Math 1220-003, Summer 2018

Exam 2 (practice)

Please write your name on the front and back of the exam. Remember to turn off your phone before starting this exam. Show all of your work for full credit. You may not use any notes or calculators during this exam.

Name: Solutions

UID: ____________________

1. (15 points) Determine whether the each of following statements is true or false. If true, write "True." If false, write "False."

   (a) If the series $\sum_{i=1}^{\infty} a_i$ converges, then $\sum_{i=1}^{\infty} |a_i|$ converges.

   False

   (b) If $\sum_{i=1}^{\infty} b_i$ converges and $\lim_{i\to\infty} a_i/b_i = \infty$, then $\sum_{i=1}^{\infty} a_i$ diverges.

   False (e.g. take $b_i = \frac{1}{i^3}$ and $a_i = \frac{1}{i^2}$)

   (c) $\int_1^{\infty} \frac{1}{x}\, dx$ diverges.

   True

   (d) It $\lim_{x\to 1} f(x) = 0$ and $\lim_{x\to 1} g(x) = 0$, then $\lim_{x\to 1} \frac{f(x)}{g(x)} = 0$.

   False

   (e) If $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

2. (15 points) Find the following limits. If the limit is infinite, write ∞ or $-\infty$ accordingly. If the limit does not exist, write "does not exist".

(a) $\lim_{x\to 0} \frac{4x}{\tan x}$

"$\frac{0}{0}$"

$$= \lim_{x\to 0} \frac{4}{\sec^2 x} = \lim_{x\to 0} 4\cos^2 x$$

$$= 4$$

(b) $\lim_{x\to \frac{\pi}{2}} \tan x - \sec x$

"$\infty - \infty$"

$$= \lim_{x\to \pi/2} \frac{\sin x}{\cos x} - \frac{1}{\cos x} = \lim_{x\to\pi/2} \frac{\sin x - 1}{\cos x} \quad \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{-\sin x} = \frac{0}{-1} = 0.$$

(c) $\lim_{x\to 0^+} (\sin x)^{1/x}$

Take $\ln$ first:

$$\lim_{x\to 0^+} \ln\left((\sin x)^{1/x}\right) = \lim_{x\to 0^+} \frac{1}{x} \ln(\sin(x)) = \lim_{x\to 0^+} \frac{\ln(\sin(x))}{x}$$

$$= \text{"}\frac{-\infty}{0}\text{"} = -\infty$$

$$\Rightarrow \lim_{x\to 0^+} (\sin x)^{1/x} = \text{"}\exp(-\infty)\text{"} = 0.$$

3. (20 points) Does the integral $\int_0^3 \frac{x}{(x-2)(x+1)}\,dx$ converge or diverge? If it converges, find the value of the integral.

$\int \frac{x}{(x-2)(x+1)}\,dx = ?$ $\quad \frac{x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$

$\Rightarrow x = A(x+1) + B(x-2)$. Plug in $x=-1$: $-1 = B\cdot(-3)$

$\Rightarrow B = 1/3$

Plug in $x=2$: $2 = A\cdot 3 \Rightarrow A = 2/3$

$\rightsquigarrow \int_0^3 \frac{x}{(x-2)(x+1)}\,dx = \lim_{t\to 2^-} \int_0^t \frac{x}{(x-2)(x+1)}\,dx + \lim_{t\to 2^+} \int_t^3 \frac{x}{(x-2)(x+1)}\,dx$

undefined at $x=2$

$= \boxed{\lim_{t\to 2^-} \int_0^t \frac{2/3}{x-2} + \frac{1/3}{x+1}\,dx} + \lim_{t\to 2^+} \int_t^3 \frac{2/3}{x-2} + \frac{1/3}{x+1}\,dx$

$= \lim_{t\to 2^-} \left[\frac{2}{3}\ln|x-2| + \frac{1}{3}\ln|x+1|\right]_0^t = \lim_{t\to 2^-} \left[\frac{2}{3}\ln|t-2| + \frac{1}{3}\ln|t+1| - 2/3\ln|-2| - \frac{1}{3}\ln|1|\right]$

$-\infty$ $\quad \frac{1}{3}\ln(3)$

$= -\infty$

$\boxed{\text{Diverges}}$

4. (15 points) Use the limit comparison test to determine whether the series $\sum_{i=1}^{\infty} \frac{i-1}{i(1+\sqrt{i})}$ converges.

$\frac{i-1}{i^{3/2}+i}$ looks like $i/i^{3/2} = 1/i^{1/2}$, so we compare to $b_i = \frac{1}{i^{1/2}}$

$$\lim_{i\to\infty} \frac{\frac{i-1}{i^{3/2}+i}}{1/i^{1/2}} = \lim_{i\to\infty} \frac{i-1}{i^{3/2}+i} \cdot \frac{i^{1/2}}{1}$$

$$= \lim_{i\to\infty} \frac{i^{3/2}-i^{1/2}}{i^{3/2}+i} = \underline{1}$$

So $\sum \frac{i-1}{i(1+\sqrt{i})}$ and $\sum \frac{1}{i^{1/2}}$ either both converge or both diverge.

$\sum \frac{1}{i^{1/2}}$ diverges (p-series with $p=1/2$)

So $\sum \frac{i-1}{i(1+\sqrt{i})}$ $\boxed{\text{diverges}}$

5. (20 points) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}e^n}{ne^n - 1}$ converge? If so, does it converge conditionally, or absolutely?

Absolute ratio test:

$$\lim_{n\to\infty} \frac{e^{n+1}/((n+1)e^{n+1}-1)}{e^n/(ne^n-1)} = \lim_{n\to\infty} \frac{e^{n+1-n}}{\cancel{e^n}} \frac{ne^n-1}{(n+1)e^{n+1}-1}$$

$$= e \lim_{n\to\infty} \frac{ne^n - 1}{(n+1)e^{n+1}-1} \to \text{"}\frac{\infty}{\infty}\text{"}$$

L'Hopital:

$$e \lim_{n\to\infty} \frac{ne^n-1}{(n+1)e^{n+1}-1} = e \lim_{n\to\infty} \frac{e^n + ne^n}{e^{n+1} + (n+1)e^{n+1}}$$

$$= e \lim_{n\to\infty} \frac{(n+1)\cancel{e^n}}{(n+2)\, e^{n+1-n}} = \lim_{n\to\infty} \frac{n+1}{n+2} = 1$$

$\boxed{\text{Converges absolutely}}$.

6. (15 points) Determine whether or not the series $\sum_{n=1}^{\infty} \frac{n^2+1}{3^n}$ converges. $\xrightarrow{}$ ratio tes!

$$\lim_{n\to\infty} \frac{(n+1)^2+1 / 3^{n+1}}{n^2+1 / 3^n} = \lim_{n\to\infty} \frac{(n+1)^2+1}{3^{n+1-n}} \frac{\cancel{3^n}}{n^2+1} = \lim_{n\to\infty} \frac{1}{3} \frac{(n+1)^2+1}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{1}{3} \frac{n^2+2n+1+1}{n^2+1} = 1/3$$

$$\boxed{\text{Converges}}$$

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Form: Practice

Name: ______________________________

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4	20	
5	15	
6	20	
7	15	
Total:	100	