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$, for all integers $n \geq 1$.

Solution: Base case: $n = 1$. Here we check that

$$\sum_{k=1}^{1} \frac{1}{k^2} = 1 \leq 2 - \frac{1}{1} = 1$$

Induction step: suppose that

$$\sum_{k=1}^{m} \frac{1}{k^2} \leq 2 - \frac{1}{m}$$

for some natural number $m \geq 1$. Then

$$\sum_{k=1}^{m+1} \frac{1}{k^2} = \sum_{k=1}^{m} \frac{1}{k^2} + \frac{1}{(m+1)^2} \leq 2 - \frac{1}{m} + \frac{1}{(m+1)^2}$$

so it suffices to show that

$$-\frac{1}{m} + \frac{1}{(m+1)^2} \leq -\frac{1}{m+1}.$$

We multiply each side of this inequality by $m(m+1)^2$, so that we just have to show

$$-(m+1)^2 + m \leq -(m+1)m$$

(note that $m(m+1)^2$ is positive, so we don't have to flip the inequality). Expanding out the left- and right-hand sides above, we've reduced the problem to showing

$$-m^2 - m - 1 \leq -m^2 - m$$

for all $m \geq 1$, which is obviously true.

   (b) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, for all integers $n \geq 1$

Solution: Base case: $n = 1$, in which case we check:

$$\sum_{k=1}^{1} k^2 = 1, \text{ and } \frac{1(1+1)(2 \cdot 1 + 1)}{6} = 1$$

Induction step: let m be a natural number with $m \geq 1$. Suppose that

$$\sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}.$$

Then

$$\sum_{k=1}^{m+1} k^2 = \frac{m(m+1)(2m+1)}{6} + (m+1)^2$$

So we wish to show that

$$\frac{m(m+1)(2m+1)}{6} + (m+1)^2 = \frac{(m+1)(m+2)(2m+3)}{6}$$

Multiplying both sides by $6/(m+1)$ (and noting that $m+1 \neq 0$), we've reduced the problem to proving that

$$m(2m+1) + 6(m+1) = (m+2)(2m+3)$$

which is easy enough to check.

(c) $\displaystyle\prod_{k=1}^{n}\left(1+\frac{1}{k}\right) = n+1$, for all integers $n \geq 1$, where $\displaystyle\prod_{i=1}^{n} a_i = a_1 a_2 \cdots a_n$ denotes the product

Solution: Base case: $n = 1$. In this case, we see $1 + \frac{1}{1} = 1 + 1$, as desired.

Induction step: suppose

$$\prod_{k=1}^{m}\left(1+\frac{1}{k}\right) = m+1$$

for some natural number $m \geq 1$. Then

$$\prod_{k=1}^{m+1} = (m+1) \cdot \left(1 + \frac{1}{m+1}\right) = (m+1)\frac{m+2}{m+1} = m+2,$$

as desired

2. The following is an argument that all cows are the same color. We prove this by induction, by setting $P(n) =$"any collection of n cows all have the same color". Clearly, $P(1)$ is true since every cow is the same color as itself. Now let $k \geq 1$ be a natural number and suppose $P(k)$ is true and let S be a set of $k+1$ cows, numbered $1, 2, \ldots, k+1$. Then cows 1 through k are all the same color, and cows 2 through $k+1$ are all the same color, by the induction hypothesis. But this means all $k+1$ of our cows are the same color, so we've proven $P(k+1)$. By induction, we've shown $P(n)$ is true or all n, and in particular when n is the number of cows on earth. So we've shown that all cows must be the same color.

   Now, a quick google search shows that there are different colors of cows in the world. What's wrong with the argument above?

   Solution: The base case is fine. The problem is with the induction step: it doesn't work if $k = 1$. Indeed, in that case, $S = \{c_1, c_2\}$ for some cows c_1 and c_2. Then $\{c_1\}$ is a set of cows that are all the same color, and $\{c_2\}$ is a set of cows that are all the same color, but there's no reason why c_1 and c_2 should be the same color as each other, since the sets $\{c_1\}$ and $\{c_2\}$ have an empty intersection in this case. (Note that $P(k)$ *does* imply $P(k+1)$ when $k > 1$!)

3. (a) Prove that any finite, non-empty subset of $\mathbb{Z}$ has a minimum.

Solution: **Solution 1:** Let $S \subseteq \mathbb{Z}$ be nonempty. If $S \subseteq N$, then we're done by the well-ordering principle, so we can assume that S contains a negative integer. Let $T = \{x \in S \mid s < 0\}$. Then $T \subseteq S$, so T is finite, and we can define N to be

$$N = \sum_{x \in T} x$$

Then let $T' = \{x - N \mid x \in T\}$. Note that, for all $y \in T$, we have

$$y - N = \sum_{x \in T, x \neq y} (-x) \geq 0,$$

since $x < 0$ for all $x \in T$. Thus $T' \subseteq \mathbb{N}$, so T' has a minimal element by the well-ordering principle. Call this minimal element y_0. Then $y_0 + N \in T$. I claim that $y_0 + N$ is the minimal element of T. To show this, suppose $x \in T$. Then $x - N \in T'$. This means $x - N \geq y_0$ since y_0 is the minimal element of T'. But this means $x \geq y_0 + N$, as desired.

Now let $x \in S$ be arbitrary. If $x < 0$, then $x \in T$, so $y_0 + N \leq x$. If $x \geq 0$, then $y_0 + N < 0 \leq x$ (remember, $y_0 + N \in T$, which is the set of negative elements of S). So $y_0 + N$ is the minimal element of S.

Solution 2 (sketch) Do induction on the size of S. If $|S| = 1$, then it only has one element, and that's the minimal element of S. For the induction step, suppose the proposition is true for sets of size n, and suppose $|S| = n + 1$. Then pick some element $x \in S$ and set $S' = S \setminus \{x\}$. Then S' has a minimal element; call it y. If $x < y$, then x is the minimal element of S. Otherwise, y is the minimal element of S. In either case, S has a minimal element.

(b) Use part (a) to show that any finite, non-empty subset of $\mathbb{Z}$ has a maximum.

Solution: Let $S \subseteq \mathbb{Z}$ be a finite, nonempty subset of $\mathbb{Z}$. Let $T = \{-x \mid x \in S\}$. Then T has a minimal element by part a; call it y. Then $-y \in S$. Now let $x \in S$ be arbitrary. Then $-x \in T$ and $y \leq -x$. But then $-y \geq x$. So $-y$ is the maximal element of S, so S has a maximal element.

(c) Use part (b) to show that if $a, b \in \mathbb{Z}$ and $a \neq 0$, then $\gcd(a, b)$ exists and is unique.

Solution: We wish to show that there is a maximal integer c such that $c|a$ and $c|b$. In other words, we wish to show that the set

$$S = \{c \in \mathbb{Z} \mid c|a, c|b\}$$

has a maximal element. By part b, it's enough to show that S is nonempty and finite. We have $1 \cdot a = a$ and $1 \cdot b = b$, so $1 \in S$ and S is nonempty. Let

$$T = \{c \in \mathbb{Z} \mid c|a\}.$$

Then $S \subseteq T$, so it's enough to show that T is finite. Now, if $c \mid a$, then $c \cdot d = a$ for some $d \in \mathbb{Z}$, and so $|c| \cdot |d| = |a|$. But $a \neq 0$, so $d \neq 0$, so $|d| \geq 1$. But this means $|c| = |a|/|d| \leq |a|$. Thus

$$T \subseteq \{c \in \mathbb{Z} \mid -a \leq c \leq a\}$$

and the set on the right is finite.

To show uniqueness, suppose S has two maximum elements x and x'. Then, by defnition, $x \geq x'$ and $x' \geq x$, so $x = x'$.

4. Compute the following gcd's using the euclidean algorithm:

(a) $\gcd(1084, 412)$

Solution:

$$\begin{aligned} 1084 &= 2 \cdot 412 + 260 \\ 412 &= 1 \cdot 260 + 152 \\ 260 &= 1 \cdot 152 + 108 \\ 152 &= 1 \cdot 108 + 44 \\ 108 &= 2 \cdot 44 + 20 \\ 44 &= 2 \cdot 20 + 4 \\ 20 &= 5 \cdot 4 \end{aligned}$$

So the gcd is 4.

(b) $\gcd(1979, 531)$

Solution:

$$\begin{aligned} 1979 &= 3 \cdot 531 + 386 \\ 531 &= 1 \cdot 386 + 145 \\ 386 &= 2 \cdot 145 + 96 \\ 145 &= 1 \cdot 96 + 49 \\ 96 &= 1 \cdot 49 + 47 \\ 49 &= 1 \cdot 47 + 2 \\ 47 &= 23 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 \end{aligned}$$

So the gcd is 1

(c) $\gcd(305, 185)$

Solution:

$$\begin{aligned} 305 &= 1 \cdot 185 + 120 \\ 185 &= 1 \cdot 120 + 65 \\ 120 &= 1 \cdot 65 + 55 \\ 65 &= 1 \cdot 55 + 10 \\ 55 &= 5 \cdot 10 + 5 \\ 10 &= 2 \cdot 5 \end{aligned}$$

So the gcd is 5.

5. Use your work for the above exercise to compute the continued fractions expansions of the following:

(a) $\frac{1084}{412}$

Solution:

$$\frac{1084}{412} = [2; 1, 1, 1, 2, 4, 1, 4]$$

(b) $\frac{1979}{531}$

Solution:

$$[3; 1, 2, 1, 1, 1, 23, 2]$$

(c) $\frac{305}{185}$

Solution:

$$[1; 1, 1, 1, 5, 2]$$

6. Find the continued fraction expansion of $\sqrt{7}$ and prove it's periodic. (Hint: we learned in class that $\sqrt{7}$ should have a periodic continued fraction. Use a computer or a calculator to guess what it should be, then see if you can prove that's the case by showing $\sqrt{7} - 2$ appears in its own continued fraction expansion, kind of like what we did in class for $\sqrt{2}$)

Solution: By running the continued fraction algorithm for a few iterations, we see that the continued fraction expansion of $\sqrt{7}$ begins $[2; 1, 1, 1, 4, 1, 1, 1, 4, \ldots]$. To prove $\sqrt{7} = [2; \overline{1, 1, 1, 4}]$, it's enough to show that $\sqrt{7} - 2 = [0; \overline{1, 1, 1, 4}]$. For this, it's enough to show that:

$$\sqrt{7} - 2 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \sqrt{7} - 2}}}}$$

We simplify the right-hand side a few times:

$$\begin{aligned}
\cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \sqrt{7} - 2}}}} &= \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2 + \sqrt{7}}{3 + \sqrt{7}}}}} \\
&= \cfrac{1}{1 + \cfrac{3 + \sqrt{7}}{5 + 2 \cdot \sqrt{7}}} \\
&= \frac{5 + 2\sqrt{7}}{8 + 3\sqrt{7}}
\end{aligned}$$

and it's easy enough to check that

$$\sqrt{7} - 2 = \frac{5 + 2\sqrt{7}}{8 + 3\sqrt{7}}$$