Birational classification of algebraic varieties

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Premise

- Humans love to categorize things

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- E.g. how many kinds of animals are there?



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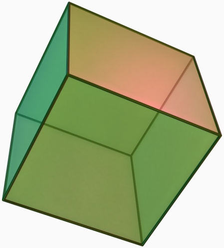

Cube

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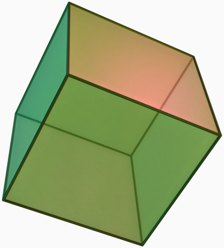

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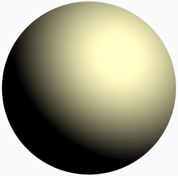

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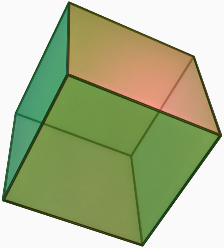

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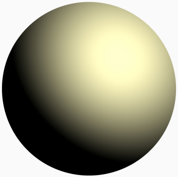

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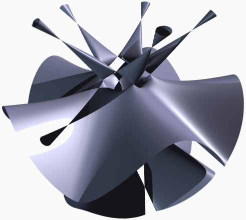
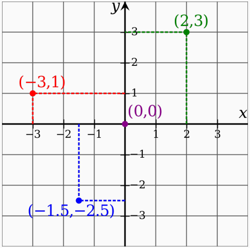


???

Polynomials

- 1630: Descartes and Fermat come up with the idea of coordinates and graphing



- This lets us talk about polynomials

Polynomials

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$$x^2 + 3y^2 - 1$$

$$w^{10} - 2y^2x^3z^5 + 1$$

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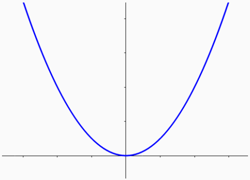
- This also allows us to work with 4-dimensional shapes and beyond
- Through abstraction, we can consider more general situations

Shapes from polynomials



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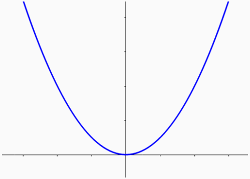

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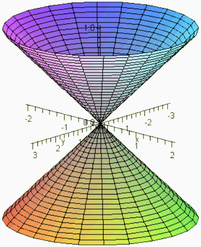

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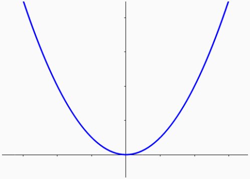

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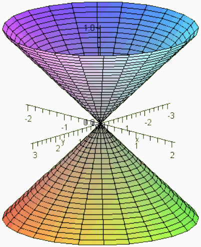

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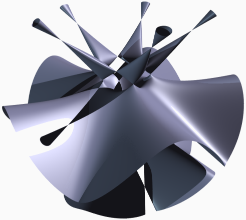

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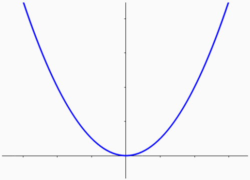

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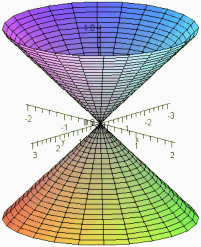

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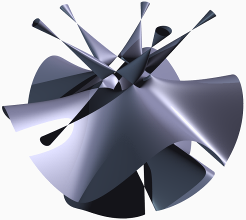

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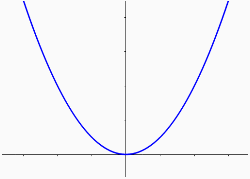

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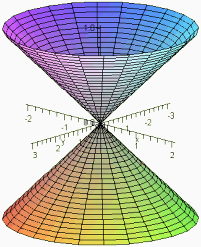

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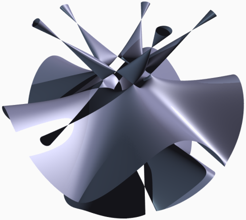

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- Algebraic variety: A shape you get by graphing a polynomial (or several)
- Algebraic geometry: The study of these shapes
- Natural question: what kinds of algebraic varieties are there?

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- If our definition of species is too narrow or too broad then our classification becomes boring
- The definition of “species” algebraic geometers use today was devised by Riemann (1851)

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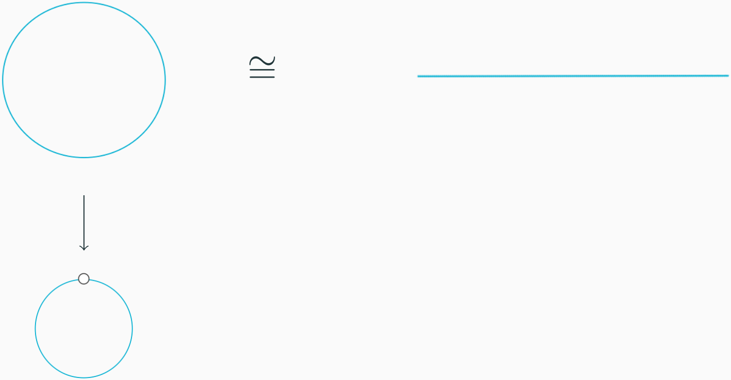

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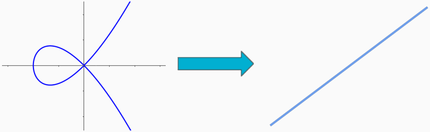

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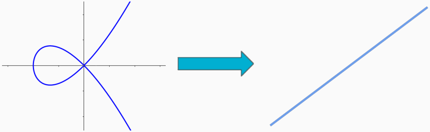
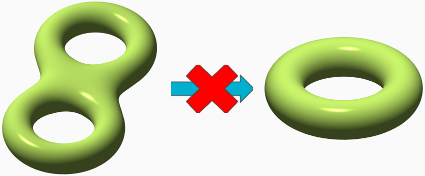
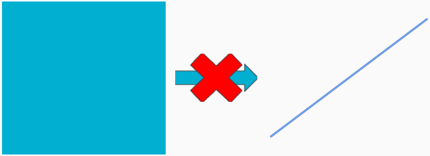


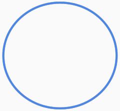
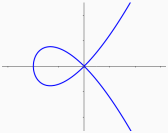
Not birationally equivalent:





Birational equivalence

Birational classification of algebraic varieties: listing all the different species ("birational equivalence classes") of algebraic varieties you can get

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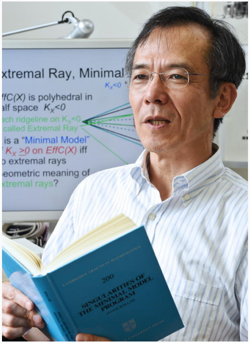

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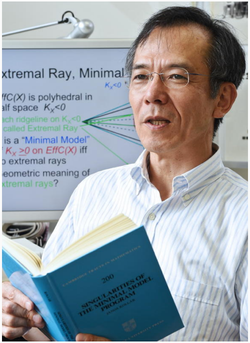
- Idea (Mori, 1985): perhaps each species of algebraic variety has a particularly simple member, called a minimal model
- We can check if two species are the same by comparing their minimal models



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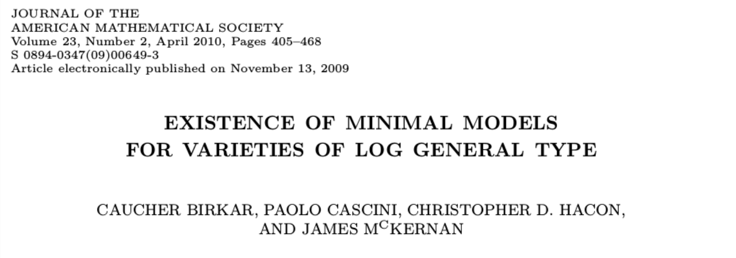

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- 4D+: an open problem!

Thanks for listening!