Marden's theorem

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GSAC Colloquium

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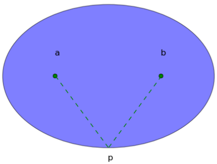
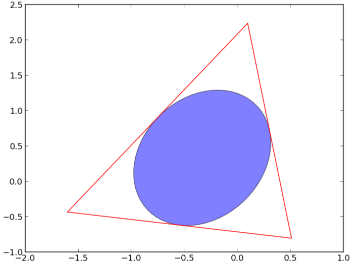
Statement of the theorem

Theorem 1

Let p be a degree-3 polynomial over $\mathbb{C}$. Suppose the roots of p form a triangle in the complex plane. Then the roots of p' are the foci of the steiner inellipse of this triangle.

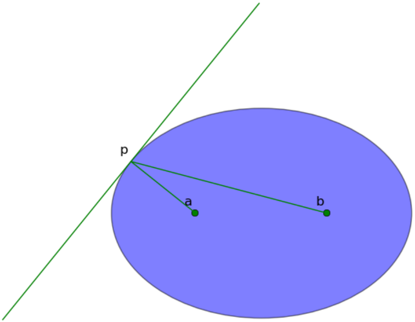
- Ellipse: $\{p : d(p,a) + d(p,b) = r\}$ for some a, b called *foci* and some r called the *major axis length*
- Steiner inellipse: the unique ellipse tangent to the three sides of a triangle at their midpoints





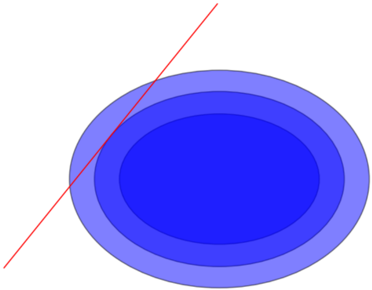
Ellipse properties

Optical property



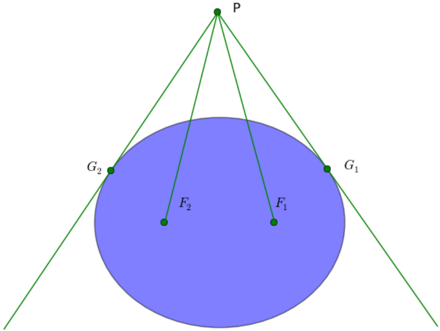
Ellipse properties

Uniqueness property: given a pair of points and a line, there is at most one ellipse with foci at those points tangent to that line



Ellipse properties

$\angle F_1 P G_1 = \angle F_2 P G_2$



Outline (following Kalman)

- Let T be the triangle defined by the roots of p and let E be an ellipse with foci at the roots of p'. If E intersects a side of T at its midpoint, then. . .

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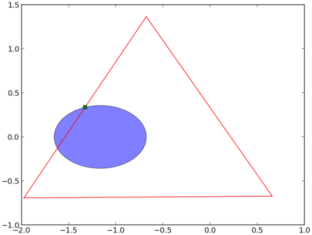
- E is tangent to every side at its midpoint

Outline in pictures

1. 

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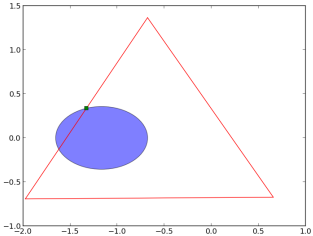
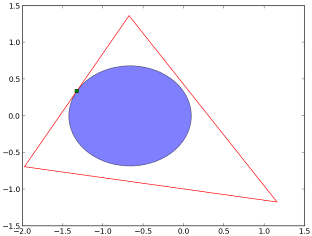
1. 

2. 

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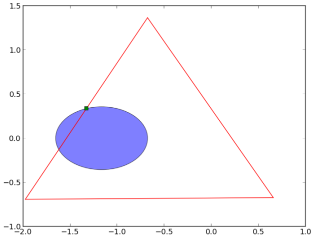
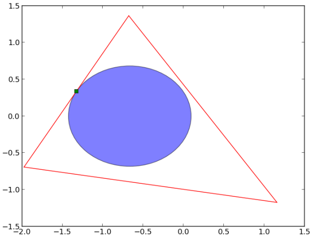
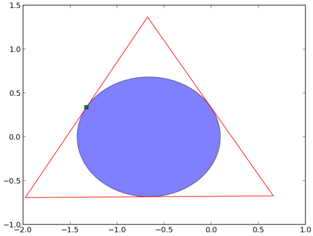
1. 

2. 

3. 

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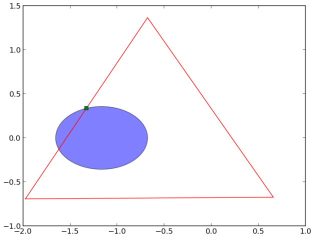

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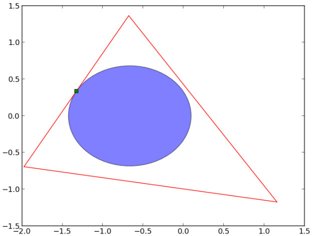

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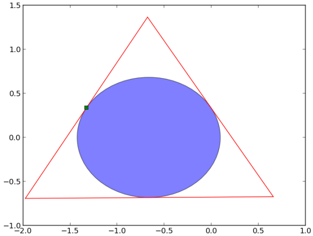

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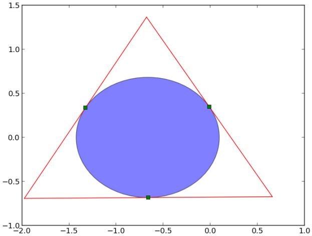

4.

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Let T be the triangle defined by the roots of p and let E be an ellipse with foci at the roots of p'. If E intersects a side of T at its midpoint, then E is tangent to that side.

- Thus, the *unique* ellipse that is tangent to that side and has foci at the roots of p' is tangent to that side at its midpoint (what we really need).

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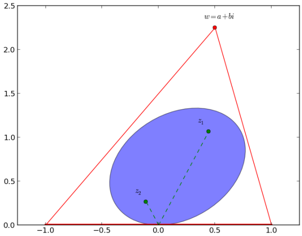

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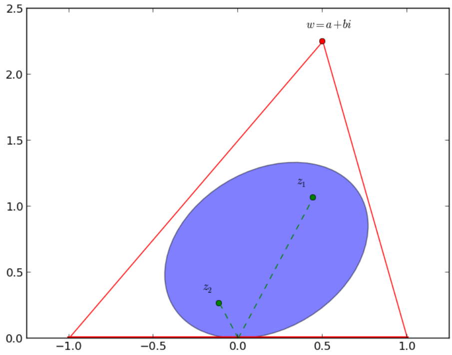
- $\Rightarrow \; p(z) = z^3 - wz^2 - z, \quad p'(z) = 3z^2 - 2wz - 1$
- Note:

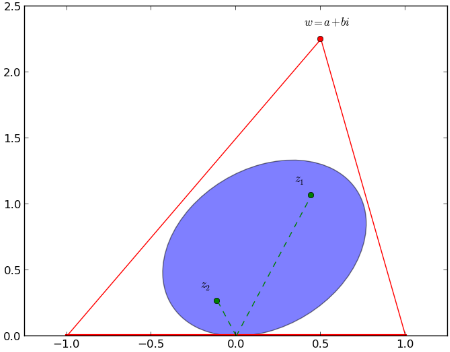
$$\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) + \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) = -\frac{b}{a},$$

$$\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) \cdot \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) = \frac{c}{a}$$

- $z_1 z_2 = -1/3 \Rightarrow \operatorname{Arg} z_1 + \operatorname{Arg} z_2 = \pi \pmod{2\pi\mathbb{Z}}$



- $z_1 z_2 = -1/3 \Rightarrow \text{Arg}\, z_1 + \text{Arg}\, z_2 = \pi \pmod{2\pi\mathbb{Z}}$
- $z_1 + z_2 = 2w/3 \Rightarrow \text{Im}\, z_1 > 0$ or $\text{Im}\, z_2 > 0$



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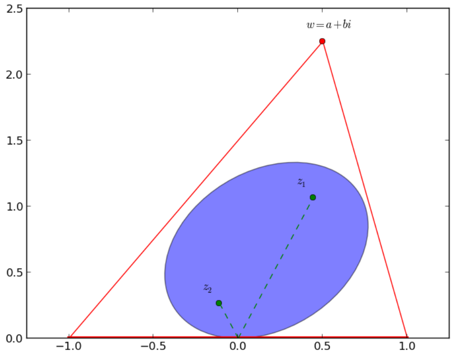

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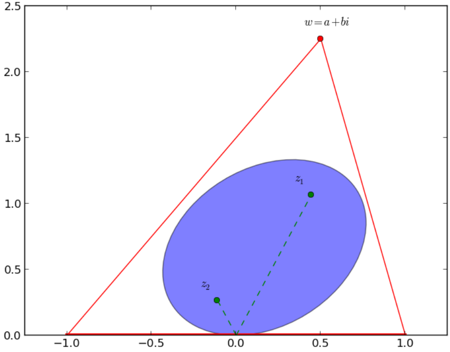
- So $0 < \operatorname{Arg} z_1, \operatorname{Arg} z_2 < \pi$ and $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 = \pi$
- By the optical property of ellipses, x-axis is tangent to our ellipse □



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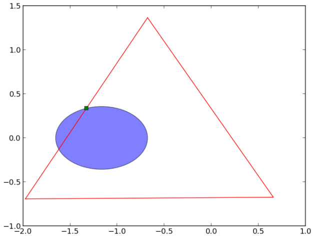
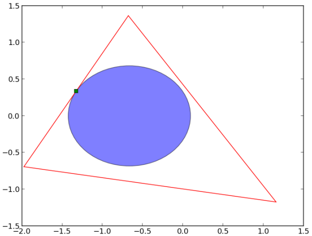
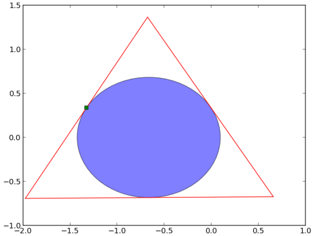
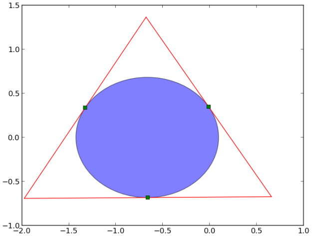
1. 

2. 

3. 

4. 

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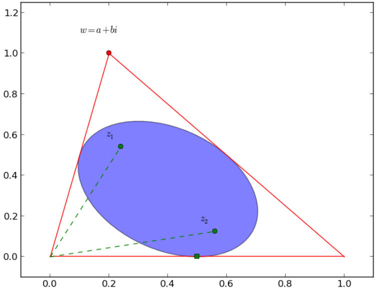

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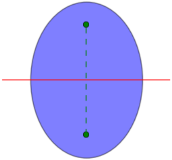
- $z_1 + z_2 = \frac{2}{3}(1+w)$, so one focus is above x-axis.
- Since ellipse tangent to x-axis, both foci on one side



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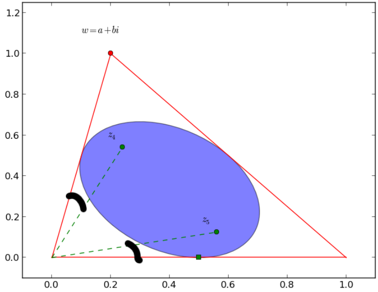
- $z_1 z_2 = w/3$



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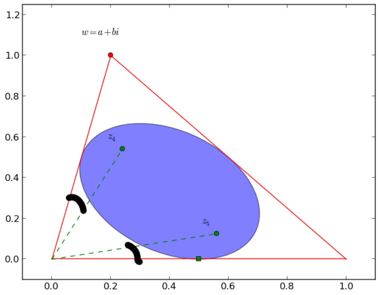
- $z_1 z_2 = w/3 \Rightarrow \operatorname{Arg} z_1 + \operatorname{Arg} z_2 = \operatorname{Arg} w$.



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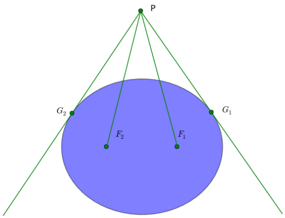
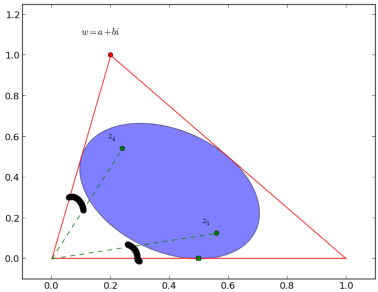
- $z_1 z_2 = w/3 \Rightarrow \operatorname{Arg} z_1 + \operatorname{Arg} z_2 = \operatorname{Arg} w$.
- The line between 0 and w is tangent to the ellipse by third ellipse property. □

$\angle F_1 P G_1 = \angle F_2 P G_2$





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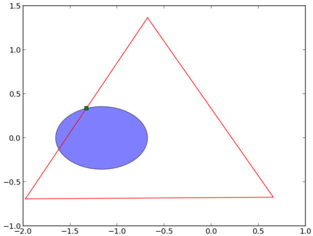

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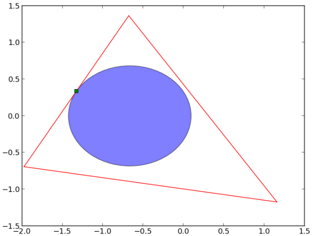

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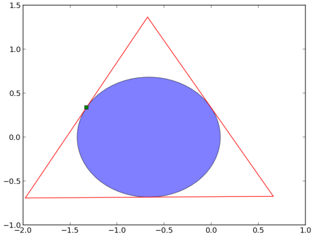

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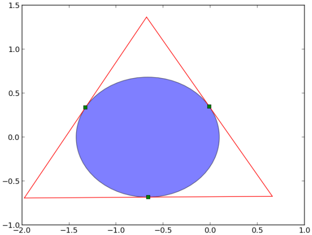

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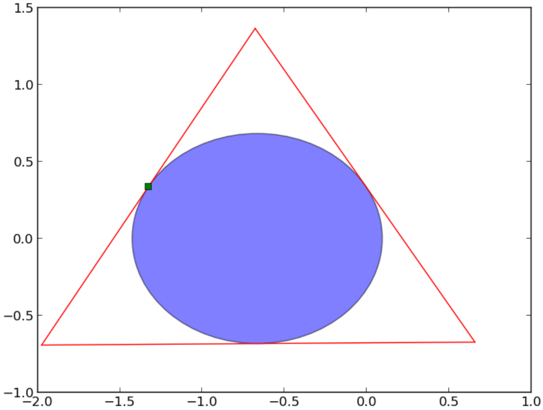

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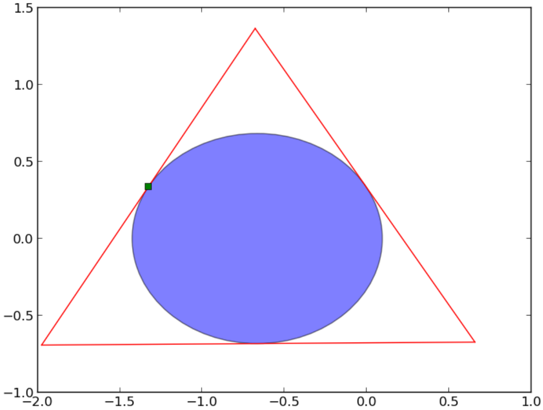
- By step 1, there is some E' with same foci tangent to another side at its midpoint
- By uniqueness property, $E = E'$ $\square$.



Empirical evidence

References

- Kalman, Dan. "An Elementary Proof of Marden's Theorem". The American Mathematical Monthly, vol. 115, no. 4, April 2008, pp. 330338.
- My website (slides and python script)
  `math.utah.edu/~smolkin/talks`