RESEARCH DESCRIPTION

FUNDA GÜLTEPE

1. INTRODUCTION

The most basic objects in any category are free objects, yet they are often not the easiest ones to understand. For the category of groups, they are abundant: in any finitely generated group of matrices, for example, there is a free non-abelian subgroup unless the group has a finite index solvable subgroup. (Tits alternative [Tit72]). Particularly important groups related to the free group F_n on n letters are its group of automorphisms $\mathrm{Aut}(F_n)$ and its quotient by inner automorphisms, the outer automorphism group $\mathrm{Out}(F_n)$. Automorphism groups are groups of bases of F_n and the works of Dehn and Nielsen on these groups showed the fundamental role of free groups in the study of all other groups. Studying these groups via complexes they act on not only lets us understand the geometry of these groups but incorporates topology into the realm of geometric group theory.

My current research topic is understanding the geometry of $\mathrm{Out}(F_n)$ by studying its subgroups and in particular, their growth rates. I am specifically interested in answering the question of whether or not subgroups of $\mathrm{Out}(F_n)$ which have exponential growth rate have a uniform exponential growth rate and classifying the ones which do. For this, I am using an approach which incorporates techniques from 3-dimensional topology into geometric group theory by taking the 3-dimensional manifold $\sharp_n(S^2 \times S^1)$ as a model space for $\mathrm{Out}(F_n)$.

The study of the structure and dynamics of $\mathrm{Out}(F_n)$ gains both motivating ideas and mathematical interest because of its close connections with other important topological and algebraic objects. One such is the homomorphism $\mathrm{MCG}(S) \to \mathrm{Out}(F_n)$, where S is a surface with one boundary component, sending a mapping class to its induced automorphism on the fundamental group. Another is the homomorphism $\mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$, sending an automorphism to its induced automorphism on the abelianization of F_n. Many results have been proven to determine the extent to which well-known properties of $\mathrm{MCG}(S)$ and $\mathrm{GL}_n(\mathbb{Z})$ pass to $\mathrm{Out}(F_n)$ via these homomorphisms.

In our work, we use another connection with mapping class groups, the homomorphism $\mathrm{MCG}(M) \to \mathrm{Out}(F_n)$ where $M = \sharp_n(S^2 \times S^1)$. It is very close to being an isomorphism, as its kernel is a finite elementary abelian 2-group (generated by "rotation" homeomorphisms in neighborhoods of finitely many disjoint 2-spheres). It enables us to utilize a considerable body of three-dimensional techniques developed over many decades.

2. Background

Given a generating set of a finitely generated group, the growth rate of a group tells us the number of elements that can be written as a product of a given number of elements from the generating set, and much about the geometry and dynamics of a group and its elements can be learned from the growth rate. The exponential growth rate $\omega(G, S)$ of such a group G with a generating set S is given by:

$$\omega(G, S) = \lim_{n \to \infty} \sqrt[n]{|B_S(n)|},$$

where

$$B_S(n) = \{g \in G : \ell_S(g) \leq n\}.$$

Here, the *length* $\ell_S(g)$ is the least integer k so that the g can be expressed as a product of k elements from S.

If $\omega(G, S) > 1$ then G said to have *exponential growth*. In particular, in a free semi group generated by two elements, the number of elements of length n is the same as the number of ways to form an n-letter word using the generating set. As a result, any finitely generated group which contains a free semi group on two generators has exponential growth. It is possible to take the infimum over all generating sets in the above formula, which is denoted by $\omega(G)$. Now, if $\omega(G) > 1$, G is said to have *uniform exponential growth*. Finitely generated subgroups of the general linear group have this property, which in that setting is equivalent to being not virtually nilpotent [EMO02].

It is also known that homotopy classes of homeomorphisms of surfaces (mapping class groups) and analogous groups of automorphisms of free groups have uniform exponential growth [AAS07]. In the mapping class group setting, the question of whether finitely generated subgroups of mapping class groups have uniform exponential growth rate was answered positively by Mangahas in [Man10]. The main theorem of Mangahas in [Man10] states that the subgroups which are not abelian have uniform exponential growth and minimal growth rate is bounded below by a constant depending only on the surface. The Tits Alternative for the mapping class groups proven by Ivanov in [Iva92] combined with the result of Birman, Lubotzky and McCarthy in [BLM83] saying that any solvable subgroup of mapping class group is virtually abelian gives an idea of where to look for free groups inside all finitely generated subgroups of mapping class groups. Mangahas uses the classification of subgroups of mapping class groups due to Ivanov [Iva92] along with concepts and techniques such as subsurface projection in the curve complex [MM00], minimal translation of pseudo Anosovs [MM99] and results of Fujiwara [Fuj08], and Hamidi-Tehrani [HT02] (completing her arguments in the details when finding a uniform number for the exponential growth of free subgroups of rank 2). Unfortunately, some of these crucial concepts are not fully developed in the $\mathrm{Out}(F_n)$ setting, and some others are far more complicated, so further techniques need to be developed and more cases need to be investigated.

Since $\mathrm{Out}(F_n)$ satisfies the Tits Alternative [BFH00] and virtually nilpotent groups have polynomial growth, it will be sufficient to look for the free groups of rank 2 in non virtually abelian subgroups. For this, we will first look for *fully irreducible elements* of $\mathrm{Out}(F_n)$ since they are the most natural analogs of *pseudo Anosov* self-homeomorphisms of a surface with one boundary component and pseudo Anosovs have a fundamental role in main lemma of Mangahas in [Man10]. Just like pseudo Anosov elements, fully irreducibles are defined to be the class of automorphisms no power of which fixes a conjugacy class of free factors of F_n. Just as pseudo Anosovs are crucial in the study of mapping class groups, so are fully irreducibles in the study of geometry and dynamics of $\mathrm{Out}(F_n)$ and spaces on which it acts (in [LL03], [CP10], [BBC10] etc.) The analogy with the pseudo Anosovs gives us a reason for investigating the applicability of the methods and algorithms used to construct pseudo Anosovs to fully irreducible elements. One of the most known methods for creating pseudo Anosov diffeomorphisms is given by Thurston in [Thu88], as a part of the process of classification of elements of mapping class groups.

Thurston in [Thu88] says that, in a group generated by two Dehn twists about two filling curves on a closed surface with genus $g \geq 2$, the groups generated by twists with powers greater than a finite number N is free of rank 2 and the elements from these groups which are not conjugate to powers of Dehn twists themselves are pseudo Anosov. Adapting this theorem to $\mathrm{Out}(F_n)$ to generate fully irreducible elements and to find rank 2 free groups, Clay and Pettet in [CP10] used an algebraic definition of a Dehn twist automorphism relative to a $\mathbb{Z}$–splitting of the free group and obtained a number N for the minimum power of twists, yet this number N depended on the choice of the twists.

To find a number N which is independent of the choice of Dehn twists, it was necessary to leave the 1-dimensional model for $\mathrm{Out}(F_n)$ since the dependence was due to the necessity for picking a basis of F_n in the proof. Instead we look at the tori in another model for $\mathrm{Out}(F_n)$, $\sharp_n(S^2 \times S^1)$, which we are calling M. For this, I first defined a concept of being normal for essential imbedded tori in M and proved the existence of such representative in a given homotopy class [Gül12]. Then, Clay, Rafi and I in [MCR] defined Dehn twists about such tori in universal cover $\widetilde{M}$ and obtained a bound for intersection number after twisting. We then applied a certain ping-pong argument on the sets of sphere systems of the manifold M. After that, a uniform value for N was achieved.

The manifold M provides a topological rendering for important algebraic constructs. First, there is a direct correspondence between (essential) spheres in M and free splittings of the fundamental group F_n. This was developed by Hatcher in [Hat95], and used in [HV04] to give the conditions on which $\mathrm{Aut}(F_n)$ has certain homological stability. Second, the tori in M correspond to equivalence classes of $\mathbb{Z}$-splittings of F_n. Note here that all tori in M are compressible, since F_n contains no free abelian subgroups of rank 2, but we use tori that are essential in the sense that the image of the fundamental group of the torus in F_n is infinite cyclic.

The tori also provide a topological version (indeed, a motivating one) of *Dehn twist automorphisms* of F_n ([RS97]). Dehn twists about tori have long been studied in three-dimensional topology as analogues of Dehn twists of 2-dimensional surfaces.

To work with tori, however, various technical challenges must be surmounted. Of fundamental importance is the need to find a "normal" torus in each homotopy class, and to understand the uniqueness of normal tori. This is somewhat analogous to the geodesic in each homotopy class of circle in a hyperbolic 2-manifold; geodesics and normal tori both minimize intersections with spheres of complementary codimension.

3. Sphere systems and normal tori

The 3-dimensional space on which $\mathrm{Out}(F_n)$ acts is $M = \sharp_n(S^2 \times S^1)$. The relation between M and $\mathrm{Out}(F_n)$ is that the latter is isomorphic to the mapping class group of M up to twists about 2-spheres in M. M can be described as follows: we remove the interiors of $2n$ disjoint 3-balls from S^3 and identify the resulting 2-sphere boundary components in pairs by orientation-reversing diffeomorphisms, creating $S^2 \times S^1$ summands.

Associated to M is a rich algebraic structure coming from the essential 2-spheres that M contains. A *sphere system* is a collection of isotopy classes of disjoint and non-trivial 2-spheres in M no two of which are isotopic. There is a simplicial complex associated to M called the *sphere complex* and denoted by $\mathbb{S}(M)$, having isotopy classes of non-trivial 2-spheres in M as vertices and sphere systems of $k+1$ spheres as k-dimensional simplices.

The sphere complex is simply connected [HM90], and has a subspace which is homeomorphic to Outer Space. This is the subspace of the sphere complex which consists of sphere systems such that all complementary components are simply connected. The outer space, which is known also as Culler Vogtmann Space, is the one dimensional model for $\mathrm{Out}(F_n)$. For a survey on Outer Space we refer to [Vog02]. For the details we refer to [Hat95] and [HV04]. It should also be noted that the sphere complex is the same as the *free splitting complex* hyperbolicity of which was proven recently by Mosher and Handel [HM12] and which is very closely related to the *complex of free factors* defined first by Hatcher and Vogtmann in [HV98].

A first translation to our new 3-dimensional language is via essential spheres in M: we define 3-dimensional versions of maximal curve systems and of pants decompositions for M as follows: We call a collection Σ of disjointly imbedded essential, non-isotopic 2-spheres in M *a maximal sphere system* if every complementary component of Σ in M is a 3-punctured 3-sphere. Here, the equivalent concept to a pair of pants in a surface is a 3-punctured 3-sphere.

Next, to be able to define a concept of Dehn twist, we need to use the correspondence between the equivalence classes of $\mathbb{Z}$–splittings of F_n and homotopy classes of essential tori in M. Since we need bounds on the intersection number after Dehn twisting, we are particularly interested in finding a representative from a homotopy class of torus which intersects the spheres of a maximal sphere system minimally. In other words, we need to define a notion of a *normal form* and prove that normal representatives exist in a given homotopy class, fairly uniquely. This is achieved in my paper [Gül12] in following steps:

Definition 3.1. *Given an imbedded torus and a maximal sphere system Σ in M, we say that the torus is in normal form with respect to Σ if each intersection of the torus with each complementary 3-punctured 3-sphere is a disk, a cylinder or a pants piece.*

The first step is finding a representative, hence an existence condition:

Theorem 3.2 (Gültepe). *Every imbedded essential torus in M is homotopic to a normal torus and the homotopy process does not increase the intersection number with any sphere of a given maximal sphere system Σ.*

Our uniqueness condition is defined as follows: two tori are said to be *normally homotopic* if there is a homotopy of M changing one of the tori to the other one without introducing new intersections on the sphere crossings, hence through normal, but possibly immersed tori at each level.

Secondly, I proved the following theorem, which is analogous to [Hat95, Proposition 1.2] of Hatcher and which is a type of uniqueness condition for finding a normal representative in a homotopy class of tori in M:

Theorem 3.3 (Gültepe). *If α and β are two homotopic tori in M, both in normal form, then they are normally homotopic.*

Using these two theorems, it was deduced:

Corollary 3.4 (Gültepe). *If a torus α is in normal form with respect to a maximal sphere system Σ, then the intersection number of α with any S in Σ is minimal among the representatives of the homotopy class $\boldsymbol{\alpha}$.*

4. Generating fully irreducibles via intersecting tori

We consider two intersecting tori in M and from now on always use normal representatives whenever a maximal sphere system has been chosen. An example of such an intersecting pair is illustrated in Figure 1.

We know that each homotopy class of tori in M gives an equivalence class of a $\mathbb{Z}$–splitting of F_n. The dual tree in $\widetilde{M}$ corresponding to such a

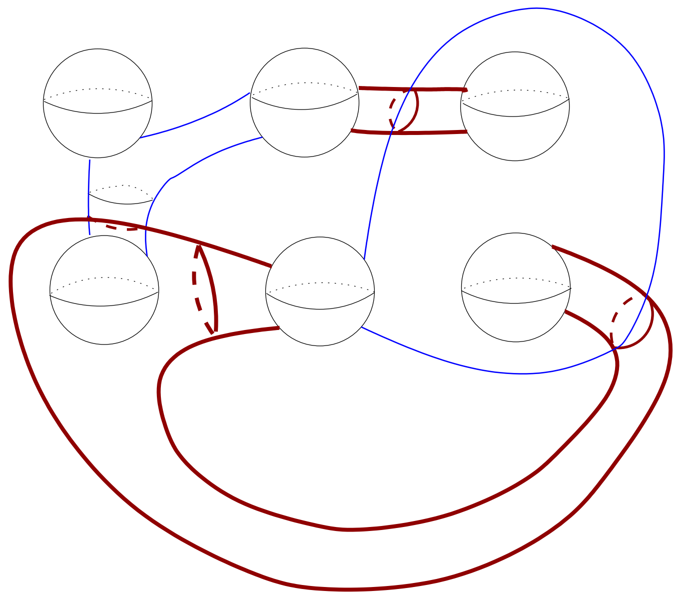

FIGURE 1. Two thrice-intersecting tori α(in black) and β (in red) in $\sharp_3(S^2 \times S^1)$. β intersects α twice nontrivially and once trivially whereas α intersects β once nontrivially and twice trivially.

splitting is called *Bass-Serre tree* and hence we will have a Bass-Serre tree corresponding to each homotopy class of tori. Given an essential imbedded torus α in M, the image of $\pi_1(\alpha)$ under the homomorphism induced by the inclusion $i\colon \alpha \to M$ is an infinite cyclic subgroup of $\pi_1(M)$, defined up to conjugacy. These are $\mathbb{Z}$–subgroups of $\pi_1(M)$. Two $\mathbb{Z}$–splittings correspond to two tori and the $\mathbb{Z}$–subgroups of $\pi_1(M)$ corresponding to these tori act on the Bass Serre trees of each other as elliptic or hyperbolic automorphisms. This action is by multiplication from the right. Recall also that given two elementary $\mathbb{Z}$–splittings $A_1 *_C B_1$ (or $A_1 *_C$) and $A_2 *_{C_2} B_2$ (or $A_2 *_{C_2}$) where $C_1 = \langle c_1 \rangle$ and $C_2 = \langle c_2 \rangle$, the element c_2 is said to be *elliptic* in the Bass-Serre tree of the first splitting if it is contained in a conjugate of A_1 or B_1 and called *hyperbolic* otherwise. These definitions also match with the way these automorphisms act on Bass-Serre trees:

Definition 4.1. *Let $A_1 *_\alpha B_1$ (or $A_1 *_\alpha$) and $A_2 *_\beta B_2$ (or $A_2 *_\beta$) be two $\mathbb{Z}$–splittings of F_n corresponding to tori α and β. The translation length of α in the Bass-Serre tree T_β of the splitting corresponding to β is defined as*

$$\min \left\{ d(\alpha(x), x) : x \in T_\beta \right\}.$$

We will denote this length by $\ell_\beta(\alpha)$.

It is clear that $\ell_\beta(\alpha) > 0$ when α is hyperbolic in T_β and zero if it is elliptic.

Depending on the action of the generator of the $\mathbb{Z}$ subgroups of $\pi_1(M)$ corresponding to each torus, we have three types of splittings: hyperbolic-hyperbolic, hyperbolic elliptic and elliptic-elliptic.

Definition 4.2. *Suppose α and β are essential embedded tori in M. A bigon is an imbedding $\phi\colon D^2 \to M$ such that $\partial D^2 = I_1 \cup I_2$ where $\phi(I_1) \subset \alpha$ and $\phi(I_2) \subset \beta$ for connected intervals I_1 and I_2 but $\phi(\partial D^2)$ does not bound a disk in $\alpha \cup \beta$. Similarly, a cap is an embedding $\phi\colon B^3 \to M$ such that $\partial B^3 = D_1 \cup D_2$ where $\phi(D_1) \subset \alpha$ and $\phi(D_2) \subset \beta$ for 2-disks D_1 and D_2.*

Let the number of intersection components of α and β which are non-trivial in β be denoted by $L_\alpha(\beta)$.

Proposition 4.3 (Gültepe)**.** *Let α and β be two imbedded tori and assume that there are no bigons between α and β. Then,*

$$\ell_\alpha(\beta) = L_\alpha(\beta)$$

where $\ell_\alpha(\beta)$ is the translation length of the generator of β in Bass-Serre tree T_α of α.

To prove this proposition, one needs to prove this following Lemma:

Lemma 4.4 (Gültepe)**.** *Each bigon can be eliminated by a local homotopy reducing the number of intersection components locally. Similarly, caps can be eliminated by a homotopy.*

The main theorem we prove in [MCR] is:

Theorem 4.5 (Clay, Gültepe, Rafi)**.** *Given a pair of hyperbolic-hyperbolic $\mathbb{Z}$–splittings α and β, and integers $k, l \geq 5$, the group $\langle D_\alpha^k, D_\beta^l \rangle$ is a free group of rank* 2*.*

From now on a bold letter will mean a homotopy class. To prove the theorem, we need the following intersection bound which was inspired by the work of Hamidi-Tehrani in [HT02]:

Theorem 4.6 (Clay, Gültepe, Rafi)**.** *For any pair of essential tori $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we have*

$$i(\Sigma, D^n_{\boldsymbol{\alpha}}(\boldsymbol{\beta})) \geq (n-3)\, \ell_{\boldsymbol{\beta}}(\boldsymbol{\alpha})\, i(\boldsymbol{\alpha}, \Sigma) - i(\boldsymbol{\beta}, \Sigma)$$

In the theorem, Σ is a fixed maximal sphere system and $i(\boldsymbol{\alpha}, \Sigma)$ is the number of intersection components of $\boldsymbol{\alpha}$ with spheres of Σ. This intersection number is calculated using the *decorated tree* associated to $\boldsymbol{\alpha}$ in $\widetilde{M}$, which is given again in [Gül12].

The main concept used in Theorem 4.6 is $D^n_{\boldsymbol{\alpha}}(\boldsymbol{\beta})$, which is the image of $\boldsymbol{\beta}$ under the n-th power of the twist about $\boldsymbol{\alpha}$.

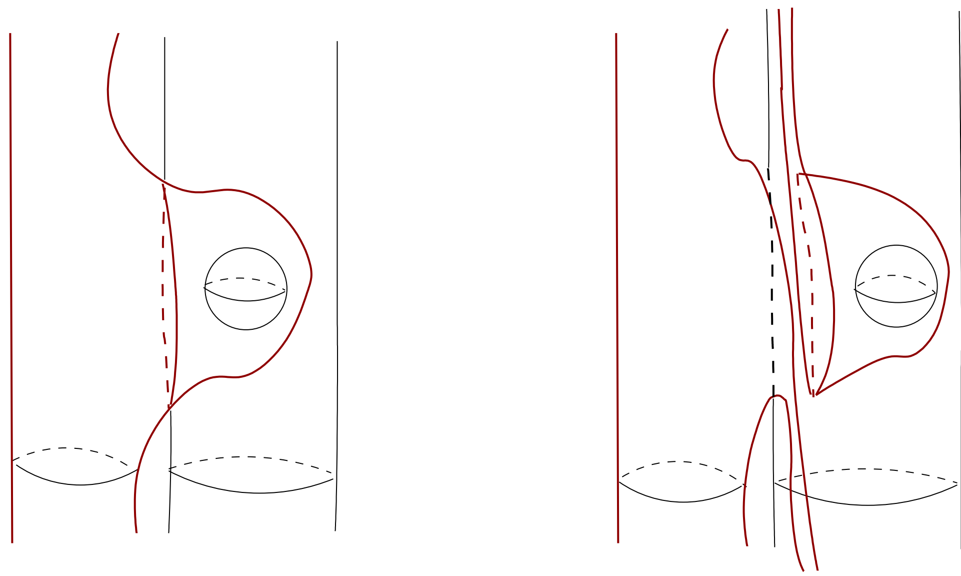

FIGURE 2. A schematic picture of the image of the red torus under a twist of the black torus in $\widetilde{M}$ (the upper tube comes from a different lift).

5. Sketch of the proof of the main results

For our purposes, we will describe a Dehn twist $D_{\boldsymbol{\alpha}}(\boldsymbol{\beta})$ in the universal cover $\widetilde{M}$. First we take two normal representatives α and β from $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively.

For each trivial intersection of β with α in M, the intersection circle bounds a disk in β. To describe the image of such intersection disk under a twist about α, we use surgery in $\widetilde{M}$. If this intersection circle is nontrivial in α, we take a lift of the intersection disk in $\widetilde{M}$ in a lift $\widetilde{\alpha}$ of α, cut it off and glue another disk to its boundary which follows $\widetilde{\alpha}$. An example for a lift of this type of intersection is the first intersection given in Figure 3 where the black torus is a lift of α and the red one is a lift of β. Images in $\widetilde{M}$ after twisting once are given in Figure 4.

For each trivial intersection circle of β which is also trivial in α, we will follow a similar procedure, given again by a surgery in $\widetilde{M}$. We first fix lifts $\widetilde{\alpha}$ and $\widetilde{\beta}$ of α and β, respectively. A twist about α will lift to a twist about the chosen lift of α. To follow the image of the lift of intersection disk under a twist about $\widetilde{\alpha}$, we first take an arc in $\widetilde{M}$ connecting the lift of the intersection disk to another representative of itself located in the next fundamental domain of $\widetilde{\alpha}$. Then take two copies of the intersection circle in $\widetilde{\beta}$, cut $\widetilde{\beta}$ along these. We cap off the one whose image in M bounds a disk in β with another disk, and attach to the second one an annulus which follows the arc and is glued to the capped off part of the next representative

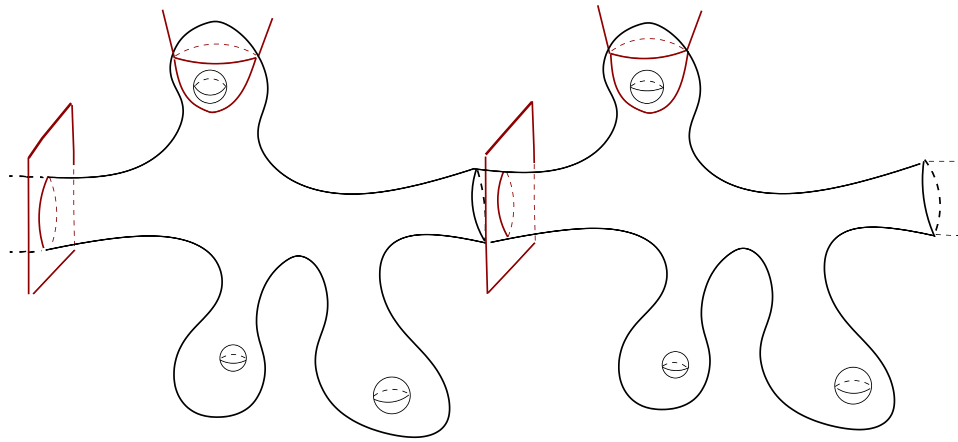

FIGURE 3. Lifts of two intersecting tori, shown in two copies of the fundamental domain of a lift of a black torus in $\widetilde{M}$.

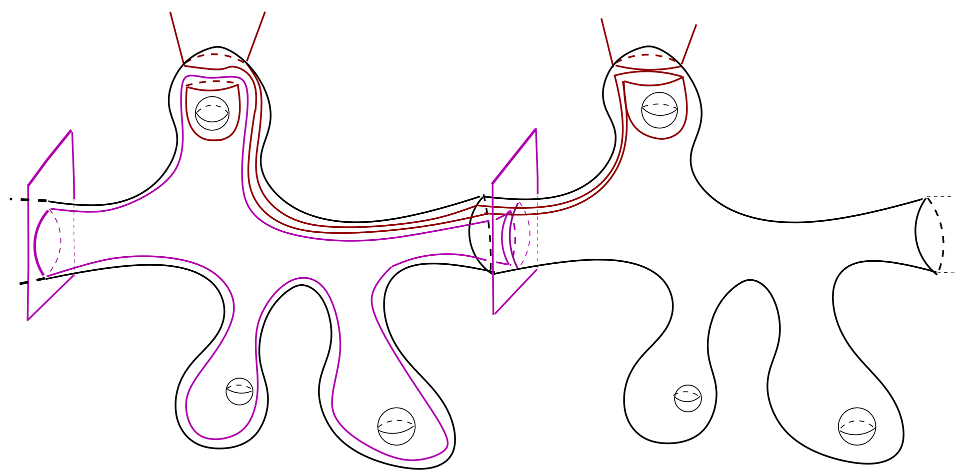

FIGURE 4. Image of the intersections given in Figure 3 under the twist about black torus once.

of the intersection circle. Observe that such annuli might intersect lifts of some spheres when they are following the arc in fundamental domain of $\widetilde{\alpha}$ but since twist about α has a support in a neighborhood of α only, so has a twist about a lift of α. Since such annuli occupy only a part of a neighborhood of $\widetilde{\alpha}$, they may not cross all sphere intersections $\widetilde{\alpha}$ makes. An example for such a (trivial-trivial) intersection circle is given in Figure 2. Another example of a same type of intersection is represented by the second intersection given in Figure 3. In this latter example, the intersection circle is trivial in both tori and $\widetilde{\alpha}$ is black and $\widetilde{\beta}$ is red. Describing its image in a fundamental domain under a twist about $\widetilde{\alpha}$ once requires the same type of surgery and this image is given in Figure 4 in purple. Sphere intersections were not depicted in these pictures.

To prove the main theorem 4.5, we use a standard ping-pong argument using the lower intersection bounds we obtained above on the following sets:

$$N_\alpha = \big\{\Sigma : i(\alpha, \Sigma) < i(\beta, \Sigma)\big\}$$
$$N_\beta = \big\{\Sigma : i(\alpha, \Sigma) > i(\beta, \Sigma)\big\}$$

If, in addition, we take two *filling* tori, the theorem will give us fully irreducible elements of $\mathrm{Out}(F_n)$ where, a pair of tori is *filling* if all sphere systems in M are intersected by at least one of them. Moreover, fully irreducible elements obtained these way are hyperbolic.

6. FURTHER CASES AND NEXT STEPS TOWARDS CONTROLLING UNIFORM GROWTH

As stated earlier, when we have two elementary $\mathbb{Z}$–splittings, we have three cases for the action of the generators of the fundamental groups of corresponding tori on each other's Bass-Serre trees: hyperbolic-hyperbolic, hyperbolic-elliptic and elliptic-elliptic. Since the notion of Dehn twist has to include all these cases, we need to prove theorem 4.5 for the remaining cases also. For this, we will need intersection bounds after twisting and because of the nature of an elliptic action, the remaining cases tend to be more complicated than hyperbolic-hyperbolic case.

This step will answer this question:

Question 1: Given two Dehn twist automorphisms a and b, is there a constant p_0 so that for $n \geq p_0$, the group $\langle D_\alpha^n, D_\beta^n \rangle$ is a free group of rank 2?

I would like to note here that a Dehn twist automorphism is only one example of an automorphisms which is not fully irreducible. So the subgroups which are not generated by fully irreducible automorphisms will need to be examined further.

The remaining finite index subgroups generated by two elements will have at least one fully irreducible generator. The fully irreducible elements can be either toroidal or atoroidal (hyperbolic), and these types also have different subtypes depending on the attracting and repelling trees corresponding to such an automorphisms ([CH12]). For the subgroups generated by at least one irreducible element, since we know by [BFH97] that they are either virtually cyclic or contain a free group of rank 2, one might ask the following question:

Question 2: Is the action of $\mathrm{Out}(F_n)$ on the sphere complex acylindrical?

The concept of acylindricity, due to Bowditch, was given first in [Bow08] and was applied to the action of mapping class group on the curve complex by Fujiwara in [Fuj08] to answer the analogous question in the mapping class group setting, where the group was generated by one or two pseudo

Anosov elements. Hence, answering the question above positively in our case would lead us to finding a uniform growth on subgroups which contain an atoroidal (hyperbolic) fully irreducible automorphism, following the steps of Fujiwara [Fuj08].

These would be steps towards answering our main question:

Question 3: Does every finitely generated non-virtually abelian subgroup of $\mathrm{Out}(F_n)$ have uniform exponential growth?

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Department of Mathematics, University of Oklahoma, Norman, OK 73019
E-mail address: fgultepe@math.ou.edu
URL: http://www.math.ou.edu/∼fgultepe/