Singularities in characteristic 0 and characteristic p

Daniel Smolkin

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These notes are based heavily on [2], and to a lesser extent on [5] and [6]. In these notes, we'll discuss how to measure singularities in characterstic 0 and then discuss how to measure them in characteristic p. Finally, we'll state some theorems showing these methods are compatible with each other.

1 Characteristic 0

We start with a basic question: given a polynomial $f \in \mathbb{C}[z_1, \cdots, z_n]$ with a singularity at 0, how can we measure the "singularness" of this polynomial in a precise way? In other words: we can look at various singularities and see intuitively that some singularities are worse than others. For instance, it feels like a transverse self-intersection is probably not as bad as a cusp. And a sharper cusp feels more singular than a rather gradual cusp:

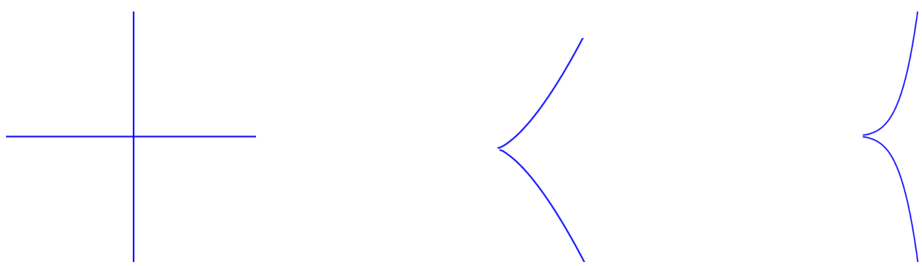


Figure 1: Graphs of the equations "$xy = 0$," "$y^2 - x^3 = 0$," and "$y^2 - x^{11} = 0$." We see that the singularities get worse from left to right. These graphs were drawn using Geogebra (`https://www.geogebra.org`)

The question becomes finding an invariant for different singularities that allows us to say that the singularity of $f = xy$ (a simple normal crossing) is more mild than that of $f = y^2 - x^3$ (a cusp).

The most naïve approach for measuring singularities is to use what's called the *multiplicity* of the polynomial f. By definition, f is singular at 0 if all of its first-order partial derivatives vanish:

$$f(0) = \frac{\partial f}{\partial z_1}(0) = \cdots = \frac{\partial f}{\partial z_n}(0) = 0.$$

We say that f has multiplicity d at 0 if all of the $(d-1)^{\text{st}}$ order partial derivatives vanish. In other words,

$$\text{Mult}_0(f) = \min \left\{ d \;\middle|\; \frac{\partial^{i_1} \cdots \partial^{i_n} f}{\partial z_1^{i_1} \cdots \partial z_n^{i_n}}(0) \neq 0 \right\}$$

This invariant is too coarse, however: the multiplicities $f = xy$ and $f = y^2 - x^3$ are both 2.

1.1 Analytic approach

The analytic approach is to examine the the integrability of $1/|f|$ around 0. If f approaches zero at the order of $x^{1/2}$, for instance, the function $1/|f|$ is integrable. But if f approaches 0 more quickly, say to the order of x, then $1/|f|$ won't be integrable. This motivates the following definition:

$$\mathrm{lct}_0(f) = \sup \left\{ \lambda \;\middle|\; \int_{B_\varepsilon(0)} \frac{1}{|f|^{2\lambda}} < \infty \text{ for } \varepsilon \text{ sufficiently small} \right\}$$

The abbreviation "lct" stands for "log canonical threshold"; this terminology comes from connections with birational geometry that we'll explore later. The nicer our singularity, the longer $1/|f|^{2\lambda}$ will be integrable, so *nicer singularities should have larger log canonical thresholds.*

The log canonical threshold is easily computed for monomials.

Lemma 1.1. *Let $f = z_1^{a_1} \cdots z_n^{a_n}$. Then* $\mathrm{lct}_0(f) = \min \left\{ \frac{1}{a_i} \right\}$.

Proof. This is easy to see by changing to polar coordinates: using $|z_i|^2 = r^2$, we see

$$\int \frac{1}{|f|^{2\lambda}} dz = \int \frac{r_1 \cdots r_n}{(r_1^{a_1} \dots r_n^{a_n})^{2\lambda}} dr \wedge d\theta$$

It follows from Fubini's theorem (and calc 2) that this integral is finite if and only if $1 - 2a_i\lambda > -1$ for all i; in other words, this integral is finite if and only if $\lambda < \min 1/a_i$. □

1.2 Algebro-geometric approach

Hironaka's theorem tells us that, for any polynomial $f \in \mathbb{C}[z_1, \cdots, z_n]$, we can reduce the computation of $\mathrm{lct}(f)$ to the case where f is a monomial:

Theorem 1.2 (Hironaka)**.** *Let $f \in \mathbb{C}[z_1, \cdots, z_n]$ be arbitrary. Then there exists a smooth variety X and a proper birational map $\pi : X \to \mathbb{C}^n$ such that $f \circ \pi$ and $\mathrm{Jac}_{\mathbb{C}}(\pi)$ are monomials locally analytically.*

Such a variety X is called a *log resolution* of the pair $(\mathbb{C}^n, f)$. Now, we know that for any f, we have

$$\int_{B_\varepsilon} \frac{1}{|f|^{2\lambda}} = \int_{\pi^{-1}(B_\varepsilon)} \frac{\mathrm{Jac}_{\mathbb{R}}(\pi)}{|f \circ \pi|^{2\lambda}}.$$

This is just the change of coordinates formula from differential topology. Further, since π is proper, the closure of $\pi^{-1}(B_\varepsilon)$ is compact, so we can check the convergence of the integral on a neighborhood of the vanishing locus of $f \circ \pi$. Now, suppose $\mathrm{Jac}_{\mathbb{C}}(\pi) = z_1^{k_1} \cdots z_m^{k_m}$ and $f \circ \pi = z_1^{a_1} \cdots z_m^{a_m}$. Since $\mathrm{Jac}_{\mathbb{R}} = |\mathrm{Jac}_{\mathbb{C}}|^2$, this integral becomes

$$\int_{\pi^{-1}(B_\varepsilon)} \frac{\mathrm{Jac}_{\mathbb{R}}(\pi)}{|f \circ \pi|^{2\lambda}} = \int_{pi^{-1}(B_\varepsilon)} \frac{\left| z_1^{k_1} \cdots z_m^{k_m} \right|^2}{|z_1^{a_1} \cdots z_m^{a_m}|^{2\lambda}}$$

so we see that this integral converges exactly when $2\lambda a_i - 2k_i > 1$. Thus we get a formula

$$\mathrm{lct}_0(f) = \min_i \frac{k_i + 1}{a_i}$$

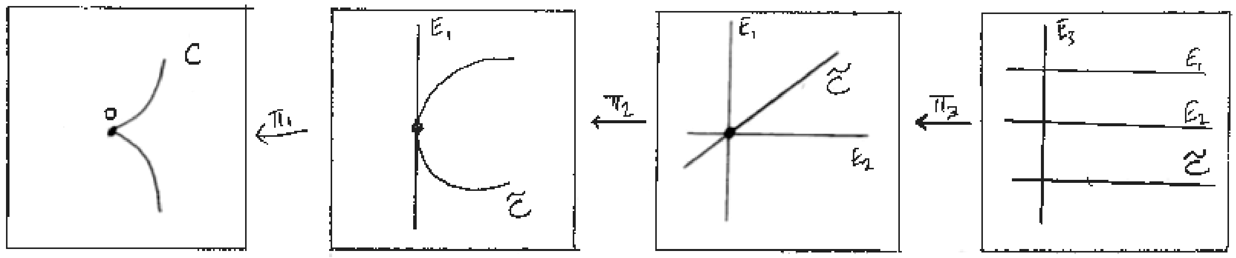
Thus we can compute log canonical thresholds by computing the log resolutions guaranteed by Hironaka's theorem.

For those in the know, the divisor of $\mathrm{Jac}_{\mathbb{C}} \pi$ is K_π, the relative canonical divisor of X over $\mathbb{C}^n$. So it's not hard to see that

$$\mathrm{lct}(f) = \sup \{ \lambda \mid \lceil K_\pi - \lambda \pi^* \operatorname{div}(f) \rceil \geq 0 \}$$

Example: Let's compute the lct of $f = x^2 - y^3$. The first step is to find a log resolution of f, which we can do by successively blowing-up the origin.

In the above figure, each π_i is a blow-up at the origin, and $\tilde{C}$ is the strict transform of the curve C. We have a log resolution precisely when the divisors K_π and $\pi^* \operatorname{div} f$ have simple-normal crossings, i.e., no more than two components intersect in one ponit, and all intersections are transverse. So we see that one blow-up is insufficient because the intersection of $\tilde{C}$ and E_1 is not transverse. A pair of blow-ups is also insufficient,



since then we have three components intersecting at one point. So we see that we need three blow-ups to get a monomialization.

By Hartshorne exercise II.8.5, we know that $K_{\pi_i} = E_i$, where E_i is the exceptional divisor of π_i. Further, it's easy to see that $K_{a \circ b} = K_b + b^* K_a$ for any two maps a, b. By repeatedly applying these rules, we find that

$$K_{\pi_1 \circ \pi_2 \circ \pi_3} = E_1 + 2E_2 + 4E_3,$$

whereas

$$(\pi_1 \circ \pi_2 \circ \pi_3)^* \operatorname{div} f = \tilde{C} + 2E_1 + 3E_2 + 6E_3$$

Then $\operatorname{lct}(f)$ is the minimum of the numbers $\frac{k_i+1}{a_i}$ as (k_i, a_i) ranges among the pairs $(0,1), (1,2), (2,3), (4,6)$. So we see that $\operatorname{lct}(f) = \frac{5}{6}$. □

Remark: Similar computations show that $\operatorname{lct}(xy) = 1$ and $\operatorname{lct}(y^2 - x^{11}) = \frac{13}{22}$, so we've succeeded in finding an invariant that distinguishes these three cases. And, as we remarked earlier, the nicer singularities have larger log canonical thresholds.

We get a richer invariant by considering multiplier ideals $\mathscr{J}(f^\lambda) := \pi_* \mathscr{O}_X (\lceil K_\pi - \lambda F_\pi \rceil)$. We recover the lct from these ideals by noticing

$$\operatorname{lct}(f) = \sup \left\{ \lambda \mid \mathscr{J}\left(f^\lambda\right) = \mathbb{C}[x_1, \cdots, x_n] \right\}.$$

2 Characteristic p

Now we have a polynomial f in $\mathbb{F}_p[x_1, \cdots, x_n]$ with a singularity at the origin and we wish to find a way of measuring this singularity. If we try to adapt the techniques used in the characteristic 0 setting, we immediately get stuck—there's no good way to integrate in characteristic p, and don't have resolution of singularities anymore (at least, not that we know!). The answer lies in the Frobenius endomorphism. We start with some

2.1 Preliminaries.

Let k be a field of characteristic p and let R be an integral domain over k. Let $F : R \to R$ denote the *Frobenius endomorphism*; that is $F(x) = x^p$ for all $x \in R$. After fixing an algebraic closure $\bar{R}$ of $\operatorname{frac} R$, and we can define $R^{1/p^e} := \left\{ x^{1/p^e} \mid x \in R \right\}$ for all $e \in \mathbb{Z}_{\geq 0}$. Here, x^{1/p^e} is the unique (p^e)th root of x in $\bar{R}$ (exercise: check this really is unique!). Note that we have obvious inclusions $R \subseteq R^{1/p} \subseteq R^{1/p^2} \subseteq \cdots$, and also that we have a natural isomorphism $R \cong R^{1/p^e}$ as *rings* (but not as R-modules!).

The R-module structure of R^{1/p^e} for various e is perhaps the main object of study in characteristic-p algebraic geometry. We have the following definition:

Definition 2.1. *A ring R is said to be F-finite if $R^{1/p}$ is a finitely-generated R-module. Equivalently, R is said to be F-finite if R^{1/p^e} is a finitely-generated R-module for all $e \geq 0$.*

Aside. We define $F^e_* R$ to be the R-module that's equal to R as a set, but with the R-module structure defined by $r \bullet x = r^{p^e} x$. In other words, $F^e_* R$ has the module structure that we get by restricting scalars along the map $R \xrightarrow{F^e} R$. The notation "F^e_*" comes from the language of $\mathscr{O}_X$-modules. Then $F^e_* R$ and R^{1/p^e} are canonically isomorphic as R-modules. This gives us another way to think about R^{1/p^e}. We get a natural inclusion $R \to F^e_* R$ by sending x to x^{p^e}.

Yet another paradigm arises by noticing that the R-module structure of R^{1/p^e} is in some sense equivalent to the natural R^{p^e}-module structure of R. Thus, many authors will refer to the three morphisms,

$$R \to R^{1/p^e}$$
$$R \to F^e_* R$$
$$R^{p^e} \to R$$

interchangeably.

Most rings one encounters in the wild are F-finite. Indeed, for any F-finite field k, all rings essentially of finite type over k are F-finite.

Example. If $R = \mathbb{F}_p[x_1, \cdots, x_n]$, then $R^{1/p^e} = \mathbb{F}_p[x_1^{1/p^e}, \cdots, x_n^{1/p^e}]$. This is a free R-module with basis

$$\left\{ x_1^{a_1/p^e} \cdots x_n^{a_n/p^e} \;\middle|\; 0 \leq a_i < p^e \right\}.$$

This shows that polynomial rings over finite fields are F-finite. More generally, for any ring S and any ideal $I \subseteq S[x_1, \cdots, x_n]$, we have

$$\left(\frac{S[x_1, \cdots, x_n]}{I} \right)^{1/p^e} = \frac{S^{1/p^e}\left[x_1^{1/p^e}, \cdots, x_n^{1/p^e}\right]}{I^{1/p^e}} \tag{1}$$

where I^{1/p^e} is defined analogously to R^{1/p^e} above, and for any multiplicative set W, we have

$$\left(W^{-1}S\right)^{1/p^e} = W^{-1}\left(S^{1/p^e}\right). \tag{2}$$

Thus, rings of essentially finite type over F-finite rings are F-finite.

2.2 F-pure thresholds

When we were working over the complex numbers, we were able to measure the singularities of polynomials f by finding the largest number $c > 0$ such that

$$\int_{B_\varepsilon(0)} \frac{1}{|f|^c} < \infty$$

for $\varepsilon < 0$ sufficiently small. Now, we don't have a good theory of integration in characteristic p. However, we observe that the above integral is certainly finite if f^c is nonzero at 0. This leads us to the following, very naïve definition: given $f \in \mathbb{F}_p[x_1, \cdots, x_n]$, we set

$$\operatorname{fpt}(f) = \{\sup c \in \mathbb{R} \mid f^c \notin \mathfrak{m}\}.$$

where $\mathfrak{m}$ is the maximal ideal of the origin. The astute reader will notice that this definition is not just naïve, but nonsensical: what does it even mean to raise f to some non-integer power? The machinery we developed in the previous section gets us close enough: we can define, for all $a, e \in \mathbb{Z}$:

$$f^{a/p^e} := (f^a)^{1/p^e} = \left(f^{1/p^e}\right)^a$$

Definition 2.2. *Let $(R, \mathfrak{m})$ be a local domain with characteristic $p > 0$ and let $f \in R$. Then the* F-pure threshold *of f is given by*

$$fpt(f) := \sup\left\{ c = \frac{a}{p^e} \;\middle|\; a, e \in \mathbb{Z},\ f^c \notin \mathfrak{m} \cdot R^{1/p^e} \right\}.$$

The ideal $\mathfrak{m} \cdot R^{1/p^e}$ is not to be confused with $\mathfrak{m}^{1/p^e}$; the former ideal is generally much smaller than the latter. The remarkable thing is that this naïve definition ends up being a good notion of the singularities of f, and is closely related to the characteristic-0 notions.

Example. Let $f = x^2 - y^3 \in \mathbb{F}_p[x, y]$. Then the F-pure threshold of f depends on the prime p in the following way:

$$\operatorname{fpt}(f) = \begin{cases} \frac{1}{2}, & p = 2 \\ \frac{2}{3}, & p = 3 \\ \frac{5}{6}, & p \equiv 1 \bmod 6 \\ \frac{5}{6} - \frac{1}{6p}, & p \equiv 5 \bmod 6 \end{cases}$$

Let's prove the case where $p \equiv 1 \bmod 6$ to get a flavor for how these calculations work. First, we expand the product $(x^2 - y^3)^a$ to get

$$(x^2 - y^3)^a = \sum_{i=0}^{a} \binom{a}{i} (-1)^{(a-i)} x^{2i} y^{3(a-i)}$$

Then we take p^e-th roots on each side. Since Frobenius is a ring endomorphism, we can just do this term-by-term. Since $\mathbb{F}_p$ is a perfect field, this doesn't affect the binomial coefficients:

$$(x^2 - y^3)^{a/p^e} = \sum_{i=0}^{a} \binom{a}{i} (-1)^{(a-i)} x^{2i/p^e} y^{3(a-i)/p^e}$$

Now, each of the terms above has a different bi-degree, so terms can only cancel out if one of those binomial coefficients vanish. So we see that $(x^2 - y^3)^{a/p^e}$ is not in $\mathfrak{m}$ precisely when there exists some i such that:

$$2i < p^e, \quad 3(a - i) < p^e, \quad \binom{a}{i} \not\equiv 0 \bmod p$$

(Otherwise, we could factor out either an x or a y from each of the terms.) This leads to a few bounds on the F-pure threshold of f: if $a/p^e < 1/2$, then we can satisfy the inequalities above by setting $i = a$. Thus, $\operatorname{fpt}(f) \geq 1/2$. On the other hand, $\operatorname{fpt}(f) \leq 5/6$, since if we have any pair of numbers (a, i) satisfying the inequalities above, then we can show:

$$a \leq \frac{p^e - 1}{3} + i \leq \frac{p^e - 1}{3} + \frac{p^e - 1}{2} = \frac{5p^e}{6} - \frac{5}{6}$$

and so

$$\frac{a}{p^e} \leq \frac{5}{6} - \frac{5}{6p^e}.$$

To prove the case when $p \equiv 1 \bmod 6$, we set $a = \frac{5p^e - 5}{6}$ (which is an integer), and $i = \frac{p^e - 1}{2}$. We see that

$$2i = p^e - 1 < p^2$$

and

$$3(a - i) = 3\left(\frac{5p^e - 5}{6} - \frac{3p^e - 3}{6}\right) = \frac{1}{2}(2p^e - 2) < p^e.$$

Now we just need to show $\binom{a}{i} \not\equiv 0 \bmod p$. By Lucas' theorem [1], it suffices to show that each digit in the base-p expansion of a is greater than or equal to the corresponding digit in the base-p expansion of i. But each digit in the base-p expansion of a is $\frac{5}{6}(p-1)$, whereas each digit of the base-p expansion of i is $\frac{1}{2}(p-1)$. This shows that $\operatorname{fpt}(x^2 - y^3) \geq \frac{a}{p^e} = \frac{5}{6} - \frac{5}{6p^e}$ for all e, which finishes the proof.

The calculation for the case $p \equiv 5 \bmod 6$ is more involved, so I'll defer to Daniel Hernandez's thesis for that calculation [3]. More generally, Daniel's thesis is the best reference I know for learning more about the F-pure threshold. □

3 Singularities intrinsically

Previously we considered the singularities of hypersurfaces in $\mathbb{F}_p[x_1, \cdots, x_n]$. Now we ask: how do we measure the singularities of an affine variety in an intrinsic way? The first step in this direction was established by Kunz in the '60s:

Theorem 3.1. *Let R be a local ring of characteristic $p > 0$. Then R is regular if and only if R^{1/p^e} is free as an R-module.*

This suggests that we can measure how singular a ring is by measuring how far R^{1/p^e} is from being a free R-module. One way of doing this is by counting the number of direct summands of R in R^{1/p^e}. Note that if R is regular, then the Cohen structure theorem, along with the same argument as in equation (1), gives us that $R^{1/p^e} \cong R^{p^{ed}}$, where $d = \dim R$. This gives us an upper bound: the number of summands of R in R^{1/p^e} is bounded above by p^{ed}. Thus, a ring R is closer to being regular if this number is close to p^{ed}. We'll denote the number of summands of R in R^{1/p^e} by a_e.

The least we can ask for is to have $a_e \geq 1$:

Definition 3.2. *R is called F-split if there exists some e such that R is a direct summand of R^{1/p^e}. In other words, the inclusion $R \hookrightarrow R^{1/p^e}$ splits.*

Exercise: If R is F-split, then the inclusion $R \hookrightarrow R^{1/p^e}$ splits for *all* e.

We can also study the asymptotic nature of a_e as e goes to infinity. This leads to another invariant called the *F-signature* of R:

$$s(R) = \lim_{e\to\infty} \frac{a_e}{p^{ed}}.$$

Of course, we must first ask whether this limit exists; this was proven by Kevin Tucker [7]. Given that, it's clear that $s(R) \geq 0$, and our work above shows that $s(R) \leq 1$. We also know that the F-signature of regular rings is 1. It turns out that the converse is true as well [4].

If the numbers a_e grow quickly enough, we call the ring R *strongly F-regular*

Definition 3.3. *R is called* strongly F-regular *if $s(R) > 0$.*

It should be noted that this definition of strong F-regularity is ahistorical: the notion of strong F-regularity first arose in the work of Hochster and Huneke on tight closure, long before the idea of F-signature was introduced.

The idea of F-signature is that Since strongly F-regular rings are supposed to be quite close to regular rings, we expect them to have some nice properties. Indeed, we have the following theorem:

Theorem 3.4 ([6], Theorem 1.18)**.** *Strongly F-regular rings are Cohen-Macaulay and normal.*

4 Relationship with characteristic 0

Let $f = x^2 - y^3 \in \mathbb{C}[x, y]$ and let $f_p = x^2 - y^3 \in \mathbb{F}_p[x, y]$ be the "mod p reduction" of f. We've already seen that $\mathrm{lct}(f) = 5/6$ and $\mathrm{fpt}(f_p)$ is given by:

$$\mathrm{fpt}(f) = \begin{cases} \frac{1}{2}, & p = 2 \\ \frac{2}{3}, & p = 3 \\ \frac{5}{6}, & p \equiv 1 \bmod 6 \\ \frac{5}{6} - \frac{1}{6p}, & p \equiv 5 \bmod 6 \end{cases}$$

Notice that we have the following relationship in this special case:

$$\lim_{p\to\infty} \mathrm{fpt}(f) = \mathrm{lct}(f)$$

This is not an accident. In general, we have a theorem saying the same thing happens for all f, suggesting our definition of the F-pure threshold is the correct one:

Theorem 4.1 ([2], Theorem 3.18)**.** *Fix* $f \in \mathbb{Z}[x_1, \cdots, x_n]$*. Then*

- $\text{fpt}(f_p) \leq \text{lct}(f)$ *for all* $p \gg 0$ *prime, and*
- $\lim_{p\to\infty} \text{fpt}(f_p) = \text{lct}(f)$

So we see that one can compute the lct of a polynomial by reducing mod p and computing the F-pure threshold. This is especially surprising if you consider that the lct has something to do with the integrability of f at the origin. As surprising as this is, it reflects a general theme in the theory of singularities in characteristic p, or F-singularities. For instance, we have a similar theorem relating strongly F-regular singularities to klt singularities, which are important to birational geometers:

Theorem 4.2. *Let R be a Gorenstein domain finitely generated over a field of characteristic 0. Then R has F-regular type if and only if R has klt singularities*

Here, "F-regular type" means, roughly, that there's a dense set of primes, modulo which you get a strongly F-regular ring. More precisely, for any finitely generated $\mathbb{C}$-alegbra R, given by

$$R \cong \mathbb{C}[x_1, \cdots, x_n]/(f_1, \cdots, f_m)$$

we let A be the $\mathbb{Z}$-algebra generated by all the coefficients of all the polynomials f_i. Then we can construct the A-algebra corresponding to R,

$$R_A := A[x_1, \cdots, x_n]/(f_1, \cdots, f_m).$$

Then R is said to have F-XXX type, where "XXX" is your favorite adjective, if there's a dense set of maximal ideals $\mu \in \operatorname{Spec} A$ such that the fiber of R_A over μ is F-XXX (note that these fibers are always rings of positive characteristic).

Similarly, we have

Theorem 4.3. *Let R be a Gorenstein domain finitely generated over a field of characteristic 0. If R has F-split type, then R has log canonical singularities. If R has F-injective type, then R has Du Bois singularities.*

The converses to the above statements are conjectured. See section 1.5 of [6] for details.

References

[1] Lucas's theorem. `https://en.wikipedia.org/wiki/Lucas%27s_theorem`. Accessed: 2016-12-08.

[2] Angélica Benito, Eleonore Faber, and Karen E. Smith. Measuring singularities with frobenius: The basics. 2013. arXiv:1309.4814.

[3] Daniel Jesús Hernández. *F-purity of hypersurfaces.* PhD thesis, University of Michigan, 2011.

[4] Craig Huneke and Graham J. Leuschke. Two theorems about maximal Cohen-Macaulay modules. *Math. Ann.*, 324(2):391–404, 2002.

[5] Karl Schwede and Kevin Tucker. A survey of test ideals. 2011. arXiv:1104.2000.

[6] Karen E. Smith and Wenliang Zhang. Frobenius splitting in commutative algebra. 2014. arXiv:1409.1169.

[7] Kevin Tucker. *F*-signature exists. *Inventiones mathematicae*, 190(3):743–765, 2012.