MATH 3333.003 Learning Objectives

Learning Objective Summary

The learning objectives below define the course curriculum. Every problem that appears on homework or the exams will assess your understanding of these learning objectives in some way. Students are enouraged to regularly review these learning outcomes to identify areas of strength and weakness in their understanding of the course material.

§1.1 System of Linear Equations and §1.2 Row Reduction of Echelon Forms

2.1 Distinguish a linear system of equations from a non-linear system of equations.
2.2 Identify the augmented matrix of a given system of linear equations.
2.3 Define a consistent/inconsistent system of linear equations.
3.1 Identify the augmented matrix of a given system of linear equations.
3.2 Identify the leading entries in a matrix.
3.3 Identify a matrix in row echelon form.
4.1 Find the general solution to a system of linear equations whose augmented matrix is in row echelon form using substitution.
4.2 Identify the basic variable(s) and free variable(s) in a system of linear equations.
4.3 Analyze how many solutions a system of linear equations based on an augmented matrix in row echelon form.
4.4 Perform row operations to transform a matrix to row echelon form .
5.1 Identify a matrix in reduced row echelon form.
5.2 Perform row operations to transform a matrix to reduced row echelon form.
5.3 Identify the pivot columns, pivot rows, and pivot positions in a matrix.
5.4 Find the general solution to a system of linear equations whose augmented matrix is in reduced row echelon form.

§1.3 Vector Equations

6.1 Manipulate vectors in $\mathbb{R}^n$ under the operations of addition, subtraction, and scalar multiplication.
6.2 Visualize vector addition in $\mathbb{R}^2$ and $\mathbb{R}^3$ using the parallelogram law.
6.3 Convert a vector equation of the form $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$ into a system of linear equations and vice versa.

6.4 Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$, define *a linear combination of* $\mathbf{v}_1, \ldots, \mathbf{v}_p$ and what it means for a vector $\mathbf{b}$ to satisfy $\mathbf{b} \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

6.5 Apply the row reduction algorithm to determine if a vector $\mathbf{b}$ is a linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$, or equivalently, to determine if $\mathbf{b} \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

§1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

7.1 Identify the coefficient matrix of a system of linear equations.

7.2 Define the product $A\mathbf{x}$ of an $m \times n$ matrix A with an $n \times 1$ column vector $\mathbf{x}$.

7.3 Convert a matrix equation of form $A\mathbf{x} = \mathbf{b}$ into a vector equation of the form $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$ and vice versa.

7.4 Given an $m \times n$ matrix A, determine whether the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution for all different possible choices of $\mathbf{b} \in \mathbb{R}^m$.

7.5 Determine if the columns of an $n \times m$ matrix span $\mathbb{R}^n$.

§1.5 Solution Sets of Linear Systems

8.1 Distinguish between a homogeneous system of linear equations from a nonhomogeneous system of linear equations.

8.2 Distinguish between trivial and non-trivial solutions to homogeneous systems of linear equations.

8.3 Present the solution to a homogeneous linear system as the span of a given set of vectors.

8.5 Present the solution set to a consistent system of linear equations in parametric vector form.

8.6 Identify the homogeneous part of the solution to a consistent system of nonhomogeneous linear equations.

8.7 Describe the solution sets to systems of linear equations in $\mathbb{R}^3$ and $\mathbb{R}^2$ geometrically.

§1.7 Linear Independence

9.1 Define linearly independent and linearly dependent sets of vectors.

9.2 Apply the row reduction algorithm to identify if a given set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ is lineary dependent and (if possible) derive a linear dependence relation amongst a set of linealy dependent vectors.

9.3 Interpret the definition of linear independence/dependence from the perspective of the matrix equation $A\mathbf{x} = \mathbf{0}$.

9.4 Characterize sets consisting of two or more linearly dependent vectors in terms of the condition that one vector in the set is a linear combination of the others.

9.5 Apply the definition of linear independence to reason about the linear independence/dependence of a given set of vectors based on incomplete information.

§1.8 Introduction to Linear Transformations

11.1 Define a linear transformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$.

11.2 Define the domain and range of a linear transformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$.
11.3 Calculate the image of a vector under a linear transformation.
11.4 Given a linear transformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$ and a vector $\mathbf{b} \in \mathbb{R}^n$, determine whether $\mathbf{b}$ is in the range of T.
11.5 Given a linear transformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$ and a vector $\mathbf{b}$ in the range of T, determine if $\mathbf{b}$ has a unique pre-image under T.
11.6 Verify that a given trasformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is linear by showing that it satisfies the axioms of a linear transformation.
11.7 Show that a given transformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is not linear by giving a counter example to one of the properties of a linear transformations.

§1.9 The Matrix of a Linear Transformation

12.1 Find the standard matrix of a linear transformation.
12.2 Define what is means for a transformation to be one-to-one.
12.3 Define what it means for a transformation to be onto.
12.6 Characterize onto linear transformations T in terms of the condition that the columns of the standard matrix of T span the codomain of the transformation.
13.1 Define what it means for a transformation to be onto.
13.2 Define what is means for a transformation to be one-to-one.
13.3 Characterize onto linear transformations T in terms of the condition that the columns of the standard matrix of T span the codomain of the transformation.
13.4 Characterize one-to-one linear transformations T in terms of the condition that $T(\mathbf{x}) = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$.
13.5 Characterize one-to-one linear transformations T in terms of the condition that the columns of the standard matrix of T has linearly independent columns.

§2.1 Matrix Operations

14.1 Identify a zero matrix an identity matrix.
14.2 Simplify matrix expressions using the operations of matrix addition and scalar multiplication.
14.3 Interpret row vectors as scalar valued linear trasformations
14.4 Define the sum of two linear transformations and interpret this operation in the setting of matrix addition.
14.4 Simplify matrix expressions using the operations of transposition and matrix multiplication.
15.1 Simplify matrix expressions using the operations of matrix addition and scalar multiplication.
15.2 Calculate the matrix product AB where A is an $n \times m$ matrix and B is an $m \times p$ matrix using the row column rule for matrix multiplication.
15.3 Simplify matrix expressions using the operations of matrix multiplication.
15.4 Define the composition $T \circ S\colon \mathbb{R}^p \to \mathbb{R}^n$ where $T\colon \mathbb{R}^n \to \mathbb{R}^m$ and $S\colon \mathbb{R}^p \to \mathbb{R}^m$ are transformations.

15.5 Interpret the matrix product AB as the standard matrix of the composition of two linear transformations.
17.4 Define the transpose A^T of a matrix A.
17.5 Apply the fact that if A and B are matrices then $(AB)^T = B^T \cdot A^T$.

§2.2 & §2.3 Invertible Matrices and Linear Transformations

16.1 Define an invertible linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$.
16.2 Apply the fact that a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if T is onto.
16.3 Apply the fact that a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if T is one-to-one.
16.4 Define what it means for an $n \times n$ matrix A to be invertible.
16.5 Characterize invertible matrices A using the Invertible Matrix Theorem.
17.1 Define the rank of a matrix and use the condition $\operatorname{rank}(A) = n$ to characterize when an $n \times n$ matrix is invertible.
17.2 Find the inverse of an $n \times n$ invertible matrix A by finding a reduced echelon form of the matrix $\begin{pmatrix} A & I_n \end{pmatrix}$.
17.3 Apply the fact that if A and B are invertible matrices then $(AB)^{-1} = B^{-1} \cdot A^{-1}$.

§2.8 & §2.9 Subspaces of $\mathbb{R}^n$, Dimension, and The Rank Theorem

18.1 Define what it means for a subset $H \subseteq \mathbb{R}^n$ to be a subspace.
18.2 Evaluate whether a given subset $H \subseteq \mathbb{R}^n$ is a subspace either by veriying that H satisfies the definition of a subspace, or giving a counter example to one of the properties of subspaces.
18.3 Define the column space $\operatorname{Col}(A)$ and the null space $\operatorname{Nul}(A)$ of a matrix A.
18.4 Define what it means for subset $\mathcal{B} \subseteq H$ to be a basis for a subspace of $\mathbb{R}^n$.
18.5 Use the row reduction to find bases for the column space and null space of a matrix.
19.1 Define the column space $\operatorname{Col}(A)$ and the null space $\operatorname{Nul}(A)$ of a matrix A.
19.2 Define what it means for subset $\mathcal{B} \subseteq H$ to be a basis for a subspace of $\mathbb{R}^n$.
19.3 Use the row reduction to find bases for the column space and null space of a matrix.
20.1 Given subspace $H \subseteq \mathbb{R}^n$ and a basis $\mathcal{B}$ for H, find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of a vector $\mathbf{x} \in H$ relative to the basis $\mathcal{B}$.
20.2 Define the dimension $\dim(H)$ of a subspace $H \subseteq \mathbb{R}^n$.
20.3 Relate the $\operatorname{rank}(A)$, $\dim \operatorname{Col}(A)$, and $\dim \operatorname{Nul}(A)$ using the rank theorem.
21.1 Given subspaces $H_1 \subseteq \mathbb{R}^n$ and $H_2 \subseteq \mathbb{R}^m$ define an isomorphism $T\colon H_1 \xrightarrow{\sim} H_2$.
21.2 Given subspaces $H_1 \subseteq \mathbb{R}^n$ and $H_2 \subseteq \mathbb{R}^m$, define what it means for H_1 and H_2 to be isomorphic.
21.3 Given a p-dimensional subspace $H \subseteq \mathbb{R}^n$ together with a basis $\mathcal{B}$ for H, interpret the coordinate map $T_B\colon H \to \mathbb{R}^p$, $T_{\mathcal{B}}(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}$ as an isomorphism.
21.4 Apply The Basis Theorem to reason about bases for subspaces.

§L.22-4 Change of Basis, Matrices of Linear Transformations, and Matrix Similarity

21.1 Given subspace H and bases $\mathcal{B}$ and $\mathcal{C}$ for H, define the change of basis matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ as the standard matrix of the composition $T_{\mathcal{B}} \circ T_{\mathcal{C}}^{-1}$.

21.2 Given bases $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_p\}$ for a subspace H, calculate the change-of-basis-matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ using the formula

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \left([\mathbf{c}_1]_{\mathcal{B}} \quad \cdots \quad \mathbf{c}_p]_{\mathcal{B}}\right).$$

21.3 Apply the fact that $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}^{-1} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

23.1 Given subspace H and bases $\mathcal{B}$ and $\mathcal{C}$ for H, find the change of basis matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$

23.2 Apply the fact that $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}^{-1} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

23.3 Apply the identity $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}[\mathbf{x}]_{\mathcal{C}} = [\mathbf{x}]_{\mathcal{B}}$.

23.4 Given a subspace H, a basis $\mathcal{B}$ for H, and a linear transformation $T\colon H \to H$, find $[T]_{\mathcal{B}}$, the matrix of T relative to the basis B.

23.5 Given a subspace H and bases $\mathcal{B}$ and $\mathcal{C}$ for H, apply the identity

$$[T]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \cdot [T]_{\mathcal{B}} \cdot \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}.$$

23.6 Apply the identity $[T]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{B}}$.

24.1 Given a subspace H, a basis $\mathcal{B}$ for H, and a linear transformation $T\colon H \to H$, find $[T]_{\mathcal{B}}$, the matrix of T relative to the basis B.

24.2 Given a subspace H and bases $\mathcal{B}$ and $\mathcal{C}$ for H, apply the identity

$$[T]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \cdot [T]_{\mathcal{B}} \cdot \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}.$$

24.3 Given $n \times n$ matrices A and B, define what it means to say that A and B are similar.

24.4 Given a subspace $H \subseteq \mathbb{R}^n$, bases $\mathcal{B}$ and $\mathcal{C}$ for a H, and linear transformation $T\colon H \to H$, apply the fact that $[T]_{\mathcal{B}}$ is similar to $[T]_{\mathcal{C}}$.

24.5 Define what it means for an $n \times n$ matrix to be diagonalizable.

§3.1-2 Introduction to Determinants and Properties of Determinants

25.1 Calculate the determinant of matrices by expanding along rows or columns.

26.1 Calculate the determinant of an REF matrix by multiplying the diagonal entries.

26.2 Track how the determinant of a matrix changes when row operations are applied.

26.3 Calculate the determinant of a matrix by row reducing A to an REF form.

26.4 Apply the fact that a matrix A is invertible if and only if $\det(A) \neq 0$.

26.5 Apply the fact that a matrix that if A and B are $n \times n$ matrices then $\det(AB) = \det(A)\det(B)$.

27.1 Calculate the determinant of an REF matrix by multiplying the diagonal entries.

27.2 Track how the determinant of a matrix changes when row operations are applied.

27.3 Calculate the determinant of a matrix by row reducing A to an REF form.
27.4 Apply the fact that a matrix A is invertible if and only if $\det(A) \neq 0$.
27.5 Apply the fact that a matrix that if A and B are $n \times n$ matrices then $\det(AB) = \det(A)\det(B)$.
27.6 Apply the fact that a matrix that if A is an invertible matrix then $\det(A) = \dfrac{1}{\det(A)}$.
27.7 Apply the fact that a matrix that if A is an $n \times n$ matrix then $\det(A^T) = \det(A)$.

§5.1-3 Eigenvalues, Eigenvectors, and Diagonalizability

27.7 Define an eigenvalue λ of a matrix A using the condition that $\det(A - \lambda I) = 0$.
27.8 Define an eigenvalue λ of a matrix A using the condition that $\mathrm{Nul}(A - \lambda I) \neq \{\mathbf{0}\}$.
27.9 Relate the eigenvalues of similar matrices.
28.1 Define the characteristic polynomial of a matrix A.
28.2 Find the eigenvalues of a matrix A by solving the characteristic equation of A.
28.3 Define what it means for a vector $\mathbf{v}$ to be an eigenvector of a matrix A with eigenvalue λ.
28.4 Assess whether a vector $\mathbf{v}$ is an eigenvector of a matrix A by calculating $A\mathbf{v}$.
28.5 Given a matrix A with eigenvalue λ, calculate a basis for the λ-eigenspace of A.
29.1 Given an $n \times n$ matrix A with eigenvalue λ, define the algebraic multiplicity of λ in A.
29.2 Given a diagonalizable linear transformation T with standard matrix A, determine a basis $\mathcal{C}$ such that $[T]_{\mathcal{C}}$ is diagonal.
29.3 Apply the diagonalization theorem to conclude that an $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors.
29.4 Given an $n \times n$ diagonalizable matrix A, diagonalize A by calculating a basis for $\mathbb{R}^n$ consising of eigenvectors of A.
29.5 Given an $n \times n$ matrix A with eigenvalue λ, define the geometric multiplicity of λ in A.
30.1 Given a diagonalized matrix $A = PDP^{-1}$, calculate powers of A using the formula $A^n = PD^nP^{-1}$.
30.2 Apply the statement that if A is a matrix and $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are eigenvectors of A with distinct eigenvalues, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent
30.3 Assess whether a matrix is diagonalizable using Theorem 7 in textbook section 5.3.
30.4 Given a diagonalizable matrix A, determine if a given matrix B is similar to A.

§L.31-36 Abstract Vector Spaces

31.1 Row reduce a matrix to REF by applying The Row Reduction Algorithm.
31.2 Row reduce a matrix to RRED by applying The Row Reduction Algorithm.
31.3 Interpret the definition of abstract vector space.
32.2 Describe the addition law and law of scalar multiplication on the vector spaces $\mathbb{R}^n$, $M_{n\times m}(\mathbb{R})$, $\mathbb{P}_n$, and $\mathcal{C}^1(\mathbb{R})$.
32.3 Given a vector space V, identify the zero vector in V.
32.4 Given a vector space V and a subset $H \subseteq V$, prove or disprove the statement that

H is a subspace of V.

33.1 Given a vector space V, define what it means for a subset $\mathcal{B} \subseteq V$ to be a basis for V.

33.2 Given a vector space V and a basis $\mathcal{B}$ for V. Define the coordinate isomorphism $T_{\mathcal{B}}(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}$ associated to the basis $\mathcal{B}$.

33.3 Given a vector space V and a basis $\mathcal{B}$ for V. Define the dimension of V.

33.4 Given a finite dimensional vector space V and a basis $\mathcal{B}$ for V, use the coordinate isomorphism $T_{\mathcal{B}}$ to test whether a given collection of vectors in V is linearly independent.

34.1 Given vector spaces V and W, define a linear transformation $T\colon V \to W$.

34.2 Given vector spaces V and W, and a mapping $T\colon V \to W$, prove or disprove the statement that T is a linear transformation.

34.3 Given a linear transformation $T\colon V \to W$ between finite dimensional vector spaces, and bases $\mathcal{B}$ and $\mathcal{C}$ for V and W respectively, define $\underset{\mathcal{C}\leftarrow\mathcal{B}}{[T]}$, the matrix of T relative to the bases $\mathcal{B}$ and $\mathcal{C}$.

34.4 Given vector spaces V and W, and a linear transformation $T\colon V \to W$, define the image $\operatorname{im}(T)$ of T, and the kernel $\ker(T)$ of T.

34.5 Let V and W be finite dimensional vector spaces, with bases $\mathcal{B}$ and $\mathcal{C}$ respectively. Given a linear transformation $T\colon V \to W$, calculate a basis for $\ker(T)$ and $\operatorname{im}(T)$.

34.6 Assess whether a linear transformation between finite dimensional vector spaces is one-to-one or onto.

35.1 Given a vector space V and a linear transformation $T\colon V \to V$, define an eigenvalue and an eigenvector of T.

35.2 Given a finite dimensional vector space V with basis $\mathcal{B}$, and a linear transformation $T\colon V \to V$, calculate the eigenvalues of T.

35.3 Given a finite dimensional vector space V with basis $\mathcal{B}$, and a linear transformation $T\colon V \to V$ with eigenvalue λ, calculate a basis for the λ-eigenspace of T.

35.4 Given a finite dimensional vector space V, and a diagonalizable linear transformation $T\colon V \to V$, calculate a basis $\mathcal{C}$ for V such that $[T]_{\mathcal{C}}$ is diagonal.

35.5 Given a finite dimensional vector space V, and a diagonalizable linear transformation $T\colon V \to V$, determine whether T is diagonalizable.