

# Characterizing Cuspidal $D_4$ Modular Forms by the Growth of their Fourier Coefficients.

arXiv:2601.21071

Finn McGlade

38th Annual Workshop on Automorphic Forms  
Manhattan, Kansas

May 2026

## Outline

- 1 Siegel Cusp Forms and the Growth of Fourier Coefficients
- 2 Characterizing Quaternionic Cusp Forms on  $\text{Spin}(4,4)$

# Contents

- 1 Siegel Cusp Forms and the Growth of Fourier Coefficients
- 2 Characterizing Quaternionic Cusp Forms on  $\text{Spin}(4, 4)$

Modular Forms on  $\text{Sp}_4$ 

Given a commutative ring  $R$ , define the **symplectic group**  $\text{Sp}_4(R)$  as

$$\text{Sp}_4(R) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GL}_4(R) : {}^t g \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \right\}.$$

The real Lie group  $\text{Sp}_4(\mathbb{R})$  acts on

$$\mathcal{H}_2 = \{Z \in M_2(\mathbb{C}) : Z = {}^t Z \text{ and } \text{Im}(Z) > 0\}$$

via the transformations  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}$ .

### Definition (Siegel Modular Forms)

Let  $\ell \in \mathbb{Z}_{\geq 0}$  and write  $M_\ell(\text{Sp}_4(\mathbb{Z}))$  for the space of **holomorphic** functions

$$F: \mathcal{H}_2 \rightarrow \mathbb{C}$$

such that  $F(\gamma \cdot Z) = \det(CZ + D)^\ell F(Z) \forall \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_4(\mathbb{Z})$  and  $Z \in \mathcal{H}_2$ .

## Cuspidality and the Fourier Expansion of Modular Forms

A Siegel Modular Form  $F \in M_\ell(\text{Sp}_4(\mathbb{Z}))$  has a **Fourier series**

$$F(Z) = \sum_T A[T] \exp(2\pi i \text{tr}(TZ)).$$

The Fourier coefficients (**FCs**)  $A[T]$  are complex **scalars** indexed by

$$T \in \left\{ \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} : n, r, m \in \mathbb{Z} \right\}.$$

### Definition

Say that  $F$  is **cuspidal** if  $A[T] = 0$  whenever  $\det(T) = 0$ .

- For  $F \in M_\ell(\text{Sp}_4(\mathbb{Z}))$  and  $T$  such that  $\det(T) \neq 0$ ,

$$A[T] \ll_F \det(T)^\ell.$$

- If  $F$  is cuspidal then we have the **Hecke bound**

$$A[T] \ll_F \det(T)^{\ell/2}.$$

## Characterizing Cusp Forms by the Growth of their Fourier Coefficients

Kohnen-Martin (14') prove the Hecke bound **characterizes** cusp forms.

**Theorem (Kohnen-Martin 14')**

Suppose  $F$  is a weight  $\ell$  Siegel modular form on  $\text{Sp}_4(\mathbb{Z})$  such that

$$A_F[T] \ll_F \det(T)^{\ell/2} \quad (\text{Hecke bound})$$

for all  $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$  satisfying  $\det(T) \neq 0$ . Then  $F$  is cuspidal.

Generalizations and refinements of the above have been obtained for various classes of **holomorphic** modular forms.

- **Böcherer-Das (14', 16', 18')** Kohnen-Martin (14') for Siegel Modular forms of higher genus ( $g > 2$ ), congruence level, and low weights ( $\ell < 2g$ ).

Today, we study an analogue of Kohnen-Martin (14') for a class of non-holomorphic, **Quaternionic** modular forms on  $\text{Spin}(4,4)$ .

## The Fourier Jacobi Expansion of Siegel Modular Forms

If  $F \in M_\ell(\mathrm{Sp}_4(\mathbb{Z}))$ , then the **Fourier-Jacobi expansion** of  $F$  is

$$F \left( \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \right) = \sum_{m \in \mathbb{Z}} \phi_m(\tau, z) e^{2\pi i m \tau'}$$

Here  $\phi_m \in J_{\ell, m}(\mathrm{SL}_2(\mathbb{Z}))$  is a **Jacobi form** with Fourier Expansion

$$\phi_m(\tau, z) = \sum_{n, r \in \mathbb{Z}} A \left[ \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \right] e^{2\pi i (n\tau + rz)}$$

Kohnen-Martin's theorem follows by combining the next two lemmas.

#### Lemma A (Kohnen-Martin 14')

Assume  $\ell \geq 4$ ,  $m \neq 0$  and the non-degenerate FCs of  $\phi \in J_{\ell, m}(\mathrm{SL}_2(\mathbb{Z}))$  satisfy the Hecke bound  $c(n, r) \ll (4nm - r^2)^{\ell/2}$ . Then  $\phi$  is cuspidal.

#### Lemma B (Kohnen-Martin 14')

Suppose  $F \in M_\ell(\mathrm{Sp}_4(\mathbb{Z}))$ . If  $\phi_m$  is cuspidal  $\forall m \neq 0$ , then  $F$  is cuspidal.

# Contents

- 1 Siegel Cusp Forms and the Growth of Fourier Coefficients
- 2 Characterizing Quaternionic Cusp Forms on  $\text{Spin}(4, 4)$

Modular Forms on  $\text{Spin}_8$ .

- Let  $\text{Spin}_8 \rightarrow \text{SO}_8$  be the simply connected **split** group of type  $D_4$ .
- Let  $K$  be a maximal compact subgroup of  $\text{Spin}_8(\mathbb{R})$ .
- $K$  contains a distinguished normal copy of  $\text{SU}(2)$ .
- Get a 3-dimensional  $K$ -representation on  $\mathbb{V} := \text{Lie}(\text{SU}(2)) \otimes \mathbb{C}$ .

## Definition


Let  $M_\ell(\text{Spin}_8(\mathbb{Z}))$  denote the space of functions

$$H: \text{Spin}_8(\mathbb{R}) \rightarrow \text{Sym}^\ell(\mathbb{V})$$

such that  $H$  is smooth, of moderate growth and satisfies

- If  $\gamma \in \text{Spin}_8(\mathbb{Z})$  and  $g \in \text{Spin}_8(\mathbb{R})$  then  $H(\gamma g) = H(g)$ .
- If  $k \in K$  and  $g \in \text{Spin}_8(\mathbb{R})$  then  $H(gk) = k^{-1}H(g)$ .
- The function  $H$  satisfies a specific differential equation  $D_\ell H \equiv 0$ .

The Fourier Expansion of Modular forms on  $\text{Spin}_8$ 

$N$ : the unipotent radical of a parabolic attached to central node 

- The **constant term** of  $H$  along  $N$  is  $H_N(g) = \int_{N(\mathbb{Z}) \backslash N(\mathbb{R})} H(ng) dn$ .

Theorem (Gross-Wallach 96', Weissman 03', Pollack 20')

A modular form  $H \in M_\ell(\text{Spin}_8(\mathbb{Z}))$  has a Fourier expansion

$$H_N(g) + \sum_{[T_1, T_2]} \Lambda[T_1, T_2] \mathcal{W}_{[T_1, T_2]}(g).$$

- The **Fourier Coefficients (FCs)**  $\Lambda[T_1, T_2]$  are scalars indexed by

$$[T_1, T_2] \in M_2(\mathbb{Z}) \oplus M_2(\mathbb{Z}).$$

- $\mathcal{W}_{[T_1, T_2]}(g): \text{Spin}_8(\mathbb{R}) \rightarrow \text{Sym}^\ell(\mathbb{V})$  is a **special function**, which depends on  $\ell$  and  $[T_1, T_2]$ , but is otherwise independent of  $H$ .

Characterizing Cusp Forms on  $\text{Spin}_8$  by the Growth of Their Fourier Coefficients

Given  $[T_1, T_2] \in M_2(\mathbb{Z}) \oplus M_2(\mathbb{Z})$ , let

$$D[T_1, T_2] = \det \begin{pmatrix} \text{tr}(T_1 T_1^\#) & \text{tr}(T_1 T_2^\#) \\ \text{tr}(T_2 T_1^\#) & \text{tr}(T_2 T_2^\#) \end{pmatrix}.$$

## Definition

$H \in M_\ell(\text{Spin}_8(\mathbb{Z}))$  is **cuspidal** if  $\Lambda[T_1, T_2] = 0$  whenever  $D[T_1, T_2] = 0$ .

## Theorem (M. 26')


If  $\ell \geq 5$  then  $H \in M_\ell(\text{Spin}_8(\mathbb{Z}))$  is cuspidal if and only if

$$\Lambda[T_1, T_2] \ll_H D[T_1, T_2]^{(\ell+1)/2} \quad (\text{Quaternionic Hecke bound})$$

for all  $[T_1, T_2] \in M_2(\mathbb{Z}) \oplus M_2(\mathbb{Z})$  satisfying  $D[T_1, T_2] \neq 0$ .

- **Gan-Gross-Savin (02')** Prove the implication “Cuspidal”  $\implies$  “Quaternionic Hecke Bound” for modular forms on  $G_2$ . Their proof generalizes to modular forms on  $\text{Spin}_8$  without modification.

Fourier-Jacobi Coefficients of Modular Forms on  $\text{Spin}_8(\mathbb{Z})$ 

$U$ : the unipotent radical of a parabolic attached to an outer node 

**Theorem (Johnson-Leung, M., Negrini, Pollack, Roy 24', M. 25')**

Suppose  $H \in M_\ell(\text{Spin}_8(\mathbb{Z}))$  is a weight  $\ell$  **quaternionic** modular form &

$$\chi: U(\mathbb{Z}) \backslash U(\mathbb{R}) \rightarrow \mathbb{C}^\times$$

is a non-degenerate character. There exists an embedding

$$\iota: \text{Sp}_4(\mathbb{R}) \rightarrow \text{Spin}_8(\mathbb{R})$$

and a linear functional  $L: \text{Sym}^\ell(\mathbb{V}) \rightarrow \mathbb{C}$  such that the function

$$\text{Sp}_4(\mathbb{R}) \rightarrow \mathbb{C}, \quad h \mapsto \int_{U(\mathbb{Z}) \backslash U(\mathbb{R})} L(H(u(h))) \chi^{-1}(u) du$$

descends to a weight  $\ell$  **holomorphic Siegel modular form**  $F_\chi$  on  $\mathcal{H}_2$ .

We refer to the modular forms  $F_\chi$  as **Fourier-Jacobi** coefficients of  $H$ .

Lemma A for Modular Forms on  $\text{Spin}_8(\mathbb{R})$ 

Write the classical Fourier expansion of the Siegel modular form  $F_\chi$  as

$$F_\chi(Z) = \sum_T A[T] \exp(2\pi i \text{tr}(TZ)).$$

## Lemma A' (M. 26')

Assume the weight  $\ell \geq 5$  and the Fourier coefficients of  $F_\chi$  satisfy

$$A[T] \ll \det(T)^{(\ell+1)/2} \quad (1)$$

for all  $T$  such that  $\det(T) \neq 0$ . Then  $F_\chi$  is cuspidal.

- The proof of Kohnen-Martin's 2014 theorem generalizes to give the proof of Lemma A'.
- For **Lemma A'**, the assumption  $\ell \geq 5$  is necessary since the FCs of the weight 4 Siegel Eisenstein series on  $\text{Sp}_4(\mathbb{Z})$  satisfy (1).

Lemma B for Modular Forms on Spin<sub>8</sub>( $\mathbb{R}$ )

## Lemma B' (M. 26')

If  $F_\chi$  is a cuspidal  $\forall$  **non-degenerate** characters  $\chi$  then  $H$  is cuspidal.

Lemma B' depends on an analogue of the classical statement

$$F \in M_\ell(\mathrm{Sp}_4(\mathbb{Z})) \text{ is } \mathbf{cuspidal} \iff \Phi F(\tau) := \lim_{t \rightarrow \infty} F \begin{pmatrix} \tau & 0 \\ 0 & it \end{pmatrix} \mathbf{vanishes}.$$

## Theorem (M, 26')

$N$ : the unipotent radical of a parabolic attached to central node



$$H \in M_\ell(\mathrm{Spin}_8(\mathbb{Z})) \text{ is } \mathbf{cuspidal} \iff H_N(g) := \int_{N(\mathbb{Z}) \backslash N(\mathbb{R})} H(ng) \, dn \mathbf{vanishes}.$$

- $H_N \leftrightarrow f_H(z_1, z_2, z_3) \in M_\ell(\mathrm{SL}_2(\mathbb{Z}))^{\otimes 3}$ , a **level 1** holomorphic modular form on three copies of the **upper-half plane** (Pollack 20')

Thank you for listening