Subadditivity and Symbolic Powers

JMM 2018: AMS Special Session on Commutative Algebra in All Characteristics

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- Answer [ELS '01, HH '02, Hara '05, MS '17]: If R is regular, then $h = \dim R$ works for all $\mathfrak{p}$!
- Question: can we find a uniform h that works for non-regular rings?

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Sketch of ELS/Hara/MS proof: set $d = \dim R$.

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Key idea: replace $\tau\left(\mathfrak{p}^{(dn)}\right)$ with an ideal so that (2) holds always, and hope (1) holds sometimes

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- An R-linear map $\varphi : F^e_* R \to R$ satisfies

$$\varphi(F^e_*(a+b)) = \varphi(F^e_* a) + \varphi(F^e_* b),$$
$$\varphi(F^e_* r^{p^e} x) = \varphi(r F^e_* x) = r\varphi(F^e_* x)$$

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- Test ideal of $\mathscr{D}$: $\tau(R, \mathscr{D}) \coloneqq$ the unique, minimal $J \neq 0$ such that $\varphi(F^e_* J) \subseteq J$ $\forall e$, $\forall \varphi \in \mathscr{D}_e$.

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    - "J, φ are compatible"

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Multiplying a Cart. Alg. by an ideal

- Given $\mathscr{D}$, $\mathfrak{a}_i \subseteq R$, construct

$$\mathscr{D}\mathfrak{a}_1 \cdots \mathfrak{a}_n := \bigcup_e \left\{ \varphi\left(F^e_* x \cdot -\right) \middle| \varphi \in \mathscr{D}_e, x \in \prod_i \mathfrak{a}_i^{p^e-1} \right\}$$

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- $\tau\left(R, \mathfrak{a}_1 \cdots \mathfrak{a}_n\right) := \tau\left(R, \mathcal{C}_R \mathfrak{a}_1 \cdots \mathfrak{a}_n\right)$

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- For any $n \in \mathbb{N}$, define $\mathscr{D}(n)_e$ as the set of $\varphi : F^e_* R \to R$ such that

$$\begin{array}{ccc} F^e_* R^{\otimes n} & \overset{\exists \widehat{\varphi}}{\dashrightarrow} & R^{\otimes n} \\ {\scriptstyle F^e_*\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ F^e_* R & \xrightarrow{\varphi} & R \end{array}$$

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Theorem (S.)

$$\tau\left(R, \mathscr{D}(n)\mathfrak{a}_1 \cdots \mathfrak{a}_n\right) \subseteq \tau(R, \mathfrak{a}_1) \cdots \tau(R, \mathfrak{a}_n)$$

Proof.

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&= \tau(R, \mathfrak{a}_1) \cdots \tau(R, \mathfrak{a}_n)
\end{aligned}$$

The set $\mathscr{D}(n)$ is constructed specifically so that

$$\tau(R, \mathscr{D}(n)\mathfrak{a}_1 \cdots \mathfrak{a}_n) \subseteq \mu\left(\tau\left(R^{\otimes n}, \mathfrak{a}_1 \otimes \cdots \otimes \mathfrak{a}_n\right)\right)$$

□

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Theorem (Carvajal-Rojas, S.)
Let k be a field of characteristic p. Then the Segre product $k[x_0, \ldots, x_r] \# k[y_0, \ldots, y_s]$ is diagonally F-regular.

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Thus, $\mathfrak{p}^{((r+s+1)n)} \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec}(k[x_0, \ldots, x_r] \# k[y_0, \ldots, y_s])$

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Theorem (S.)
If R affine toric, then $\mathscr{D}(2)_e$ is generated by

$$\{\pi_a \mid P_R \cap (a - P_R) \textit{ is ``big''}\}$$

$\mathcal{Z} \subseteq \mathbb{R}^d$ is big if $\forall v \in \frac{1}{p^e}\mathbb{Z}^d \; \exists s \in \mathbb{Z} : v + s \in \mathcal{Z}$

Example

$$R = k[x,y,u,v]/(xy - uv) \cong k[x,y,u,xyu^{-1}]$$