ON THE VANISHING AND CUSPIDALITY OF D_4 MODULAR FORMS

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Abstract. We develop vanishing and cuspidality criteria for quaternionic modular forms on $G = \mathrm{Spin}(4,4)$ using a theory of scalar Fourier coefficients. By analyzing a Fourier-Jacobi expansion for these forms, we prove that a level one quaternionic modular form on G vanishes if and only if its primitive Fourier coefficients are zero. Using this criterion, we characterize Pollack's quaternionic Saito-Kurokawa subspace by imposing equalities between certain primitive Fourier coefficients. This characterization strengthens earlier work of the author with Johnson-Leung, Negrini, Pollack, and Roy. We also study quaternionic modular forms in the more general setting of a group G_J associated to a cubic norm structure J. Here we establish a new relationship between the degenerate Fourier coefficients of quaternionic modular forms, and the Fourier coefficients of the holomorphic modular forms associated to their constant terms. As a consequence, we prove that in weights $\ell \geq 5$, a level one quaternionic modular form on $\mathrm{Spin}(4,4)$ is cuspidal if and only if its non-degenerate Fourier coefficients satisfy a polynomial growth condition.

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1. Introduction

Let G be the split spin group of rank 4 over the rational numbers. Given an automorphic function φ on G, one can ask the following two fundamental global questions; (i) does φ vanish identically, and (ii) is φ cuspidal? The purpose of this paper is to apply a theory of scalar Fourier coefficients to address problems (i) and (ii) when φ is a quaternionic modular form on G. Before we can formulate our results, we need to review some preliminary ideas.

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1.1. **Quaternionic Modular Forms on G.** In [Wei06], Weissman studies D_4 modular forms, which are certain automorphic functions on G. Weissman's work is influenced by that of Gan-Gross-Savin [GGS02], who studied an analogous class of modular forms on G_2. Pollack [Pol20] later studied a generalization of this class of automorphic forms in the setting of a reductive group G_J associated to a cubic norm structure J. In Pollack's conventions, a weight $\ell \in \mathbf{Z}_{>0}$ quaternionic modular form on G_J is a vector-valued automorphic function that is annihilated by the Schmid differential [Sch] associated to a quaternionic $G_J(\mathbf{R})$ representation in the sense of [GW96].

One notable feature on a D_4 modular form φ is the set of scalar Fourier coefficients,

$$\{\Lambda_\varphi[B] \in \mathbf{C} \colon B = [T_1, T_2] \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2} \backslash \{[0,0]\}\}.$$

The coefficients $\Lambda_\varphi[B]$ arise from taking Fourier coefficients of φ along a Heisenberg group N_P, which is the unipotent radical of a maximal parabolic subgroup P in G. The existence of $\Lambda_\varphi[B]$ as scalar is non-trivial, and relies on results regarding the generalized Whittaker vectors of quaternionic $G(\mathbf{R})$-representations ([Wal03, Theorem 16] and [Pol20, Theorem 1.2]). Another important feature of a D_4 modular form φ, is the structure of their constant term along N_P. In [Pol20, Theorem 11.1.1], it is shown that the constant term φ_{N_P} gives rise to a holomorphic modular form Φ on a Levi factor M_P of P. More specifically, the derived subgroup M_P^{der} is isomorphic to SL_2^3, and Φ is the automorphic form associated to a holomorphic modular form on three copies of the upper half plane.

1.2. **Primitivity in the Fourier Expansion of D_4 Modular Forms.** Given an automorphic form φ, it is generally important to develop criteria to assess whether φ is identically zero. When φ has scalar Fourier coefficients, these may be applied to study the vanishing of φ in unique ways. For example, let φ be the automorphic function on $\mathrm{Sp}_4(\mathbf{A_Q})$ associated to a Siegel modular form F_φ of level one, and genus 2. Then F_φ admits a Fourier expansion

$$F_\varphi(Z) = \sum_{T \geq 0} A_\varphi[T] \exp(2\pi i \mathrm{tr}(TZ)).$$

Here Z is an element in the Siegel upper half space of genus 2, and $T = [a, b, c]$ runs over the set of positive semi-definite, integral, binary quadratic forms. The complex scalars $A_\varphi[T]$ are the Fourier coefficients of φ. Say that the coefficient $A_\varphi[T]$ is primitive if $\gcd(a, b, c) = 1$.

Zagier [Zag81, pg. 387] proves that φ vanishes identically if and only if each of the primitive coefficients $A_\varphi[T]$ is 0. The interplay between the vanishing of the coefficients $A_\varphi[T]$, and the vanishing properties of φ has since developed into an area of significant interest. For example, Saha [Sah13] proves a refinement of Zagier's Theorem, which in the case of cuspidal φ, characterizes the vanishing of φ using a sparser subset of *fundamental* Fourier coefficients. The result of (loc. cit.) is generalized to vector valued, possibly non-cuspidal, Siegel modular forms of arbitrary genus in [BD22]. We refer the reader to the introductions in [AD19; Sah13] for a survey of similar vanishing results. Our first main theorem, is an analogue of Zagier's Theorem for D_4 modular forms.

Theorem 1.1. *Let φ be quaternionic modular forms on G of weight $\ell > 0$ and level one such that $\Lambda_\varphi[B] = 0$ for all primitive $B \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$. Then $\varphi = 0$. Here we say that a pair $B \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$ is primitive if $\mathbf{Q}$-span$\{B\} \cap \mathrm{M}_2(\mathbf{Z})^{\oplus 2} = \mathbf{Z}$-span$\{B\}$.*

In the setting of quaternionic modular forms on G_2, [GGS02, Theorem 16.12], gives an analogue of Theorem 1.1, showing that a level one cuspidal quaternionic Hecke eigenform on

G_2 is zero if and only if its primitive Fourier coefficients are zero. Our proof of Theorem 1.1, which does not generalize to modular forms on G_2, utilizes certain *Fourier-Jacobi coefficients* for D_4 modular forms. To define these coefficients, let $R = M_R N_R$ be a maximal parabolic subgroup of G corresponding to an outer vertex in the Dynkin diagram of G, and write N_R for the abelian unipotent radical of R. Given a character $\chi\colon N_R(\mathbf{Q})\backslash N_R(\mathbf{A}) \to \mathbf{C}$, the χ Fourier-Jacobi coefficient of φ is

$$\mathcal{F}(\varphi;\chi)(g) = \int_{N_R(\mathbf{Q})\backslash N_R(\mathbf{A})} \varphi(ng)\chi^{-1}(n)\,dn.$$

When φ is cuspidal, the coefficient $\mathcal{F}(\varphi;\chi)$ is studied for a specific non-degenerate character χ in [Joh+24, Corollary 7.6]. In [Pol24, Theorem 7.12], a generalization of $\mathcal{F}(\varphi;\chi)$ is analyzed for non-degenerate χ, when φ is a cuspidal quaternionic modular on certain exceptional groups in types F_4, E_6, E_7, and E_8. In the present work, we study the coefficients $\mathcal{F}(\varphi;\chi)$ for a general non-trivial character χ, without assuming φ is cuspidal.

To prove Theorem 1.1, we apply the coefficients $\mathcal{F}(\varphi;\chi)$ to adapt the original proof of Zagier's Theorem [Zag81, pg. 387] into the setting of D_4 modular form. The implementation of this proof strategy on G presents several novelties.

Firstly, the analytic properties of the non-degenerate Fourier Jacobi coefficients are significantly more difficult to analyze in the case of D_4-modular forms. Indeed, the classical Fourier-Jacobi coefficients of a Siegel modular form F easily inherent their holomorphy properties from those of F. Similarly, the non-degenerate Fourier-Jacobi coefficients $\mathcal{F}(\varphi;\chi)$ obey certain holomorphy properties, however, these properties are more difficult to prove since φ is not holomorphic (see Proposition 6.7). In particular, our analysis of the non-degenerate coefficients $\mathcal{F}(\varphi;\chi)$ depends on the explicit formula for the generalized Whittaker function of quaternionic $G(\mathbf{R})$-representation proven in [Pol20, Theorem 1.2].

Secondly, the unipotent radical N_R supports non-trivial degenerate characters, and the degenerate coefficients $\mathcal{F}(\varphi;\chi)$ must be analyzed separately from the non-degenerate coefficients. This makes the proof of Theorem 1.1 more challenging in the case when φ is non-cuspidal. More specifically, to prove Theorem 1.1 for non-cuspidal φ, we establish a new relationship between the degenerate Fourier coefficients $\Lambda_\varphi[B]$ of φ, and the Fourier coefficients of the holomorphic modular form Φ (see Proposition 6.5). In Subsection 8.2, we show that this relationship is true in the generality of quaternionic modular forms on the group G_J associated to a cubic norm structure J.

Remark 1.2. Yamana [Yam09] has shown that the proof of Zagier's Theorem is broadly applicable to holomorphic modular forms on classical groups. Likewise, by making use of the orthogonal Fourier Jacobi expansion studied in [Pol24], the forthcoming work [McG26] generalizes our proof of Theorem 1.1 to the quaternionic modular forms on the group G_J. It is natural to consider Theorem 1.1 first in the setting of the group G on account of Theorem 1.5, which applies to the quaternionic Maass Spezialschar of [Joh+24], and has no known analogue outside of type D_4 (see Remark 1.6). Another reason to focus on type D_4 is because the arithmetic invariant theory of $\mathrm{SL}_2(\mathbf{Z})^3$ acting $\mathrm{M}_2(\mathbf{Z})^{\oplus 2}$ is particularly rich. More specifically, we have the following conjecture, which is unique to type D_4.

Conjecture 1.3. *Let φ be a cuspidal quaternionic modular forms on G of weight $\ell > 0$ and level one such that $\Lambda_\varphi[B] = 0$ for all projective $B \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$. Then $\varphi = 0$. Here the term projective has the same meaning as in [Bha04, §2.3, pg. 221].*

1.3. **An Application to The Quaternionic Maass Spezialschar.** Let SO_8 denote the split special orthogonal group of rank 4 over $\mathbf{Q}$. Then SO_8 supports a theory of quaternionic modular forms which is closely related to the theory of modular forms on G. In fact, if φ is a quaternionic modular form on SO_8 of level 1, then φ is uniquely determined by its pull back to G (see Lemma 7.2), which is a quaternionic modular form on G. As such, a level one quaternionic modular form φ on SO_8 has an associated set of Fourier coefficients $\{\Lambda_\varphi[B] \colon B \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2} \backslash \{[0,0]\}\}$, for which the statement of Theorem 1.1 holds true.

In [Pol21], Pollack studies the theta correspondence arising from the dual pair $\mathrm{Sp}_4 \times \mathrm{O}_8$, and describes the lifting of a level one holomorphic modular form F on Sp_4, to a level one quaternionic modular form $\theta^*(F)$ on SO_8. The Fourier coefficients of the *quaternionic Saito-Kurokawa lift* $\theta^*(F)$ are given as linear combinations of the Fourier coefficients of F [Pol21, Theorem 4.1.1]. In [Joh+24], the authors characterize the quaternionic modular forms $\theta^*(F)$ using a system of linear equations. This result is analogous to the characterization of the classical Saito-Kurokawa subspace via the *Maass Relations*; see for example [EZ85, §6, (9)].

In Section 7, we refine [Joh+24, Theorem 1.3], to characterize the modular forms $\theta^*(F)$ in the style of [Zag81, Theorem 1 (iii)], which states that a level one, genus 2 Siegel modular form F_φ is a Saito-Kurokawa lift if and only if each primitive Fourier coefficient $A_\varphi[T]$ only depends on the discriminant $\mathrm{disc}(T)$. Our characterization uses a construction of [Bha04], which associates $B = [T_1, T_2]$ with the binary quadratic form in variables x and y given by,

$$T(B) = \det(xT_1 - yT_2).$$

We also require a more refined notion of primitivity (see [Joh+24, Definition 5.4]).

Definition 1.4. Say that $[T_1, T_2] \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$ is *strongly primitive* or *slice primitive* if

$$\mathbf{Q}\text{-span}\{T_1, T_2\} \cap \mathrm{M}_2(\mathbf{Z}) = \mathbf{Z}\text{-span}\{T_1, T_2\}.$$

As progress toward Conjecture 1.3, we establish Corollary 6.11, showing that a cuspidal D_4 modular form φ vanishes if and only if the slice primitive Fourier coefficients of φ are zero. Combining [Joh+24, Theorem 1.3] and Corollary 6.11 yields the following application.

Theorem 1.5. *Suppose $\ell \geq 16$ is even and let φ be a level one, cuspidal, quaternionic modular form on SO_8 of weight ℓ. The following are equivalent:*

(i) There exists a level one, weight ℓ, Siegel modular form F on Sp_4 such that $\varphi = \theta^*(F)$.
(ii) If $B, B' \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$ are slice primitive and satisfy $T(B) = T(B')$, then $\Lambda_\varphi[B] = \Lambda_\varphi[B']$.

The problem of characterizing theta lifts is well studied within the general theory of automorphic forms. For example, in the case when φ is a certain type of quaternionic Hecke eigenform on G, condition (a) above is equivalent to the condition that φ admits a non-zero period along an embedded copy of SO_6 in SO_8 [Joh+24, Corollary 9.8]. As such, Theorem 1.5 posits an indirect connection between the slice primitive Fourier coefficients $\Lambda_\varphi[B]$, and the study of automorphic period integrals. Further connections to the properties of automorphic L-functions are available via the Rallis inner product formula; see for example [Yam14].

Remark 1.6. Theorem 1.5 seems to be particular to the case of quaternionic modular forms on SO_8. Indeed, if $n \geq 2$, then [Pol21, Theorem 4.1.1] describes a more general situation, involving a class of quaternionic modular forms on $\mathrm{SO}(4, n+2)$ that are theta lifts from Sp_4. It appears that these theta lifts on $\mathrm{SO}(4, n+2)$ only admit a characterization in the style of Theorem 1.5 in the special case when $n = 2$. Indeed, if χ is a non-degenerate character of N_R, then the stabilizer of χ in M_R^{der} is identified with $\mathrm{Spin}(2,3)$, and $\mathrm{Spin}(2,3) \simeq \mathrm{Sp}_4$ via an exceptional isomorphism. The proof of Theorem 1.5 makes essential use of this identification.

1.4. **Characterizing Quaternionic Cusp Forms via a Hecke Bound.** A central question in the theory of automorphic forms is the problem of determining whether a given automorphic function is cuspidal. This question interacts in fascinating ways with theories of scalar Fourier coefficients. For example, assume F_φ is a level one, genus 2 Siegel modular form of weight $\ell \geq 4$. Then, Kohnen-Martin [KM14, Theorem 2.1] prove φ is cuspidal if and only if $A_\varphi[T] \ll_F |\mathrm{disc}(T)|^{\ell/2}$ for all positive definite forms T. The aforementioned theorem has an antecedent in [Koh10], where it shown that certain elliptic cusp forms are similarly characterized by the sizes of their Fourier coefficients. These results relate two fundamental properties of modular forms; the growth of their Fourier coefficients, and cuspidality. As such, they are of broad interest. For example, in [BD14], the authors reprove [KM14, Theorem 2.1] by a different method, and extend the result to Siegel modular forms of higher genus. We recommend the introduction to [Das25] for an overview of related work. In Section 8, we establish an analogue of [KM14, Theorem 2.1].

Theorem 1.7. *Suppose φ is a level one, quaternionic modular form on G of weight $\ell \geq 5$. Then φ is cuspidal if and only if, for all $B \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$ satisfying $\mathrm{disc}(T(B)) < 0$,*

$$\Lambda_\varphi[B] \ll_\varphi |\mathrm{disc}(T(B))|^{\frac{\ell+1}{2}}. \tag{1.1}$$

To the author's knowledge, Theorem 1.7 is the first instance of a result characterizing the cuspidality of quaternionic modular forms, on any group, via the growth of their Fourier coefficients. With that having been said, in [GGS02, Proposition 8.6], the authors establish the bound (1.1) for cuspidal quaternionic modular forms on G_2. In Proposition 5.8, we generalize the technique of (loc. cit.) to establish the bound (1.1) for all cuspidal D_4 modular forms. To prove the converse implication of Theorem 1.7, the Fourier-Jacobi coefficients $\mathcal{F}(\varphi; y)$ are of indispensable use, as are the ideas underlying the proof of [KM14, Theorem 2.1]. The proof of Theorem 1.7 also requires a general cuspidality criterion for quaternionic modular forms (Theorem 1.8), which we believe is of independent interest.

To set up Theorem 1.8, let C denote a composition algebra over $\mathbf{Q}$, and write $H_3(C)$ for the cubic norm structure consisting of 3×3 Hermitian matrices with entries in C. Let $J = \mathbb{G}_a^3$ or $J = H_3(C)$, and write G_J for the $\mathbf{Q}$-rational algebraic group associated to J in [Pol20]. So, G_J is an adjoint group of real rank 4 and type D_4, F_4, E_6, E_7, or E_8. Let P_J be the Heisenberg parabolic subgroup of G_J, and write N_J for the unipotent radical of P_J.

Theorem 1.8. *Suppose φ is a quaternionic modular form on G_J. Write φ_{N_J} for the constant term of φ along the unipotent radical N_J. If $\varphi_{N_J} \equiv 0$, then φ is cuspidal.*

Theorem 1.8 has application to the study of algebraicity results. For example, in the forthcoming work [Hu+26], the authors apply Theorem 1.8 to prove that the Fourier coefficients of certain degenerate Heisenberg series on G_J are algebraic numbers.

For the proof of Theorem 1.8, we assume J is any cubic norm structure with a positive definite trace pairing. As a consequence of [Pol20, Proposition 11.1.1.], if φ is a quaternionic modular form on G_J, then φ_{N_J} gives rise to a holomorphic modular form Φ on a Levi subgroup of P_J. In Proposition 8.3, we show that the Fourier coefficients of Φ are given as finite sums of the degenerate Fourier coefficients of φ. Conversely, if J is a cubic norm structure of the type appearing in Theorem 1.8, then Lemma 8.5 and Lemma 8.6 imply that every degenerate Fourier coefficient of φ is a finite sum of Fourier coefficients of Φ. Hence, in the setting of Theorem 1.8, the vanishing of φ_{N_J} implies that all degenerate Fourier coefficients of φ are zero, which can used to deduce the cuspidality of φ (see Subsection 8.3).

Remark 1.9. Our proof of Theorem 1.7 does not directly generalize to quaternionic modular forms in other Dynkin types. This is because we use a unique features of D_4 modular form. Namely, in Proposition 8.1, we show that if φ is a level one, non-cuspidal, D_4 modular form, then φ has a non-zero, primitive, rank 3 Fourier coefficient. We refer the reader to [Pol18, Definition 4.3.2] for the definition of rank in this setting. In forthcoming work [McG26], we combine Theorem 1.8, with the results of [Pol24], to give a uniform proof of Theorem 1.7 in types D_4, F_4, E_6, E_7, and E_8. The benefit of Proposition 8.1 is that it implies a stronger version of Theorem 1.7. More specifically, in Theorem 8.8 we show that a level one D_4 modular form φ is cuspidal if, for all primitive $B \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$ satisfying $\mathrm{disc}(T(B)) < 0$, $\Lambda_\varphi[B] \ll_\varphi |\mathrm{disc}(T(B))|^{\frac{\ell+1}{2}}$. In other words, the cuspidality of φ is characterized by the growth of the primitive, non-degenerate Fourier coefficients $\Lambda_\varphi[B]$.

1.5. **The structure of paper.** Section 3 fixes notation and recalls basic properties of the group G. In Section 4 we review preliminaries on holomorphic modular forms. Section 5 introduces quaternionic modular forms on G and recalls the structure of their scalar Fourier coefficients. The orthogonal Fourier-Jacobi expansion for quaternionic modular forms on G is developed in Section 6, where the vanishing criterion of Theorem 1.1 is proven. Section 7 applies this result to yield Theorem 1.5. Finally, in Section 8, we analyze the degenerate Fourier coefficients of quaternionic modular forms on G_J, and prove Theorem 1.8 and Theorem 1.7 .

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3. Algebraic Preliminaries

3.1. **Standard Notation.** The symbols $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{C}$, and $\mathbf{R}$ denote the rings of integers, rational numbers, complex numbers, and real numbers respectively. Given a prime number p, $\mathbf{Z}_p$ and $\mathbf{Q}_p$ are the rings of p-adic integers and p-adic numbers respectively. Let $\mathbf{A}$ denote the adele ring of $\mathbf{Q}$, and $\psi\colon \mathbf{A}/\mathbf{Q} \to \mathbf{C}^\times$ be a fixed non-trivial additive character of $\mathbf{A}/\mathbf{Q}$.

3.2. **The Quadratic Space V.** Throughout, (V, q) denotes a non-degenerate split quadratic space over $\mathbf{Q}$ of dimension 8. Write $(x, y) = q(x + y) - q(x) - q(y)$ for the bilinear form associated to q and let

$$\{b_1, b_2, b_3, b_4, b_{-4}, b_{-3}, b_{-2}, b_{-1}\}$$

be a basis of V consisting of isotropic vector $b_{\pm i}$, $i = 1, \ldots 4$. We choose this basis so that $(b_{\pm i}, b_{\pm j}) = 0$ and $(b_{\pm i}, b_{\mp j}) = \delta_{i,j}$ for all $i, j = 1, \ldots, 4$. Here $\delta_{i,j}$ denotes the Kronecker delta symbol. Next, we let $U \subseteq V$ be the isotropic two-plane spanned by $\{b_1, b_2\}$, and $U^\vee$

denote the isotropic two plane spanned by $\{b_{-1}, b_{-2}\}$. Write $V_{2,2}$ to denote the orthogonal complement of $U + U^\vee$ in V. So $(V_{2,2}, q)$ is a split quadratic space of signature $(2, 2)$ and

$$V = U \oplus V_{2,2} \oplus U^\vee. \tag{3.1}$$

Later, it will be convenient to identify $V_{2,2}$ with the space of 2×2 matrices M_2 over $\mathbf{Q}$ with $q|_{V_{2,2}}$ equal to the determinant of M_2. We form this identification $V_{2,2} \xrightarrow{\sim} \mathrm{M}_2$ via the map

$$b_3 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b_{-3} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad b_4 \mapsto \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad b_{-4} \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.2}$$

Similarly, we let $V_{3,3}$ denote the orthogonal complement of $\mathbf{Q}b_1 + \mathbf{Q}b_{-1}$ inside V. So V admits an orthogonal decomposition

$$V = \mathbf{Q}b_1 \oplus V_{3,3} \oplus \mathbf{Q}b_{-1}, \tag{3.3}$$

where $V_{3,3} = (\mathbf{Q}b_1 + \mathbf{Q}b_{-1})^\perp$ is a split quadratic space of signature $(3, 3)$.

3.3. **The Group G.** Throughout the paper, $G = \mathrm{Spin}(V)$ denotes the spin group associated to the split 8-dimensional quadratic space V. We fix a covering homomorphism $\pi \colon G \to \mathrm{SO}(V)$. Consider embedding of $\mathrm{SO}(V)$-modules,

$$\bigwedge^2 V \to \mathrm{End}(V) \tag{3.4}$$

sending $v_1 \wedge v_2 \in \bigwedge^2 V$ to the endomorphism $v_1 \wedge v_2 \cdot x = (x, v_2)v_1 - (x, v_1)v_2$ where $x \in V$. Then (3.4) induces an identification between $\wedge^2 V$ and $\mathrm{Lie}(\mathrm{SO}(V))$. Furthermore, (3.4) promotes to an identification of Lie algebras where the bracket on $\wedge^2 V$ is defined by

$$[v \wedge w, v' \wedge w'] = (v \wedge w \cdot v') \wedge w' + v' \wedge (v \wedge w \cdot w'). \tag{3.5}$$

As such, $\wedge^2 V$ gives a model for the Lie algebra of G, in which the adjoint action is described by $g \cdot (v \wedge w) = gv \wedge gw$ for $g \in G$ and $v, w \in V$.

The Lie algebra $\mathrm{Lie}(G)$ admits another model $\mathfrak{g}_E$, defined via an algebraic structure E known as a cubic norm structure. We refer the reader to [Pol18, §4.2] for the general definition of cubic norm structures. For our purposes, an important example will be the cubic norm structure, $E = \mathbf{G}_a^3$, which consists of 3 by 3 diagonal matrices (z_1, z_2, z_2).

As a subset of the space of 3 by 3 matrices, E naturally carries a multiplicative structure. The cubic norm structure on E is specified by the choice of base point $1_E = (1, 1, 1)$, the norm $N_E \colon E \to \mathbf{G}_a$, given by $N_E(Z) = \det(Z)$, and the adjoint on E, which is the quadratic map $(\cdot)^\# \colon E \to E$ given by $(z_1, z_2, z_3)^\# = (z_2z_3, z_3z_1, z_1z_2)$. For $Z, Z' \in E$ one sets $Z \times Z' = (Z + Z')^\# - Z^\# - (Z')^\#$. We define a trace pairing $(Z, Z')_E = z_1z_1' + z_2z_2' + z_3z_3'$, which induces an identification $E^\vee = E$.

Let $V_3 = \mathrm{Span}\{e_1, e_2, e_3\}$ be the standard representation of SL_3, and $V_3^\vee = \mathrm{Span}\{\delta_1, \delta_2, \delta_3\}$ its dual. Then V_3 and $V_3^\vee$ are endowed with an action of the Lie algebra $\mathfrak{sl}_3$. Let E^0 be the trace 0 subspace of E. So E^0 consists of elements $Z \in E$ such that $(Z, 1_E)_E = 0$. Equivalently, we view elements of E^0 as endomorphism of E, where an element $u \in E^0$ corresponds to the endomorphism $\Psi_u \colon E \to E$, which is given by $x \mapsto ux$.

As a vector space, $\mathfrak{g}_E$ is given as

$$\mathfrak{g}_E = (\mathfrak{sl}_3 \oplus E^0) \oplus (V_3 \otimes E) \oplus (V_3^\vee \otimes E^\vee) \tag{3.6}$$

We refer the reader to [Pol20, §4] for the definition of the Lie bracket on $\mathfrak{g}_E$.

3.4. **The Heisenberg Parabolic Subgroup P.** Let U (resp. $U^\vee$) denote the isotropic subspace of V with basis $\{b_1, b_2\}$ (resp. $\{b_{-2}, b_{-1}\}$). The Heisenberg parabolic subgroup $P \leq G$ is defined as the stabilizer

$$P = \mathrm{Stab}_G(U).$$

The Levi decomposition of P takes the form $P = M_P N_P$. Here N_P is the unipotent radical of P and M_P is the Levi factor $M_P = \mathrm{Stab}_P(U^\vee)$. Equivalently, P is the parabolic subgroup associated to the grading on $\mathrm{Lie}(G)$ defined by the element $h_P := b_1 \wedge b_{-1} + b_2 \wedge b_{-2}$. That is, $\mathrm{ad}(h_P)$ has eigenvalues $-2, -1, 0, 1, 2$ on $\mathrm{Lie}(G)$, $\mathrm{Lie}(M_P)$ is the 0 eigenspace of $\mathrm{ad}(h_P)$, and $\mathrm{Lie}(N_P)$ is the direct sum of the 1 and 2 eigenspaces of $\mathrm{ad}(h_P)$.

Using the definition (3.5), $\mathrm{Lie}(M_P) = U \wedge U^\vee + V_{2,2} \wedge V_{2,2}$, and so

$$\mathrm{Lie}(M_P) = \mathrm{Lie}(\mathrm{GL}(U)) \oplus \mathrm{Lie}(\mathrm{SO}(V_{2,2})).$$

Since G is semi-simple and simply connected, the same is true of the derived subgroup M_P^{der} [Spr98, 8.4.6(6)]. Therefore, since $\mathrm{Lie}(M_P^{\mathrm{der}}) = \mathrm{Lie}(\mathrm{SL}(U)) \oplus \mathrm{Lie}(\mathrm{SO}(V_{2,2})$,

$$M_P^{\mathrm{der}} = \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 .$$

Since $\mathrm{Lie}(N_P) = U \wedge U + U \wedge V_{2,2}$, the center $Z \leq N_P$ equals the $\lambda = 2$ eigenspace of h_P. Thus $\mathrm{Lie}(Z) = \mathbf{Q}\text{-span}\{b_1 \wedge b_2\}$, $[N_P, N_P] = Z$, and the Lie algebra of $N_P^{\mathrm{ab}} := N_P/Z$ is $\mathrm{Lie}(N_P^{\mathrm{ab}}) = \{b_1 \wedge v + b_2 \wedge v' + \mathrm{Lie}(Z) \colon v, v' \in V_{2,2}\}$. If $T_1, T_2 \in V_{2,2}$, let

$$\varepsilon_{[T_1,T_2]} \colon N_P(\mathbf{Q})\backslash N_P(\mathbf{A}) \to \mathbf{C}^\times \tag{3.7}$$

be the character associated to the functional $b_1 \wedge v + b_2 \wedge v' \mapsto (T_1, v) + (T_2, v')$.

As a module over $M_P(\mathbf{Q})$, the character lattice of $N_P(\mathbf{Q})\backslash N_P(\mathbf{A})$ is

$$U^\vee \otimes V_{2,2} \xrightarrow{\sim} \mathrm{Hom}(N_P(\mathbf{Q})\backslash N_P(\mathbf{A}), \mathbf{C}^1), \qquad b_{-1} \otimes T_1 + b_{-2} \otimes T_2 \mapsto \varepsilon_{[T_1,T_2]}. \tag{3.8}$$

Via the isomorphism [Joh+24, Theorem A.8], $\mathrm{Lie}(N_P^{\mathrm{ab}})$ is identified with the subalgebra of $\mathfrak{g}_E$ given by $\mathrm{Lie}(N_P^{\mathrm{ab}}) = \mathbf{Q}E_{12} + e_1 \otimes E + \delta_3 \otimes E^\vee + \mathbf{Q}E_{21}$, and $\mathrm{Lie}(Z) = \mathbf{Q}E_{13}$.

Consider, $W_E := \mathbf{Q} \otimes E \oplus E^\vee \oplus \mathbf{Q}$ as a symplectic vector via the form

$$\langle (a, b, c, d), (a', b', c', d') \rangle_{W_E} = ad' - (b, c')_E + (c, b')_E - a'd,$$

where $a, a', d, d' \in \mathbf{Q}$, $b, b' \in E$, and $c, c' \in E^\vee$. Given $w \in W_E$, let ε_w denote the character of N_P associated to the functional $aE_{12} + e_1 \otimes b + \delta_3 \otimes c + dE_{21} \mapsto \langle w, (a, b, c, d) \rangle_{W_E}$. Then $w \mapsto \varepsilon_w$ induces a second identification

$$W_E \xrightarrow{\sim} \mathrm{Hom}(N_P(\mathbf{Q})\backslash N_P(\mathbf{A}), \mathbf{C}^1). \tag{3.9}$$

By [Joh+24, §7.4], under the isomorphism $W_E \simeq U^\vee \otimes V_{2,2}$ induced by (3.8) and (3.9), if $w = (a, \mathrm{diag}(\beta_1, \beta_2, \beta_3), \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3), d) \in W_E$ then, ε_w is identified with $\varepsilon_{[T_1,T_2]}$ where

$$[T_1, T_2] = [\gamma_1 b_{-4} - \beta_2 b_{-3} + d b_3 + \gamma_3 b_4, -\beta_3 b_{-4} + a b_{-3} - \gamma_2 b_3 - \beta_1 b_4]. \tag{3.10}$$

3.5. **The Orthogonal Parabolic Subgroup R.** Let R denote the parabolic subgroup of G that stabilizes the line $\mathbf{Q}b_1$. Then R admits a Levi decomposition $R = M_R N_R$ where M_R is the subgroup of R stabilizing the line $\mathbf{Q}b_{-1}$, and $N_R \trianglelefteq R$ denotes the unipotent radical of R. Equivalently, R is the parabolic subgroup of G associated to the element $h_R = b_1 \wedge b_{-1}$, which acts on $\mathrm{Lie}(G)$ with eigenvalues $-1, 0, 1$. Since $\mathrm{Lie}(M_R) = b_1 \wedge b_{-1} + \bigwedge^2 V_{3,3}$ equals to the $\lambda = 0$ eigenspace of $\mathrm{ad}(h_R)$,

$$\mathrm{Lie}(M_R) = \mathrm{Lie}(\mathrm{GL}(\mathbf{Q}b_1)) \oplus \mathrm{Lie}(\mathrm{SO}(V_{3,3})). \tag{3.11}$$

Hence, $M_R^{\mathrm{der}} \simeq \mathrm{Spin}(V_{3,3})$, and the identification $\mathrm{Lie}(N_R) = b_1 \wedge V_{3,3}$ gives an M_R^{der}-module isomorphism between $V_{3,3}$ and the character lattice of $N_R(\mathbf{Q})\backslash N_R(\mathbf{A})$. Precisely, we have

$$V_{3,3}(\mathbf{Q}) \xrightarrow{\sim} \mathrm{Hom}(N_R(\mathbf{Q})\backslash N_R(\mathbf{A}), \mathbf{C}^\times), \quad y \mapsto \chi_y \tag{3.12}$$

where $\chi_y(\exp(b_1 \wedge v)) = \psi((v, y))$ for all $v \in V_{3,3}(\mathbf{A})$ and $y \in V_{3,3}(\mathbf{Q})$.

3.6. **The Siegel Parabolic Subgroup Q'.** In this subsection we assume that $y \in V_{3,3}$ is orthogonal to the plane $\mathbf{Q}$-span$\{b_2, b_{-2}\}$ and satisfies $(y, y) \neq 0$. Define $V'_{2,3}$ to be the complement of $\mathbf{Q}y$ inside $V_{3,3}$, so that $V'_{2,3}$ is a rational quadratic space of signature $(2, 3)$. The group $M' = M'_y$ is defined as

$$M' = \mathrm{Stab}_{\mathrm{Spin}(V_{3,3})}(y) \simeq \mathrm{Spin}(V'_{2,3}). \tag{3.13}$$

Since y is orthogonal to $\mathbf{Q}b_2$, M' contains a parabolic subgroup Q' defined as the stabilizer in M' of the line $\mathbf{Q}b_2$. Write $Q' = M_{Q'}N_{Q'}$ for the Levi decomposition of Q' where

$$M_{Q'} = \mathrm{Stab}_{Q'}(\mathbf{Q}b_{-2}).$$

We let $V'_{1,2}$ denote the orthogonal complement of $\mathbf{Q}b_2 + \mathbf{Q}y + \mathbf{Q}b_{-2}$ inside of $V_{3,3}$. Then $M_{Q'}$ acts on $V'_{2,3}$ preserving the decomposition $V'_{2,3} = \mathbf{Q}b_2 + V'_{1,2} + \mathbf{Q}b_{-2}$.

The unipotent radical $N_{Q'} \trianglelefteq Q'$ is abelian, and may be identified as a module over the derived subgroup $M_{Q'}^{\mathrm{der}} \simeq \mathrm{Spin}(V'_{1,2})$ according to the map $V'_{1,2} \xrightarrow{\sim} N_Q$, $v \mapsto \exp(b_2 \wedge v)$. As such, $\mathrm{Hom}(N_{Q'}(\mathbf{Q})\backslash N_{Q'}(\mathbf{A}), \mathbf{C}^1)$ is identified with $V'_{1,2}(\mathbf{Q})$ via

$$V'_{1,2}(\mathbf{Q}) \xrightarrow{\sim} \mathrm{Hom}(N_{Q'}(\mathbf{Q})\backslash N_{Q'}(\mathbf{A}), \mathbf{C}^1), \qquad S \mapsto \varepsilon_{[0,S]}. \tag{3.14}$$

Here the character $\varepsilon_{[0,S]}$ is defined in (3.7).

3.7. **Compact Subgroups.** Let V^+ be the definite 4 plane in $V(\mathbf{R})$ spanned by the vectors

$$u_1 = \frac{1}{\sqrt{2}}(b_1 + b_{-1}), \quad u_2 = \frac{1}{\sqrt{2}}(b_2 + b_{-2}), \quad v_1 = \frac{1}{\sqrt{2}}(b_3 + b_{-3}), \quad v_2 = \frac{1}{\sqrt{2}}(b_4 + b_{-4})$$

Set $V^- := (V^+)^\perp$ so that V^- is the negative definite subspace spanned by

$$u_{-1} = \frac{1}{\sqrt{2}}(b_1 - b_{-1}), \quad u_{-2} = \frac{1}{\sqrt{2}}(b_2 - b_{-2}), \quad v_{-1} = \frac{1}{\sqrt{2}}(b_3 - b_{-3}), \quad v_{-2} = \frac{1}{\sqrt{2}}(b_4 - b_{-4}).$$

The maximal compact subgroup $K_\infty \leq G(\mathbf{R})$ is defined as

$$K_\infty = \mathrm{Stab}_{G(\mathbf{R})}(V^+).$$

Then the Lie algebra $\mathfrak{k}_0 = \mathrm{Lie}(K_\infty)$ is the 1 eigenspace of the Cartan involution $\theta\colon \wedge^2 V \to \wedge^2 V$ given by $\theta(b_i \wedge b_{\pm j}) = b_{-i} \wedge b_{\mp j}$. Set $\mathfrak{p}_0$ to be the -1 eigenspace of θ. Then if $\mathfrak{k} = \mathfrak{k}_0 \otimes \mathbf{C}$ and $\mathfrak{p} = \mathfrak{p}_0 \otimes \mathbf{C}$, we obtain a Cartan decomposition of $\mathfrak{g} = \mathrm{Lie}(G) \otimes_{\mathbf{Q}} \mathbf{C}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

At the level of groups, we have a homomorphism

$$\mathrm{Spin}(V^+) \times \mathrm{Spin}(V^-) \to K_\infty, \quad (g, h) \mapsto gh, \tag{3.15}$$

which is induced by the inclusions of Clifford algebras $\mathrm{Cl}(V^\pm) \hookrightarrow \mathrm{Cl}(V(\mathbf{R}))$. The kernel of (3.15) is the diagonally embedded $\mu_2 = \{(1, 1), (-1, -1)\}$. By comparing dimensions, it follows that the derivative of (3.15) is an isomorphism. Hence, $K_\infty^0 = (\mathrm{Spin}(V^+) \times \mathrm{Spin}(V^-))/\mu_2$. Since G is simply connected, and simple, $G(\mathbf{R})$ is connected. Hence, $K_\infty = K_\infty^0$, and $K_\infty = (\mathrm{Spin}(V^+) \times \mathrm{Spin}(V^-))/\mu_2$.

When we define quaternionic modular forms on G in Subsection 5.1, it will be necessary to speak about a distinguished three dimensional representation $\mathbf{V}$ of K_∞. To define $\mathbf{V}$, we make a Lie algebra argument. Following [Joh+24, §3.4], consider the SL_2 triple in $\mathfrak{g}$:

(i) $e^+ = \frac{1}{2}(u_1 - iu_2) \wedge (v_1 - iv_2)$
(ii) $h^+ = i(u_1 \wedge u_2 + v_1 \wedge v_2) = \frac{1}{2}(u_1 - iu_2) \wedge (u_1 + iu_2) + \frac{1}{2}(v_1 - iv_2) \wedge (v_1 + iv_2)$
(iii) $f^+ = -\frac{1}{2}(u_1 + iu_2) \wedge (v_1 + iv_2)$.

Then K_∞ acts on $\mathfrak{sl}_2 = \mathbf{C}\text{-span}\{e^+, h^+, f^+\}$ via the adjoint representation, which defines the representation $\mathbf{V}$. We choose a basis x, y of $\mathbf{C}^2 = V_2$ so that $\mathbf{V}$ is identified with $\mathrm{Sym}^2 V_2$ via $e^+ = -x^2$, $h^+ = 2xy$, $f^+ = y^2$. For $\ell \in \mathbf{Z}_{\geq 1}$, we write $\mathbf{V}_\ell$ for the ℓ^{th} symmetric power of $\mathbf{V}$. So

$$\mathbf{V}_\ell = \mathrm{Sym}^{2\ell}(V_2), \tag{3.16}$$

which has a basis $x^{2\ell}, x^{2\ell-1}y, \ldots, xy^{2\ell-1}, y^{2\ell}$. The Lie algebra $\mathfrak{so}(V^+)$ contains a second SL_2 triple $\mathfrak{sl}_2'$, which is obtained by replace v_2 with $-v_2$ in the definition of $\mathfrak{sl}_2$. Then $\mathfrak{so}(V^+) = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2'$. Similarly, $\mathfrak{so}(V^-)$ admits an orthogonal decomposition as $\mathfrak{so}(V^-) = \mathfrak{sl}_2'' + \mathfrak{sl}_2'''$ where $\mathfrak{sl}_2''$ (resp. $\mathfrak{sl}_2'''$) is obtain by replacing u_i with u_{-i} and v_i with v_{-i} in the definition of $\mathfrak{sl}_2$ (resp. $\mathfrak{sl}_2'$). In this way, we present $\mathfrak{k} = \mathfrak{k}_0 \otimes_{\mathbf{R}} \mathbf{C}$ as

$$\mathfrak{k} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2' \oplus \mathfrak{sl}_2'' \oplus \mathfrak{sl}_2'''. \tag{3.17}$$

Write $L \leq K_\infty$ to denote the subgroup of K_∞ with Lie algebra $\mathfrak{sl}_2' + \mathfrak{sl}_2'' + \mathfrak{sl}_2'''$. Then

$$K_\infty = (\mathrm{SU}(2) \times L)/\mu_2.$$

and $L = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$. The embedding $\mu_2 \hookrightarrow \mathrm{SU}(2) \times L$ sends -1 to $(-1, -1, -1, -1)$. Thus if V_2 denotes the standard two dimensional representation of $\mathrm{SU}(2)$ and $W = V_2 \otimes V_2 \otimes V_2$, then $V_2 \boxtimes W$ gives a representation representation of K_∞, and as K_∞-module, we have

$$\mathfrak{p} = V_2 \boxtimes W. \tag{3.18}$$

4. Preliminaries on Holomorphic Modular Forms

4.1. **Holomorphic modular forms on M'.** Suppose $y \in V_{3,3}$ satisfies $(y, y) > 0$. Recall the orthogonal parabolic subgroup $R = M_R N_R$ and the subgroup $M' = \mathrm{Spin}(V_{2,3}')$ of Subsection 3.5. Let $\widehat{y} = y/\sqrt{(y,y)}$ and let $g_y \in G(\mathbf{R})$ be any element satisfying

$$g_y \cdot v_1 = \widehat{y}, \quad g_y \cdot u_{\pm 1} = u_{\pm 1}, \quad g_y \cdot u_{\pm 2} = u_{\pm 2}, \quad g_y \cdot v_{\pm 2} = v_{\pm 2}.$$

Then $g_y K_\infty g_y^{-1}$ is the maximal compact subgroup in $G(\mathbf{R})$ stabilizing the 4-plane spanned by $\{u_1, u_2, \widehat{y}, v_2\}$. Let $K_y := (g_y K_\infty g_y^{-1}) \cap M'(\mathbf{R})$ be the maximal compact subgroup of $M'(\mathbf{R})$ stabilizing the subspace $\mathbf{R}\text{-span}\{u_2, v_2\}$ and write $\mathbf{V}_\ell^y$ to denote the representation of K_y on the vector space $\mathbf{V}_\ell$ obtained by $k \cdot v = g_y^{-1} k g_y \cdot v$.

Recall that $V_{1,2}'$ denotes the complement of $\mathbf{Q}b_2 + \mathbf{Q}y + \mathbf{Q}b_{-2}$ in $V_{3,3}$. Then the symmetric domain $M'(\mathbf{R})/K_y$ is identified with the complex manifold

$$\mathfrak{h}_y' = \{X + iY \in V_{1,2}' \otimes_{\mathbf{Q}} \mathbf{C} : (Y, -\sqrt{2}v_2) > 0 \text{ and } (Y, Y) > 0\}.$$

To identify $M'(\mathbf{R})/K_y$ with $\mathfrak{h}_y'$, recall that $V_{2,3}'$ denotes the complement of $\mathbf{Q}\text{-span}\{y\}$ in $V_{3,3}$. Then ${\mathfrak{h}'}_y$ maps into the subspace of isotropic elements in $V_{2,3}' \otimes_{\mathbf{Q}} \mathbf{C}$ via the map

$$\mathfrak{h}_y' \to V_{2,3}' \otimes_{\mathbf{Q}} \mathbf{C}, \qquad Z \mapsto r(Z) := -q(Z)b_2 + Z + b_{-2}. \tag{4.1}$$

This yields an action of the identity component $M'(\mathbf{R})$ on $\mathfrak{h}'_y$ as follows: If $g \in M'(\mathbf{R})$, then there exists a unique element $j_y(g, Z) \in \mathbf{C}^\times$ and a unique element $gZ \in \mathfrak{h}'_y$ so that

$$gr(Z) = j_y(g, Z)r(gZ).$$

Observe that $j_y(g, Z) = (gr(Z), b_2)$ and K_y is the stabilizer of $i(-\sqrt{2}v_2)$ in $M'(\mathbf{R})$. Moreover, $j_y(\cdot, -i\sqrt{2}v_2) : K_y \to \mathbf{C}^\times$ is a character.

Later, we will specialize to the case when $y = b_3 + \alpha b_{-3}/2$ with $\alpha \in 2\mathbf{Z}_{>0}$. In this case we have explicit coordinates on $\mathfrak{h}'_y$ given as follows. Let $y_\alpha^\vee = b_3 - \alpha b_{-3}/2$ so that

$$V'_{1,2} = \mathbf{Z}\text{-span}\{b_4, y_\alpha^\vee, b_{-4}\}.$$

Then a general vector $Z \in V'_{1,2} \otimes_{\mathbf{Q}} \mathbf{C}$ takes the form

$$Z = -\tau' b_4 + z y_\alpha^\vee - \tau b_{-4} \tag{4.2}$$

where $\tau', \tau, z \in \mathbf{C}$. A short computation shows that with notation as in (4.2), $Z \in \mathfrak{h}'_y$ if and only if $\mathrm{Im}(\tau) > 0$, $\mathrm{Im}(\tau') > 0$, and $\mathrm{Im}(\tau) \cdot \mathrm{Im}(\tau') - \dfrac{\alpha}{2}\mathrm{Im}(z)^2 > 0$.

Definition 4.1. Suppose $\ell \in \mathbf{Z}$, and $\Gamma \subseteq M'(\mathbf{R})$ is a discrete subgroup. We say that $f : \mathfrak{h}_y \to \mathbf{C}$ is a holomorphic modular form of weight ℓ and level Γ if:

(1) f is holomorphic,
(2) $f(\gamma Z) = j_y(\gamma, Z)^\ell f(Z)$ for all $Z \in \mathfrak{h}_y$ and $\gamma \in \Gamma$, and
(3) $\xi(g) := j_y(g, -i\sqrt{2}v_2)^{-\ell} f(g \cdot (-i\sqrt{2}v_2)) : M'(\mathbf{R}) \to \mathbf{C}$ is of moderate growth.

Definition 4.2. Suppose $\ell \in \mathbf{Z}$, and $\Gamma \subseteq M'(\mathbf{R})$ is a discrete subgroup. We say that a function $\xi : M'(\mathbf{R}) \to \mathbf{C}$ is the automorphic function associated to a holomorphic modular form of weight ℓ and level Γ if ξ is of moderate growth and satisfies the following conditions:

(1) if $g_\infty \in M'(\mathbf{R})$ and $\gamma \in \Gamma$ then $\xi(\gamma g_\infty) = \xi(g_\infty)$,
(2) if $g_\infty \in M'(\mathbf{R})$ and $k \in K_y$ then $\xi(g_\infty k) = j_y(k, -i\sqrt{2}v_2)^{-\ell}\xi(g_\infty)$, and
(3) if $g_\infty \in M'(\mathbf{R})$ and $Z := g_\infty \cdot (-i\sqrt{2}v_2) \in \mathfrak{h}_y$ then the formula

$$f_\xi(Z) = j_y(g_\infty, -i\sqrt{2}v_2)^\ell \xi(g_\infty)$$

is holomorphic in Z.

4.2. **The Fourier Expansions of Modular Forms on M'.** We continue with the notation of the previous subsection. Thus $y \in V_{3,3}$ denotes a vector in the orthogonal complement to $\mathbf{R}$-span$\{u_2, v_2\}$ satisfying $(y, y) > 0$. Furthermore, we assume that y is orthogonal to b_2, in which case M' contains the parabolic subgroup $Q' = M_{Q'}N_{Q'}$ of Subsection 3.6.

Since the unipotent radical $N_{Q'}$ is abelian, we may apply the identification (3.14) to Fourier expand an automorphic function $\xi\colon M'(\mathbf{Q})\backslash M'(\mathbf{A}) \to \mathbf{C}$ in characters of $N_{Q'}(\mathbf{Q})\backslash N_{Q'}(\mathbf{A})$ as

$$\xi(g) = \xi_{N_{Q'}}(g) + \sum_{S \in V'_{1,2}(\mathbf{Q}):\ S \neq 0} \xi_S(g). \tag{4.3}$$

Here $g \in M'(\mathbf{A})$ and $\xi_S(g) = \displaystyle\int_{N_{Q'}(\mathbf{Q})\backslash N_{Q'}(\mathbf{A})} \xi(ng)\varepsilon_{[0,S]}(n)^{-1}\, dn.$

For the remainder of this subsection, we specialize to the case when $y = nb_3 + n\alpha b_{-3}/2$, where $n \in \mathbf{Z}_{\geq 1}$ and $\alpha \in 2\mathbf{Z}_{>0}$. For this choice of y, the complement of $\mathbf{Q}y$ in $V_{3,3}$ is

$$V'_{2,3} = \mathbf{Q}\text{-span}\,\{b_2, b_4, b_{-4}, y_\alpha^\vee, b_{-2}\}$$

where

$$y_\alpha^\vee = b_3 - \frac{\alpha}{2} b_{-3}.$$

Let $M'(\mathbf{Z}) = M'(\mathbf{Q}) \cap G(\widehat{\mathbf{Z}})$ viewed as a discrete subgroup of $M'(\mathbf{R})$. Regarding the stucture of $M'(\mathbf{Z})$, we record the following elementary result.

Lemma 4.3. *Let* $V'_{1,2}(\mathbf{Z}) = \mathbf{Z}\text{-span}\,\{b_4, y_\alpha^\vee, b_{-4}\}$. *Then*

$$N_{Q'}(\mathbf{Q}) \cap M'(\mathbf{Z}) = \{\exp(b_2 \wedge v) \colon v \in V'_{1,2}(\mathbf{Z})\}. \tag{4.4}$$

Suppose $\xi \colon M'(\mathbf{R}) \to \mathbf{C}$ is the automorphic functions associated to a holomorphic modular form of weight ℓ and level $M'(\mathbf{Z})$. Writing $N_{Q'}(\mathbf{Z}) = N_{Q'}(\mathbf{R}) \cap M'(\mathbf{Z})$, the character lattice of $N_{Q'}(\mathbf{Z}) \backslash N_{Q'}(\mathbf{R})$ is identified with the $\mathbf{Z}$-linear dual of $V'_{1,2}(\mathbf{Z})$, i.e.

$$V'_{1,2}(\mathbf{Z})^\vee = \mathbf{Z}\text{-span}\left\{b_4, \frac{1}{\alpha} y_\alpha^\vee, b_{-4}\right\}.$$

More precisely, we have an $M_{Q'}(\mathbf{Z})$ equivariant identification

$$V'_{1,2}(\mathbf{Z})^\vee \xrightarrow{\sim} \mathrm{Hom}(N_{Q'}(\mathbf{Z}) \backslash N_{Q'}(\mathbf{R}), \mathbf{C}^1), \quad S \mapsto \varepsilon_{[0,S]}. \tag{4.5}$$

Thus if $g_\infty \in M'(\mathbf{R})$, $Z = g_\infty \cdot (-i\sqrt{2}v_2)$, and

$$f_\xi(Z) = j_y(g_\infty, -i\sqrt{2}v_2)^\ell \xi(g_\infty),$$

then f_ξ Fourier expands along $N_{Q'}(\mathbf{Z}) \backslash N'(\mathbf{R})$ as

$$f_\xi(Z) = \sum_{S \in V'_{1,2}(\mathbf{Z})^\vee_{\geq 0}} A_\xi[S] \exp(2\pi i (S, Z)). \tag{4.6}$$

Here the scalars $A_\xi[S]$ are the Fourier coefficients of ξ and

$$V'_{1,2}(\mathbf{Z})^\vee_{\geq 0} = \{S \in V'_{1,2}(\mathbf{Z})^\vee \colon (S,S) \geq 0, (S, -\sqrt{2}v_2) \geq 0\}.$$

The condition $S \in V'_{1,2}(\mathbf{Z})^\vee_{\geq 0}$ is a consequence of the Koecher principle.

Using (4.2), and writing

$$S = -nb_4 - mb_{-4} - \frac{r}{\alpha} y_\alpha^\vee$$

with $n, r, m \in \mathbf{Z}$, the expansion (4.6) takes the form

$$f_\xi(\tau', z, \tau) = \sum_{\substack{n, r, m \in \mathbf{Z} \\ n, m, 2\alpha nm - r^2 \geq 0}} A_\xi[-nb_4 - mb_{-4} - \frac{r}{\alpha} y_\alpha^\vee] e^{2\pi i (\tau' m + n\tau + rz)}. \tag{4.7}$$

Grouping terms in (4.7) gives the classical Fourier-Jacobi expansion (see for example [EZ85]), $f_\xi(\tau', z, \tau) = \sum_{m \in \mathbf{Z}_{\geq 0}} \phi_m(\tau, z) e^{2\pi i m \tau'}$, where

$$\phi_m(\tau, z) = \sum_{\substack{n, r \in \mathbf{Z} \\ n, 2\alpha nm - r^2 \geq 0}} A_\xi[-nb_4 - mb_{-4} - \frac{r}{\alpha} y_\alpha^\vee] e^{2\pi i (n\tau + rz)}. \tag{4.8}$$

The coefficients ϕ_m satisfy the following well known transformation law.

Lemma 4.4. *Suppose* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$, $z \in \mathbf{C}$ *and* $\tau \in \mathbf{C}$ *satisfies* $\mathrm{Im}(\tau) > 0$. *Then*

$$\phi_m\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = \exp\left(\frac{2\pi i m c z^2 \alpha / 2}{c\tau + d}\right) (c\tau + d)^\ell \phi_m(\tau, z). \tag{4.9}$$

Proof. The transformation law (4.9) follows from property 2 of Definition 4.1. In more detail, recall the **Q**-rational morphism from M' to $\mathrm{SO}(V'_{2,3})$ defined via the action of M' on $V'_{2,3}$. Writing elements of $\mathrm{SO}(V'_{2,3})$ as matrices relative to the basis $\{b_2, b_4, y_\alpha^\vee, b_{-4}, b_{-2}\}$, we consider

$$\iota\colon \mathrm{SL}_2 \to \mathrm{SO}(V'_{2,3}), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \iota(\gamma) := \begin{pmatrix} \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & & \\ & 1 & \\ & & \begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix} \end{pmatrix}. \tag{4.10}$$

Then ι lifts to give a map $\tilde{\iota}\colon \mathrm{SL}_2 \to M'$.

Suppose $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$. Applying the description of the action of M' on $\mathfrak{h}'_{y_\alpha}$ given in (4.1), one calculates that with respect to the coordinates (4.2), $j_{y_\alpha}(\tilde{\iota}(\gamma), Z) = c\tau + d$ and

$$\tilde{\iota}(\gamma) \cdot (\tau', z, \tau) = \left(\frac{c(\tau\tau' - z^2\alpha/2) + d\tau'}{c\tau + d}, \frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right). \tag{4.11}$$

Using (4.11) in tandem with the Fourier-Jacobi expansion of f_ξ, we obtain

$$f_\xi(\tilde{\iota}(\gamma) \cdot (\tau', z, \tau)) = \sum_{m \geq 0} \phi_m \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) \exp\left(\frac{-2\pi i m c z^2 \alpha/2}{c\tau + d} \right) e^{2\pi i m \tau'}. \tag{4.12}$$

Since ξ is of level $M'(\mathbf{Z})$, Definition 4.1 implies$f_\xi(\tilde{\iota}(\gamma) \cdot (\tau', z, \tau)) = (c\tau + d)^\ell f_\xi(\tau', z, \tau)$ for all $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ and $Z \in \mathfrak{h}'_{y_\alpha}$. Thus, (4.9) follows by equating the coefficient of $e^{2\pi i m \tau'}$ in (4.12) with the coefficient of $e^{2\pi i m \tau'}$ in $(c\tau + d)^\ell f_\xi(\tau', z, \tau)$. □

4.3. A Primitivity Theorem for Modular Forms on M'.

In this subsection, we continue with the notation of the preceding subsection. Our goal is to establish Theorem 4.5, which has its origins in a result of Zagier [Zag81, pg. 387]. Theorem 4.5 is closely related to a special case of [Yam09, Theorem 3], except for the fact that the level subgroup $M'(\mathbf{Z})$ is different from the level subgroups considered in (loc. cit.). In spite of this difference, the proof of Theorem 4.5 is essentially a special case of the proof given in (loc. cit.).

Theorem 4.5. *Suppose $\xi\colon M'(\mathbf{R}) \to \mathbf{C}$ is the automorphic functions associated to a holomorphic modular form of weight $\ell > 0$ and level $M'(\mathbf{Z})$. Suppose $A_\xi[S] = 0$ for all vectors $S \in V'_{1,2}(\mathbf{Z})^\vee$ such that $\mathbf{Q}$-span$\{S\} \cap V'_{1,2}(\mathbf{Z})^\vee = \mathbf{Z}$-span$\{S\}$. Then $\xi \equiv 0$.*

Proof. For the sake of contradiction, we suppose $\xi \not\equiv 0$. Then there exists $m_0 > 0$ such that the Fourier-Jacobi coefficient $\phi_{m_0}(\tau, z) \neq 0$. Since ϕ_{m_0} is holomorphic in the variable z, we may develop $\phi_{m_0}(\tau, z)$ into a Taylor series as $\phi_{m_0}(\tau, z) = \sum_{\nu \geq 0} \lambda_\nu(\tau) z^\nu$ where

$$\lambda_\nu(\tau) = \sum_{n \geq 0} \left(\sum_{\substack{r \in \mathbf{Z} \text{ such that} \\ 2\alpha n m_0 - r^2 \geq 0}} \frac{(2\pi i)^\nu A_\xi \left[-n b_4 - m_0 b_{-4} - \frac{r}{\alpha} y_\alpha^\vee \right]}{\nu!} \right) e^{2\pi i n \tau}. \tag{4.13}$$

Applying the transformation law (4.9), we conclude that for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$,

$$\sum_{\nu \geq 0} \sum_{j \geq 0} \frac{1}{j!} \left(\frac{2\pi i c \alpha/2}{c\tau + d} \right)^j \lambda_\nu(\tau) z^{\nu + 2j} = \sum_{\nu \geq 0} \frac{1}{(c\tau + d)^{\nu + \ell}} \lambda_\nu \left(\frac{a\tau + b}{c\tau + d} \right) z^\nu. \tag{4.14}$$

Let $\nu_0 \geq 0$, be minimal such that $\lambda_{\nu_0}(z) \not\equiv 0$. By equating the coefficients of z^{ν_0} on the left and right hand sides of (4.14), we deduce that

$$\lambda_{\nu_0}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{\nu_0+\ell}\lambda_{\nu_0}(z).$$

It follows that $\lambda_{\nu_0}(\tau)$ is an elliptic modular form of weight $\ell+\nu_0$ and level $\mathrm{SL}_2(\mathbf{Z})$. Inspecting (4.13), we determine that the numbers

$$a(n) := \sum_{\substack{r \in \mathbf{Z} \text{ such that} \\ 2\alpha n m_0 - r^2 \geq 0}} \frac{(2\pi i)^{\nu_0} A_\xi \left[-nb_4 - m_0 b_{-4} - \frac{r}{\alpha} y_\alpha^\vee\right]}{\nu_0!}$$

are the Fourier coefficients of an elliptic modular form on $\mathrm{SL}_2(\mathbf{Z})$ of weight $\ell+\nu > 0$. The proof now follows from the argument given in the proof of [Yam09, Theorem 3]. □

4.4. **The Hecke Bound Characterization of Cusp Forms on M'.** Continuing with the notation of the previous subsection, we assume

$$y = y_\alpha = b_3 + \frac{\alpha}{2} b_{-3}.$$

The purpose of this subsection is to prove the following cuspidality criterion.

Theorem 4.6. *Suppose $\ell \geq 5$ and let f_ξ be a weight ℓ holomorphic modular form of level $M'(\mathbf{Z})$. Then f_ξ is cuspidal if and only if, for all $S \in V'_{1,2}(\mathbf{Z})^\vee_{\geq 0}$ satisfying $(S,S) > 0$,*

$$A_\xi[S] \ll_{f_\xi} (S,S)^{\frac{\ell+1}{2}}. \tag{4.15}$$

Remark 4.7. As we shall see, the proof of Theorem 4.6 is essentially the same as the proof of [KM14, Theorem 2.1]. With that said, the statement of Theorem 4.6 differs from that of (loc. cit.) in at least two respects. For one, [KM14, Theorem 2.1] pertains to modular forms on Sp_4 of level one. However, though there is a connection between modular forms on $M'(\mathbf{Z})$ and level one Siegel modular forms in the case when $\alpha = 2$, these two families of modular forms are, in general, distinct from one another. Secondly, in Theorem 4.6, $\ell \geq 5$ and the exponent in (4.15) equals $(\ell+1)/2$. However, in (loc. cit.), the authors assume $\ell \geq 4$ and their bound takes the form

$$A_\xi[S] \ll_{f_\xi} (S,S)^{\frac{\ell}{2}}. \tag{4.16}$$

For our application to quaternionic modular forms, we require the additional flexibility afforded by the bound (4.15), though this comes at the cost of excluding $\ell = 4$. In the setting of modular forms on Sp_{2g}, the tradeoff between the exponent in the Hecke bound, and the range of weights is well understood (see [BD14, Theorem 4.1]).

The proof of Theorem 4.6 requires a close analogue of [KM14, Theorem 2.2].

Proposition 4.8. *Suppose $\ell \geq 5$ and let ϕ be a Jacobi form of level one, weight ℓ and index $m > 0$. Following [EZ85], we record the Fourier expansion of ϕ as*

$$\phi(\tau, z) = \sum_{\substack{n \geq 0, r \in \mathbf{Z} \\ \text{such that } r^2 \leq 4mn}} c(n,r) e^{2\pi i(n\tau + rz)}.$$

Then ϕ is cuspidal if and only if the Fourier coefficients $c(n,r)$ satisfy the following condition: if $n \geq 0$, $r \in \mathbf{Z}$, and $D := r^2 - 4mn < 0$, then

$$c(n,r) \ll_\phi |D|^{\frac{\ell+1}{2}}. \tag{4.17}$$

Proof. For the proof of the "only if" implication see [KM14, Lemma 4.1]. Conversely, suppose the non-degenerate Fourier coefficient of ϕ satisfy the bound (4.17). Let $f \in \mathbf{Z}_{>0}$ be such that $f^2 \mid m$ and write $m/f^2 = ab^2$ with positive integers a, b such that a is the square-free part of m/f^2. Then applying the argument in the proof of [KM14, Theorem 2.2], we may assume that ϕ is a linear combination of Eisenstein series, and there exists a primitive Dirichlet character $\chi\colon (\mathbf{Z}/f\mathbf{Z})^\times \to \mathbf{C}$ such that

$$c(n,r) = \sum_{l \mid b} \lambda_l \left(\sum_{\substack{d \mid (n, r, ab^2/l^2) \\ r/d \equiv 0 \pmod{l}}} d^{\ell-1} c^{\chi}_{\ell, f^2}\left(\frac{nab^2/l^2}{d^2}, \frac{r}{dl} \right) \right). \tag{4.18}$$

Here $\lambda_l \in \mathbf{C}$ and the numbers $c^{\chi}_{k,f^2}(n,r)$ are Fourier coefficients of the Eisenstein series $E^{(\chi)}_{\ell,f^2}$ in [EZ85, pg. 26]. By [KM14, Lemma 4.3], if $r \geq 0$ is coprime to f, and $D = r^2 - 4nf^2$ is a fundamental discriminant, then there exists a constant $A_{\ell,f} > 0$ such that

$$|c^{\chi}_{\ell,f^2}(n,r)| > A_{\ell,f}|D|^{\ell-3/2}.$$

On the other hand, since $\ell \geq 5$, $\frac{\ell+1}{2} < \ell - \frac{3}{2}$, and so the inductive argument of [KM14, pg. 1330] implies $\lambda_\ell = 0$ for all $\ell \mid b$. Hence, ϕ is necessarily zero, which completes the proof. □

Proof of Theorem 4.6: The "only if" implication follows directly from the Hecke bound (4.16), which is satisfied for all holomorphic cusp forms on M'. The converse implication is proven by the argument presented in [KM14, §3]. In more detail, assuming the Fourier coefficients $A_\xi[S]$ satisfy (4.15) for $(S,S) > 0$, the results of Proposition 4.8 and Lemma 4.4 imply that $A_\xi[S] = 0$ whenever $S = -nb_4 - mb_{-4} - \frac{r}{\alpha} y_\alpha^\vee$ satisfies $m > 0$ and $(S,S) = 0$. Since $M_{Q'}$ acts transitively on the isotropic lines in $V'_{1,2}$, it follows that $A_\xi[S] = 0$ for all non-zero $S \in V'_{1,2}(\mathbf{Z})$ satisfying $(S,S) = 0$. It remains to show that the constant term of f_ξ vanishes, which is achieved by examining the $m = 0$ term in (4.12). □

4.5. **Relation to Modular Forms on Sp_4.** In this subsection we review the relationship between the theory of modular forms on $M' = M'_{y_\alpha}$ in the case when $\alpha = 2$, and the theory of genus two Siegel modular forms. This relationship is also explained [Joh+24, §7.3].

Let Sp_4 denote the split symplectic group of rank 2 over $\mathbf{Q}$, and let $\pi\colon \mathrm{Sp}_4 \to \mathrm{SO}(V'_{2,3})$ denote the map constructed in (loc. cit.). Since Sp_4 and M' are simply connected, π lifts to give an isomorphism $\widetilde{\pi}\colon \mathrm{Sp}_4 \to M'$. Hence, if ξ is the automorphic function on M' associated to a holomorphic modular form, we define

$$\xi^* := \varphi \circ \widetilde{\pi}.$$

Proposition 4.9. [Joh+24, §7.3] *If ξ is the automorphic function corresponding to a weight ℓ holomorphic modular form on M' of level $M'(\mathbf{Z})$, then ξ^* is the automorphic function on Sp_4 corresponding to a genus 2, Siegel modular form F_{ξ^*} of weight ℓ and level 1. Moreover, the Fourier expansion of F_{ξ^*} takes the form*

$$F_{\xi^*}(Z) = \sum_{T \geq 0} B_{\xi^*}[T] \exp(2\pi i \operatorname{tr}(TZ)), \qquad (Z \in \mathfrak{h}_{\mathrm{Sp}_4}). \tag{4.19}$$

Here T runs over half-integral positive semi-definite matrices $T = \left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right)$, and the Fourier coefficients $B_{\xi^*}[T]$ satisfy

$$B_{\xi^*}\left[\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\right] = A_\xi[-cb_4 - \frac{b}{2}(b_3 - b_{-3}) - ab_{-4}]. \tag{4.20}$$

4.6. **Holomorphic Modular Forms on M_P^{der}.** In Subsection 3.4, we gave an identification between the derived subgroup of the Heisenberg Levi factor M_P, and the group $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$. In this subsection we give preliminaries regard the holomorphic modular forms on the derived subgroup M_P^{der}.

Let $\mathfrak{h}_{\mathrm{SL}_2}$ be the complex upper half-plane and write $\mathfrak{h}_{M_P^{\mathrm{der}}} = \mathfrak{h}_{\mathrm{SL}_2} \times \mathfrak{h}_{\mathrm{SL}_2} \times \mathfrak{h}_{\mathrm{SL}_2}$. Following [Pol20, Proposition 2.3.1], we define an automorphy factor

$$j_{M_P^{\mathrm{der}}} \colon M_P^{\mathrm{der}}(\mathbf{R}) \times \mathfrak{h}_{M_P^{\mathrm{der}}} \to \mathbf{C}, \quad ((g_1, g_2, g_3), (z_1, z_2, z_3)) \mapsto j_{\mathrm{SL}_2}(g_1, z_1) j_{\mathrm{SL}_2}(g_2, z_2) j_{\mathrm{SL}_2}(g_3, z_3)$$

where $j_{\mathrm{SL}_2} \colon \mathrm{SL}_2(\mathbf{R}) \times \mathfrak{h}_{\mathrm{SL}_2} \to \mathbf{C}$ is the standard factor of automorphy $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z) = cz + d$.

Definition 4.10. Write $\mathfrak{h}_{M_P^{\mathrm{der}}} = \mathfrak{h}_{\mathrm{SL}_2} \times \mathfrak{h}_{\mathrm{SL}_2} \times \mathfrak{h}_{\mathrm{SL}_2}$ so that $M_P^{\mathrm{der}}(\mathbf{R})$ acts on $\mathfrak{h}_{M_P^{\mathrm{der}}}$. Let $\ell \in \mathbf{Z}_{>0}$ and suppose $\Gamma \leq M_P^{\mathrm{der}}(\mathbf{R})$ is a discrete subgroup. A holomorphic modular form on $\mathfrak{h}_{M_P^{\mathrm{der}}}$ of weight ℓ and level Γ is a function $f \colon \mathfrak{h}_{M_P^{\mathrm{der}}} \to \mathbf{C}$ such that

(1) f is holomorphic,
(2) if $z \in \mathfrak{h}_{M_P^{\mathrm{der}}}$ and $\gamma \in \Gamma$ then $f(\gamma \cdot z) = j_{M_P^{\mathrm{der}}}(\gamma, z)^\ell f(z)$, and
(3) $\xi \colon M_P^{\mathrm{der}}(\mathbf{R}) \to \mathbf{C}$, $\quad \xi(g) := j_{M_P^{\mathrm{der}}}(\gamma, z)^{-\ell} f(g \cdot (i, i, i))$ is of moderate growth.

Lemma 4.11. *Assume $\ell \in \mathbf{Z}_{>0}$. Suppose $f \colon \mathfrak{h}_{M_P^{\mathrm{der}}} \to \mathbf{C}$ is a weight ℓ holomorphic modular form of level $M_P^{\mathrm{der}}(\mathbf{Z})$. Write the classical Fourier expansion of f as*

$$f(z_1, z_2, z_3) = \sum_{n_1, n_2, n_3 \geq 0} a(n_1, n_2, n_3) e^{2\pi i(n_1 z_1 + n_2 z_2 + n_3 z_3)}. \tag{4.21}$$

If $a(n_1, n_2, n_3) = 0$ for all $(n_1, n_2, n_3) \in \mathbf{Z}_{>0}^3$ satisfying $\gcd(n_1, n_2, n_3) = 1$, then $f \equiv 0$.

Proof. Suppose for a contradiction that $f \not\equiv 0$. Then there exists $w = (w_1, w_2, w_3) \in \mathfrak{h}_{M_P^{\mathrm{der}}}$ such that $f(w) \neq 0$. Therefore,

$$f_1(z) = \sum_{n_1 \geq 0} \left(\sum_{n_2, n_3 \geq 0} a(n_1, n_2, n_3) e^{2\pi i(n_2 w_2 + n_3 w_3)} \right) e^{2\pi i n_1 z.}$$

is a non-zero elliptic modular form of weight $\ell > 0$. So there exists $m > 0$, such that $g(z_2, z_3) = \sum_{n_2, n_3 \geq 0} a(m, n_2, n_3) e^{2\pi i(n_2 z_2 + n_3 z_3)}$ is a non-zero, weight ℓ holomorphic modular form on $\mathfrak{h}_{\mathrm{SL}_2} \times \mathfrak{h}_{\mathrm{SL}_2}$ of level $\mathrm{SL}_2(\mathbf{Z}) \times \mathrm{SL}_2(\mathbf{Z})$. Applying the same logic to g, there exists an integer $m' > 0$ such that $h(z_3) = \sum_{n_3 \geq 0} a(m, m', n_3) e^{2\pi i n_3 z_3}$ is non-zero, weight $\ell > 0$ modular form on $\mathfrak{h}_{\mathrm{SL}_2}$ of level $\mathrm{SL}_2(\mathbf{Z})$. Thus if $d = \gcd(m, m')$, then there exists an integer $m'' > 0$ such that $\gcd(d, m'') = 1$ and $a(m, m', m'') \neq 0$, completing the proof. □

5. Preliminaries on Quaternionic Modular Forms

5.1. **Quaternionic Modular Forms on G.** Let $\ell \in \mathbf{Z}_{\geq 1}$ and recall the representation $\mathbf{V}_\ell$ defined in (3.16). Following [Pol20], we now define quaternionic modular forms on G.

We begin by reviewing the construction of a differential operator D_ℓ which goes back to the work of Schmid [Sch]. To specify D_ℓ, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the complexified Lie algebra $\mathfrak{g}$ of $G(\mathbf{R})$ from Subsection 3.7. Let $\{X_\alpha\}$ be a basis of $\mathfrak{p}$ and $\{X_\alpha^\vee\}$ the dual basis of $\mathfrak{p}^\vee$. One has that, as a representation of K_∞, $\mathfrak{p} \simeq \mathfrak{p}^\vee \simeq V_2 \boxtimes W$, where the distinguished SU_2 acts trivially on W (see (3.18)).

For a K_∞-equivariant function $F : G(\mathbf{R}) \to \mathbf{V}_\ell$, set $\widetilde{D}_\ell F = \sum_\alpha X_\alpha F \otimes X_\alpha^\vee$. The sum is independent of the choice of basis and $\widetilde{D}_\ell F$ takes values in

$$\mathbf{V}_\ell \otimes \mathfrak{p}^\vee \simeq (\mathrm{Sym}^{2\ell-1}(V_2) \otimes W) \oplus (\mathrm{Sym}^{2\ell+1}(V_2) \otimes W).$$

Let pr be the projection $\mathbf{V}_\ell \otimes \mathfrak{p}^\vee \to \mathrm{Sym}^{2\ell-1}(V_2) \otimes W$. Then $D_\ell = \mathrm{pr} \circ \widetilde{D}_\ell$.

Definition 5.1. Suppose $\ell \in \mathbf{Z}_{\geq 1}$. The space of weight ℓ (quaternionic) modular forms M_ℓ on G is the space of smooth, moderate growth functions $\varphi \colon G(\mathbf{A}) \to \mathbf{V}_\ell$ such that:

1. if $\gamma \in G(\mathbf{Q})$ and $g \in G(\mathbf{A})$ then $\varphi(\gamma g) = \varphi(g)$,
2. there exists an open compact subgroup $K_f \leq G(\mathbf{A}_f)$ such that φ is right K_f-invariant,
3. if $k \in K_\infty$, and $g \in G(\mathbf{A})$, then $\varphi(gk) = k^{-1}\varphi(g)$,
4. $D_\ell \varphi \equiv 0$, and
5. φ is $Z(\mathfrak{g})$ finite.

Let S_ℓ be the subspace of M_ℓ consisting of cusp forms. So S_ℓ consists of forms $\varphi \in M_\ell$ such that if $\mathcal{N} \leq G$ is the unipotent radical of a proper $\mathbf{Q}$-rational parabolic subgroup then

$$\varphi_{\mathcal{N}}(g) := \int_{\mathcal{N}(\mathbf{Q})\backslash\mathcal{N}(\mathbf{A})} \varphi(ng)\, dn$$

is identically zero. Write $M_\ell(1)$ (resp. $S_\ell(1)$) to denote the subspace of M_ℓ (resp. S_ℓ) consisting of forms φ such that $\varphi(gk) = \varphi(g)$ for all $g \in G(\mathbf{A})$ and $k \in G(\widehat{\mathbf{Z}})$.

5.2. **The Fourier Expansion of Quaternionic Modular Forms on G.** The Heisenberg parabolic subgroup $P = M_P N_P$ is defined in Subsection 3.4, and the unipotent radical N_P of P is two-step nilpotent with center Z.

Recall the identification of the character lattice $\mathrm{Hom}(N_P(\mathbf{Q})\backslash N_P(\mathbf{A}), \mathbf{C}^1)$ given in (3.8). If $T_1, T_2 \in V_{2,2}(\mathbf{Q})$, and $\varphi \in M_\ell$, we define the Fourier coefficient $\varphi_{[T_1,T_2]}$ through the formula

$$\varphi_{[T_1,T_2]} \colon G(\mathbf{A}) \to \mathbf{V}, \qquad \varphi_{[T_1,T_2]}(g) = \int_{N_P(\mathbf{Q})\backslash N_P(\mathbf{A})} \varphi(ng)\varepsilon^{-1}_{[T_1,T_2]}(n)\, dn. \tag{5.1}$$

Then since $Z = [N_P, N_P]$, we may Fourier expand the constant term φ_Z as

$$\varphi_Z(g) = \varphi_{N_P}(g) + \sum_{T_1,T_2 \in V_{2,2}(\mathbf{Q})\colon [T_1,T_2]\neq[0,0]} \varphi_{[T_1,T_2]}(g). \tag{5.2}$$

Write $\langle \cdot, \cdot \rangle$ for the $\mathrm{GL}(U) \times \mathrm{SO}(V_{2,2})$-invariant pairing between $U^\vee \otimes_{\mathbf{Q}} V_{2,2}$ and $U \otimes_{\mathbf{Q}} V_{2,2}$. So if $T_1, T_1', T_2, T_2' \in V_{2,2}(\mathbf{R})$, then

$$\langle b_{-1} \otimes T_1 + b_{-2} \otimes T_2, b_1 \otimes T_1' + b_2 \otimes T_2' \rangle = (T_1, T_1') + (T_2, T_2').$$

For $T_1, T_2 \in V_{2,2}(\mathbf{R})$, define $\beta_{[T_1,T_2]} \colon M_P(\mathbf{R}) \to \mathbf{C}$ by the formula

$$\beta_{[T_1,T_2]}(r) := \sqrt{2}i\langle r^{-1} \cdot (b_{-1} \otimes T_1 + b_{-2} \otimes T_2), b_1 \otimes (v_1 + iv_2) + b_2 \otimes i(v_1 + iv_2)\rangle. \tag{5.3}$$

For the convenience of the reader we recall that $v_1 = (b_3 + b_{-3})/\sqrt{2}$ and $v_2 = (b_4 + b_{-4})/\sqrt{2}$.

Definition 5.2. Let $[T_1, T_2] \in V_{2,2}(\mathbf{R})^{\oplus 2}$. We say $[T_1, T_2]$ is positive semi-definite if $\beta_{[T_1,T_2]}(r) \neq 0$ for all $r \in M(\mathbf{R})^0$. We write $[T_1, T_2] \succeq 0$ to mean that the pair $[T_1, T_2]$ is positive semi-definite. Moreover, we write $[T_1, T_2] \succ 0$ if $[T_1, T_2] \succeq 0$ and $(T_1, T_1)(T_2, T_2) - (T_1, T_2)^2 > 0$.

Given $r \in M_P(\mathbf{R})$ we write the image of m under the projection $p\colon G(\mathbf{R}) \to \mathrm{SO}(V)(\mathbf{R})^0$ as $p(r) = (m, h)$ with the understanding that $m \in \mathrm{GL}(U)(\mathbf{R})$ and $h \in \mathrm{SO}(V_{2,2})(\mathbf{R})^0$.

Modulo a technicality regarding the difference between G and G^{ad}, the following result is proven in [Pol20]. With that said, a version of Theorem 5.3 is available in [Wal03, Theorem 16], though (loc. cit.) does not establish (5.4), which is of crucial importance to us.

Theorem 5.3. *Fix $\ell \in \mathbf{Z}_{\geq 1}$ and suppose $[T_1, T_2] \in V_{2,2}(\mathbf{R})^{\oplus 2}$. Then up to scalar multiple, there is a unique moderate growth function $\mathcal{W}_{[T_1,T_2]}\colon G(\mathbf{R}) \to \mathbf{V}_\ell$ such that:*

(1) If $g \in G(\mathbf{R})$ and $k \in K_\infty$ then $\mathcal{W}_{[T_1,T_2]}(gk) = k^{-1}\mathcal{W}_{[T_1,T_2]}(g)$.

(2) If $g \in G(\mathbf{R})$ and $n \in N_P(\mathbf{R})$ satisfy $\log(n) = b_1 \wedge w_1 + b_2 \wedge w_2 + zb_1 \wedge b_2$ for $w_1, w_2 \in V_{2,2}(\mathbf{R})$ and $z \in \mathbf{R}$, then $\mathcal{W}_{[T_1,T_2]}(ng) = e^{i(T_1,w_1)+i(T_2,w_2)}\mathcal{W}_{[T_1,T_2]}(g)$.

(3) If $g \in G(\mathbf{R})$ then $D_\ell \mathcal{W}_{[T_1,T_2]}(g) = 0$.

Moreover $\mathcal{W}_{[T_1,T_2]}(g) \equiv 0$ unless $[T_1, T_2] \succeq 0$, and if $[T_1, T_2] \succeq 0$ then the function $\mathcal{W}_{[T_1,T_2]}(g)$ is uniquely characterized by requiring that for all $r \in M_P(\mathbf{R})^0$,

$$\mathcal{W}_{[T_1,T_2]}(r) = \det(m)^\ell |\det(m)| \sum_{-\ell \leq v \leq \ell} \left(\frac{\beta_{[T_1,T_2]}(r)}{|\beta_{[T_1,T_2]}(r)|}\right)^v K_v(|\beta_{[T_1,T_2]}(r)|)\frac{x^{\ell+v}y^{\ell-v}}{(\ell+v)!(\ell-v)!}. \tag{5.4}$$

Here $K_v\colon \mathbf{R}_{>0} \to \mathbf{R}$ denotes the modified K-Bessel function $K_v(x) = \dfrac{1}{2}\displaystyle\int_0^\infty t^{v-1}e^{-x(t+t^{-1})}\,dt$.

Proof. The statement is identical to that of [Joh+24, Theorem 4.5] except for the fact $G = \mathrm{Spin}(V)$ as opposed to $\mathrm{SO}(V)$. However, since the central μ_2 in $\mathrm{Spin}(V)$ is contained in K_∞ and this subgroup acts trivially on $\mathbf{V}_\ell$, property (1) of the theorem statement implies that $\mathcal{W}_{[T_1,T_2]}$ factors across $\mathrm{SO}(V)(\mathbf{R})^0$. Hence, the result follows directly from (loc. cit.). ☐

As a corollary to Theorem 5.3 we have the following.

Corollary 5.4. *Suppose $\ell \in \mathbf{Z}_{\geq 1}$ and let $\varphi\colon G(\mathbf{R}) \to \mathbf{V}_\ell$ be a weight ℓ quaternionic modular form on $G(\mathbf{A})$. If $[T_1, T_2] \in V_{2,2}(\mathbf{Q})^{\oplus 2}$ is non-zero and not positive semi-definite then $\varphi_{[T_1,T_2]} \equiv 0$. Moreover, there exists a a unique family of locally constant functions*

$$\{a_{[T_1,T_2]}(\varphi, \cdot)\colon G(\mathbf{A}_f) \to \mathbf{C}\colon [T_1, T_2] \in V_{2,2}(\mathbf{Q})^{\oplus 2} \text{ such that } [T_1, T_2] \succeq 0\}$$

such that $\varphi_{[T_1,T_2]}(g_f g_\infty) = a_{[T_1,T_2]}(\varphi, g_f)\mathcal{W}_{[2\pi T_1, 2\pi T_2]}(g_\infty)$ for all $[T_1, T_2] \succeq 0$. In particular, the Fourier expansion of φ_Z along $Z(\mathbf{A})N_P(\mathbf{Q})\backslash N_P(\mathbf{A})$ takes the form

$$\varphi_Z(g_f g_\infty) = \varphi_{N_P}(g_f g_\infty) + \sum_{T_1,T_2 \in V_{2,2}(\mathbf{Q})\colon [T_1,T_2] \succeq 0} a_{[T_1,T_2]}(\varphi, g_f)\mathcal{W}_{[2\pi T_1, 2\pi T_2]}(g_\infty). \tag{5.5}$$

Definition 5.5. Suppose $\varphi \in M_\ell$ and $B = [T_1, T_2] \in V_{2,2}(\mathbf{Q})^{\oplus 2}$ satisfies $B] \succeq 0$. The Fourier coefficients of φ indexed by B is defined as $\Lambda_\varphi(B) = \Lambda_\varphi[T_1, T_2] = a_{[T_1,T_2]}(\varphi, 1)$.

The constant term φ_{N_P} is essentially a holomorphic modular form on M_P. More precisely, following [Pol20, Proposition 11.1.1], the function

$$\Phi\colon M_P^{\mathrm{der}}(\mathbf{R}) \to \mathbf{C}, \quad \Phi(m) = j_{M_P^{\mathrm{der}}}(m, (i,i,i))^\ell \left\{\varphi_{N_P}(m), y^{2\ell}\right\}_{K_\infty} \tag{5.6}$$

descends to a holomorphic modular form on $\mathfrak{h}_{M_P^{\mathrm{der}}}$ in the sense of Definition 4.10. Here $\{\,,\,\}_{K_\infty}$ denotes the unique K_∞ invariant symmetric bilinear form on $\mathbf{V}_\ell$ satisfying

$$\{x^{\ell+v}y^{\ell-v}, x^{\ell-w}y^{\ell+w}\}_{K_\infty} = (-1)^{\ell+v}\delta_{v,w}(\ell+v)!(\ell-v)!$$

where $\delta_{v,w}$ denotes the Kronecker delta function.

Next we recall a result of [Joh+24, §4.4], which establishes a positive semi-definiteness support property for the Fourier coefficients of quaternionic modular forms on G.

Proposition 5.6. [Joh+24, Proposition 4.9] *Let $T_1, T_2 \in V_{2,2}(\mathbf{R})$, $W = \mathbf{R}\text{-span}\{T_1, T_2\}$, and $V_2^+(\mathbf{R}) = \mathbf{R}\text{-span}\{v_1, v_2\}$.*

(i) If $\mathbf{R}\text{-span}\{T_1, T_2\}$ is an indefinite two plane, a negative definite two plane, or a negative definite line, then there exists $r \in M_P(\mathbf{R})^0$ such that $\beta_{[T_1,T_2]}(r) = 0$.

(ii) If $|\beta_{[T_1,T_2]}(r)|$ is bounded away from zero on $M_R^{\mathrm{der}}(\mathbf{R})$ then $(T_1,T_1)(T_2,T_2) - (T_1,T_2)^2 > 0$. In particular T_1 and T_2 span a two plane in $V_{2,2}(\mathbf{R})$.

We conclude with an application of Proposition 5.6 to the study of cusp forms $\varphi \in S_\ell$, which will be applied during the proof of Corollary 6.11.

Corollary 5.7. *Suppose $\ell \in \mathbf{Z}_{\geq 1}$ and $\varphi \colon G(\mathbf{A}) \to \mathbf{V}_\ell$ is a weight ℓ quaternionic modular form. Then φ is a cusp form if and only if the Fourier expansion (5.5) takes the form*

$$\varphi_Z(g_f g_\infty) = \sum_{T_1, T_2 \in V_{2,2}(\mathbf{Q}) \colon [T_1,T_2] \succ 0} a_{[T_1,T_2]}(\varphi, g_f) \mathcal{W}_{[2\pi T_1, 2\pi T_2]}(g_\infty). \tag{5.7}$$

Proof. For the proof of the forward implication, we refer the reader to [Joh+24, Corollary 4.10]. To handle the backward implication, suppose φ is such that $\varphi_{N_P} \equiv 0$ and $a_{[T_1,T_2]}(\varphi, g_f) \equiv 0$ whenever $[T_1, T_2]$ does not satisfy $[T_1, T_2] \succ 0$.

The group G contains 4 conjugacy classes of maximal parabolic subgroups. Two of these conjugacy classes are represented by the Heisenberg parabolic subgroup P, and the orthogonal parabolic subgroup R. Since R corresponds to an outer node in the Dynkin diagram of G, representatives for the remaining two conjugacy classes may be obtain by translating R by the triality outer automorphisms described in [Joh+24, Theorem A.8]. By [Joh+24, Theorem A.1], the triality outer automorphism of G stabilize N_P, and induce an action of the symmetric group S_3 on the space M_ℓ. Moreover, the induced action of S_3 on M_ℓ preserves the subspace consisting of forms whose Fourier coefficients are supported on positive definite indices. Hence, in order to prove that φ is cuspidal, it suffices to show that the constant term $\varphi_{N_P \cap N_R} \equiv 0$.

Writing $g \in G(\mathbf{A})$ as $g = g_f g_\infty$, a standard manipulation gives that

$$\varphi_{N_P \cap N_R}(g_f g_\infty) = \sum_{w \in V_{2,2}(\mathbf{Q})^{\oplus 2} \colon \varepsilon_w|_{N_P \cap N_R} = 1} a_w(\varphi, g_f) \mathcal{W}_{2\pi w}(g_\infty). \tag{5.8}$$

Since $\mathrm{Lie}(N_P \cap N_R)$ contains the subspace $b_1 \wedge V_{2,2}$, if $w = [T_1, T_2]$ satisfies $\varepsilon_w|_{N_P \cap N_R} = 1$, then $T_1 = 0$. Hence, the only terms appearing in (5.8) are those for which $w = [T_1, T_2]$ does not satisfy $[T_1, T_2] \succ 0$. Therefore, $\varphi_{N_P \cap N_R} \equiv 0$, and φ is cuspidal. □

5.3. **The Hecke Bound for Quaternionic Modular Forms on G.** If $B = [T_1, T_2] \in V_{2,2}(\mathbf{Q})^{\oplus 2}$, we define

$$Q(B) = \det\left(\begin{pmatrix} (T_1,T_1) & (T_1,T_2) \\ (T_2,T_2) & (T_2,T_2) \end{pmatrix}\right).$$

Then $r \in M_P$ acts on $V_{2,2}(\mathbf{Q})^{\oplus 2}$ preserving Q up to a similitude character $\nu \colon M_P \to \mathbf{G}_m$ i.e.

$$Q(r \cdot B) = \nu(r)^2 Q(B).$$

To define ν, write the image of $r \in M_P$ under the projection $M_P \to \mathrm{GL}(U) \times \mathrm{SO}(V_{2,2})$ as (m, h). Then $\nu(r) = \det(m)$. The purpose of this subsection is to prove an analogue of the Hecke bound for quaternionic cusp forms on G. Our proof is adapted from [GGS02, Proposition 8.6], in which the authors prove an analogous bound for modular forms on G_2.

Proposition 5.8. *Suppose $\varphi \in S_\ell$ is a weight ℓ cuspidal quaternionic modular form on G. If $B \in V_{2,2}(\mathbf{Q})^{\oplus 2}$ satisfies $B \succ 0$, then*

$$\Lambda_\varphi[B] \ll_\varphi Q(B)^{\frac{\ell+1}{2}}.$$

Proof. Assume $B \in V_{2,2}(\mathbf{Q})^{\oplus 2}$ satisfies $B \succ 0$. Independent of the choice of B, we fix an element $B_0 \in V_{2,2}(\mathbf{R})^{\oplus 2}$ such that $Q(B_0) = 1$. The group $M_P(\mathbf{R})$ acts transitively on the elements $B' \in V_{2,2}(\mathbf{R})^{\oplus 2}$ such that $Q(B') > 0$. Therefore, we may fix an element $m_0 \in M_P(\mathbf{R})$ such that $m_0 \cdot B_0 = B$. It follows that $Q(B) = \nu(m_0)^2$.

By definition, if $m_\infty \in M_P(\mathbf{R})$, then

$$\int_{[N_P]} \varphi(nm_\infty)\varepsilon_B(n)^{-1}\, dn = \Lambda_\varphi[B] \cdot \mathcal{W}_B(m_\infty) \tag{5.9}$$

where $\mathcal{W}_B$ is given by (5.4). Fix an element $u_0 \in \mathbf{V}_\ell$ so that if $\{\cdot,\cdot\}_{K_\infty}$ is the K_∞ invariant bilinear form on $\mathbf{V}_\ell$ then

$$\{x^{\ell-v}y^{\ell+v}, u_0\}_{K_\infty} = \begin{cases} 2\ell! & \text{if } v = 0, \\ 0 & \text{else.} \end{cases}$$

Since M_P is a split, connected, and reductive, M_P is an almost direct product of M_P^{der} and the radical of M_P, $R(M_P)$, which is a central torus [Spr98, Corollary 8.1.6]. As M_P has rank 4, and $M_P^{\mathrm{der}} = \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$, $R(M_P)$ is isomorphic to the multiplicative group $\mathbf{G}_m$. We choose this isomorphism so that the natural action of $R(M_P)$ on $U^\vee \otimes V_{2,2}$ is identified with the action of $\mathbf{G}_m$ on $U^\vee \otimes V_{2,2}$ by scalar multiplication. In this way, we may write $\nu(m_0)^{-1}m_0 \in M_P(\mathbf{R})$ and

$$\begin{aligned}\left\{\mathcal{W}_B(\nu(m_0)^{-1}m_0), u_0\right\}_{K_\infty} &= \nu(\nu(m_0)^{-1}m_0)^\ell |\nu(\nu(m_0)^{-1}m_0)| K_0\left(|\beta_B(\nu(m_0)^{-1}m_0)|\right) \\ &= Q(B)^{\frac{-\ell-1}{2}}\left\{\mathcal{W}_{B_0}(1), u_0\right\}_{K_\infty}.\end{aligned}$$

Since φ is cuspidal, $|\varphi(g)|$ is bounded on $G(\mathbf{Q})\backslash G(\mathbf{A})$. Therefore, since the domain of integration in (5.9) is compact, we may set $m_\infty = \nu(m_0)^{-1}m_0$ to obtain

$$|\Lambda_\varphi[B]| \leq \frac{\|\varphi\|_\infty}{\{\mathcal{W}_{B_0}(1), u_0\}_{K_\infty}} Q(B)^{\frac{\ell+1}{2}}.$$

□

6. The Fourier-Jacobi Expansion of Modular Forms on G

Given an automorphic form φ on $G(\mathbf{A})$, and a vector $y \in V_{3,3}(\mathbf{Q})$, define

$$\mathcal{F}(\varphi; y) \colon G(\mathbf{A}) \to \mathbf{V}_\ell, \qquad g \mapsto \int_{N_R(\mathbf{Q})\backslash N_R(\mathbf{A})} \varphi(ng)\chi_y(n)^{-1}\, dn.$$

Here the the character χ_y is as defined in (3.12).

6.1. **Soft Analysis of $\mathcal{F}(\varphi; y)$: The case of isotropic y.** Suppose φ is a vector valued automorphic function of $G(\mathbf{A})$ and let $y \in V_{3,3}(\mathbf{Q})$ be non-zero and isotropic. In this subsection, we establish Lemma 6.1, which gives a relationship between the degenerate Fourier-Jacobi coefficient $\mathcal{F}(\varphi; y)$, and Fourier coefficients of φ along the Heisenberg unipotent radical N_P.

We are considering the case of isotropic y, and since $M_R^{\mathrm{der}} \simeq \mathrm{Spin}(V_{3,3})$, we restrict to considering the case of $y = nb_2$ where $n \in \mathbf{Z}\backslash\{0\}$. Then $\mathcal{F}(\varphi; y)|_{G(\mathbf{R})}$ is left invariant under the rational points of the parabolic subgroup in $\mathrm{Spin}(V_{3,3})$ stabilizing b_2. Let N denote the unipotent radical of this parabolic subgroup. So,

$$N \simeq \left\{ = \begin{pmatrix} 1 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 1 & v & * & 0 \\ \mathbf{0} & \mathbf{0} & I_4 & * & \mathbf{0} \\ 0 & 0 & \mathbf{0} & 1 & 0 \\ 0 & 0 & \mathbf{0} & 0 & 1 \end{pmatrix} \in \mathrm{SO}(V) \colon v \in V_{2,2} \right\},$$

and $\mathrm{Hom}(N(\mathbf{Q})\backslash N(\mathbf{A}), \mathbf{C}^\times) = \{\varepsilon_{[0,T]} \colon T \in V_{2,2}(\mathbf{Q})\}$, where $\varepsilon_{[0,T]}$ is defined in (3.7). Since N is abelian, we may Fourier expand $\mathcal{F}(\varphi; nb_2)$ in terms of the coefficients

$$\mathcal{F}_T^N(\varphi; nb_2)(g) := \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \mathcal{F}(\varphi; nb_2)(rg)\varepsilon_{[0,T]}^{-1}(r)\, dr. \tag{6.1}$$

Here $g \in G(\mathbf{A})$ and $T \in V_{2,2}(\mathbf{Q})$. Regarding $\mathcal{F}_T^N(\varphi; nb_2)$ we have the following.

Lemma 6.1. *Suppose $g \in G(\mathbf{A})$ and $T \in V_{2,2}(\mathbf{Q})$. Let $\varepsilon_{[0,T]}$ be the character of $N_P(\mathbf{Q})\backslash N_P(\mathbf{A})$ defined in (3.7) and write $\varphi_{[0,T]}(g) = \int_{N_P(\mathbf{Q})\backslash N_P(\mathbf{A})} \varphi(ng)\varepsilon_{[0,T]}(n)^{-1}\, dn$. Then*

$$\mathcal{F}_T^N(\varphi; nb_2)(g) = \int_{\mathbf{Q}\backslash\mathbf{A}} \psi^{-1}(ns)\varphi_{[0,T]}(\exp(sb_1 \wedge b_{-2})g)\, ds. \tag{6.2}$$

Proof. Throughout the proof we abbreviate notation by writing $[\mathcal{G}] := \mathcal{G}(\mathbf{Q})\backslash\mathcal{G}(\mathbf{A})$ to denote the adelic quotient of an algebraic group $\mathcal{G}$. Plugging in definitions, one obtains

$$\mathcal{F}_T^N(\varphi; nb_2)(g) = \int_{[N]} \int_{[N_R]} \varphi(urg)\chi_{nb_2}^{-1}(u)\varepsilon_{[0,T]}^{-1}(r)\, du\, dr. \tag{6.3}$$

The character χ_{nb_2} is trivial on $Z \leq N_R$. Since N_R is abelian, we may thus factor the inner integral in (6.3) across Z to obtain

$$\mathcal{F}_T^N(\varphi; nb_2)(g) = \int_{[N]} \int_{[N_R/Z]} \varphi_Z(urg)\chi_{nb_2}^{-1}(u)\varepsilon_{[0,T]}^{-1}(r)\, du\, dr. \tag{6.4}$$

Let $X = M_P \cap N_R = \{\exp(tb_1 \wedge b_{-2}) \colon t \in \mathbf{G}_a\}$. Since N_R is abelian, the integral over $[N_R/Z]$ in (6.4) is an iterated integral over $X(\mathbf{Q})\backslash X(\mathbf{A})$ and $[N_R/(ZX)]$. Moreover, the commutator $[X, N]$ is contained in Z. Hence, moving the integral over $[X]$ to the left of the integration over $[N]$, one arrives at the expression

$$\mathcal{F}_T^N(\varphi; nb_2)(g) = \int_{[X]} \chi_{nb_2}^{-1}(x) \int_{[N]} \int_{[N_R/(ZX)]} \varphi_Z(xurg)\varepsilon_{[0,T]}^{-1}(r)\, du\, dr\, dx. \tag{6.5}$$

Since $N \times N_R/(ZX) = N_P^{\mathrm{ab}}$, (6.5) may be rewritten as

$$\mathcal{F}_T^N(\varphi; nb_2)(g) = \int_{[X]} \chi_{nb_2}^{-1}(x) \int_{[N_P^{\mathrm{ab}}]} \varphi_Z(xng)\varepsilon_{[0,T]}^{-1}(n)\, dn\, dx. \tag{6.6}$$

Fourier expanding φ_Z along $N_P^{\mathrm{ab}}(\mathbf{Q})\backslash N_P^{\mathrm{ab}}(\mathbf{A})$, we apply the inclusion $X \leq M_P$ and the fact that M_P normalized N_P to simplify the inner integral in (6.6) as

$$\int_{[N_P^{\mathrm{ab}}]} \varphi_Z(xng)\varepsilon_{[0,T]}^{-1}(n)\,dn = \int_{[N_P^{\mathrm{ab}}]} \sum_{T_1,T_2\in V_{2,2}(\mathbf{Q})} \varphi_{[T_1,T_2]}(xng)\varepsilon_{[0,T]}^{-1}(n)\,dn$$

$$= \sum_{T_1,T_2\in V_{2,2}(\mathbf{Q})} \varphi_{[T_1,T_2]}(xg)\int_{[N_P^{\mathrm{ab}}]} \varepsilon_{[T_1,T_2]}(xnx^{-1})\varepsilon_{[0,T]}^{-1}(n)\,dn. \tag{6.7}$$

Fix $s \in \mathbf{A}$, $x = \exp(sb_1 \wedge b_{-2})$, and suppose $n = \exp(b_1 \wedge v + b_2 \wedge v')$ with $v, v' \in V_{2,2}(\mathbf{A})$. Then $xnx^{-1} = \exp(b_1 \wedge (v + sv') + b_2 \wedge v)$ and so if $T_1, T_2 \in V_{2,2}(\mathbf{Q})$,

$$\int_{[N_P^{\mathrm{ab}}]} \varepsilon_{[T_1,T_2]}(xnx^{-1})\varepsilon_{[0,T]}^{-1}(n)\,dn = \int_{[V_{2,2}]}\int_{[V_{2,2}]} \psi((T_1, v+sv') + (T_2, v) - (v', T))\,dv\,dv'. \tag{6.8}$$

The double integral above is non-zero if and only if $(T_1, v+sv') + (T_2, v') - (v', T) = 0$ for all $v, v' \in V_{2,2}(\mathbf{A})$. Setting $v' = 0$ this condition reduces to $(T_1, v) = 0$ for all $v \in V_{2,2}(\mathbf{A})$. Since $V_{2,2}$ is non-degenerate, it follows that the integral (6.8) is non-vanishing if and only if $T_1 = 0$ and $T_2 = T$. Thus the sum in (6.7) reduces to a single term equal to $\varphi_{[0,T]}(xg)$. Therefore, (6.6) simplifies to (6.2) as required. $\square$

6.2. **Soft Analysis of $\mathcal{F}(\varphi; y)$: The case of non-isotropic y.** Let φ is a vector valued automorphic function on $G(\mathbf{A})$, and suppose $y \in V_{3,3}(\mathbf{Q})$ is non-zero and non-isotropic. The goal of this subsection is to give companion result to Lemma 6.1, that analyses $\mathcal{F}(\varphi; y)$ in the case when y is non-isotropic. By the result of [Joh+24, Lemma B.2], the orbits of $M_R^{\mathrm{der}}(\mathbf{Z})$ on the space of non-isotropic vectors in $V_{3,3}(\mathbf{Z})$ are exhausted by representatives

$$ny_\alpha := nb_3 + \frac{n\alpha}{2}b_{-3} \tag{6.9}$$

where $n \in \mathbf{Z}_{\geq 1}$ and $\alpha \in 2\mathbf{Z}\backslash\{0\}$. For ease of notation we write $y_\alpha = b_3 + \alpha b_{-3}/2$

To analyze $\mathcal{F}(\varphi; ny_\alpha)$, we study its Fourier expansion in characters of the $[N_{Q'}]$. Here Q' denotes the Siegel parabolic subgroup of $M' = \mathrm{Stab}_{M_R^{\mathrm{der}}}(y)$ (see Subsection 3.6).

Since M' stabilizes ny_α, the coefficient $\mathcal{F}(\varphi; ny_\alpha)$ is left invariant by $N_{Q'}(\mathbf{Q})$. Hence, we may Fourier expand $\mathcal{F}(\varphi; ny_\alpha)$ in characters of $N_{Q'}(\mathbf{Q})\backslash N_{Q'}(\mathbf{A})$ as

$$\mathcal{F}(\varphi; ny_\alpha)(g) = \sum_{S\in V'_{1,2}(\mathbf{Q})} \mathcal{F}_S^{Q'}(\varphi; ny_\alpha)(g)$$

where $g \in G(\mathbf{A})$ and

$$\mathcal{F}_S^{Q'}(\varphi; ny_\alpha)(g) := \int_{N_{Q'}(\mathbf{Q})\backslash N_{Q'}(\mathbf{A})} \mathcal{F}(\varphi; ny_\alpha)(wg)\varepsilon_{[0,S]}^{-1}(w)\,dw.$$

In analogy with Lemma 6.1, we have the following relationship between the Fourier coefficients $\mathcal{F}_S^{Q'}(\varphi; ny_\alpha)(g)$ and Fourier coefficients of φ along the N_P.

Lemma 6.2. *Let $\alpha \in 2\mathbf{Z}\backslash\{0\}$ and $n \in \mathbf{Z}_{\geq 1}$. Suppose $S \in V'_{1,2}(\mathbf{Q})$ and $g \in G(\mathbf{A})$. Let $\varphi_{[ny_\alpha,S]}$ denote the Fourier coefficient of φ along N_P corresponding to the character $\varepsilon_{[ny_\alpha,S]}$. Then*

$$\mathcal{F}_S^{Q'}(\varphi; ny_\alpha)(g) = \int_{\mathbf{A}} \varphi_{[ny_\alpha,S]}(\exp(sb_1 \wedge b_{-2})g)\,ds, \qquad (g \in G(\mathbf{A})).$$

Proof. The statement is a mild generalization of a [Joh+24, Lemma 7.1] with essentially the same proof. The details of the general proof are available in [McG25, Lemma 4.0.3]. ☐

6.3. **The Fourier-Jacobi Expansion of Quaternionic Modular Forms on G.** In this subsection we combine Lemma 6.1 and Lemma 6.2 with the results of Subsection 5.1 to refine the Fourier-Jacobi expansion,

$$\varphi(g) = \varphi_{N_R}(g) + \sum_{y \in V_{3,3}(\mathbf{Z}):\, y \neq 0} \mathcal{F}(\varphi; y)(g)., \tag{6.10}$$

in the case when $\varphi \in M_\ell(1)$.

Proposition 6.3. *Suppose $\varphi \in M_\ell(1)$ is a weight ℓ quaternionic modular form of level 1. Then the Fourier-Jacobi expansion (6.10) takes the form*

$$\varphi(g_\infty) = \varphi_{N_R}(g_\infty) + \sum_{y \in V_{3,3}(\mathbf{Z}) \setminus \{0\}:\, (y,y) \geq 0} \mathcal{F}(\varphi; y)(g_\infty) \tag{6.11}$$

for all $g_\infty \in G(\mathbf{R})$. Moreover, if $\varphi \in S_\ell(1)$, then for $g_\infty \in G(\mathbf{R})$, (6.11) takes the form

$$\varphi(g_\infty) = \sum_{y \in V_{3,3}(\mathbf{Z}):\, (y,y) > 0} \mathcal{F}(\varphi; y)(g_\infty). \tag{6.12}$$

Proof. Since φ has level one, to prove that $\mathcal{F}(\varphi; y)|_{G(\mathbf{R})} \equiv 0$ for a given $y \in V_{3,3}(\mathbf{Z})$, it suffices to show that $\mathcal{F}(\varphi; y')|_{G(\mathbf{R})} \equiv 0$ for any y' in the same $M_R^{\mathrm{der}}(\mathbf{Z})$ orbit of y.

Initially, we suppose $\varphi \in M_\ell(1)$ and $y \in V_{3,3}(\mathbf{Z})$ satisfies $\langle y, y \rangle < 0$. Then by [Joh+24, Lemma B.2], there exist $\alpha \in 2\mathbf{Z}_{<0}$ and $n \in \mathbf{Z}_{\geq 1}$ such that y is in the same $M_R^{\mathrm{der}}(\mathbf{Z})$ orbit as $ny_\alpha = nb_3 + n\alpha b_{-3}/2$. Combining Lemma 6.2 with Corollary 5.4, if $S \in V'_{1,2}(\mathbf{Q})$ then $\mathcal{F}_S^{Q'}(\varphi; ny_\alpha)(g_\infty)$ is equal to

$$\int_{\mathbf{A}_f} a_{[ny_\alpha, S]}(\varphi, \exp(s_f b_1 \wedge b_{-2}))\, ds_f \cdot \int_{\mathbf{R}} \mathcal{W}_{[2\pi n y_\alpha, 2\pi S]}(\exp(s_\infty b_1 \wedge b_{-2}) g_\infty)\, ds_\infty. \tag{6.13}$$

Since ny_α is a vector of negative norm, Proposition 5.6(a) implies that the pair $[ny_\alpha, S]$ is not positive semi-definite. Therefore, $\mathcal{W}_{[2\pi n y_\alpha, 2\pi S]} \equiv 0$ according to Theorem 5.3, and $\mathcal{F}(\varphi; ny_\alpha)$ vanishes identically on $G(\mathbf{R})$. Hence, we have proven expression (6.11).

We now address the proof of (6.12). Thus suppose φ is cuspidal. It remains to show that $\mathcal{F}(\varphi; y)$ vanishes identically on $G(\mathbf{R})$ whenever $y \in V_{3,3}(\mathbf{Z})$ is isotropic. For this, it suffices to show that $\mathcal{F}(\varphi; nb_2)$ vanishes identically on $G(\mathbf{R})$ for all $n \in \mathbf{Z}_{\neq 0}$. Applying Lemma 6.1, $\mathcal{F}(\varphi; nb_2)$ vanishes identically on $G(\mathbf{R})$ provided $\varphi_{[0,T]}$ vanishes identically on $G(\mathbf{R})$ for all $T \in V_{2,2}(\mathbf{Q})$. When $T = 0$, $\varphi_{[0,T]} = \varphi_{N_P}$ vanishes since φ is cuspidal. If $T \neq 0$, then a consequence of definition 5.2, $[0, T]$ does not satisfy $[0, T] \succ 0$. Hence, the vanishing of $\varphi_{[0,T]}$ follows from Corollary 5.7. ☐

6.4. **Hard Analysis of $\mathcal{F}(\varphi; y)$: The case of isotropic y.** In this subsection, we further refine Lemma 6.1 in the case when φ is a weight ℓ quaternionic modular form of level 1. The main results are Proposition 6.4 and Corollary 6.5. The proofs of these result use Lemma 8.2 and Proposition 8.3, for which the reader should consult Section 8.

Proposition 6.4. *Let $n \in \mathbf{Z}_{\neq 0}$, and $g_\infty \in G(\mathbf{R})$. Recall the Fourier coefficients $\mathcal{F}_T^N(\varphi; nb_2)(g)$ defined in (6.1). If $T \in V_{2,2}(\mathbf{Q})\backslash\{0\}$ then $\mathcal{F}_T^N(\varphi; nb_2)(g_\infty) = 0$ for all $g_\infty \in G(\mathbf{R})$, and*

$$\mathcal{F}(\varphi; nb_2)(g_\infty) = \int_{\mathbf{Q}\backslash\mathbf{A}} \varphi_{N_P}(\exp(sb_1 \wedge b_{-2})g_\infty)\psi^{-1}(ns)\, ds. \tag{6.14}$$

Proof. In light of Lemma 6.1 and the fact that S is abelian, expression (6.14) follows immediately from the vanishing of $\mathcal{F}_T^N(\varphi; nb_2)(g_\infty)$ for all non-zero $T \in V_{2,2}(\mathbf{Q})$. Thus suppose $T \in V_{2,2}(\mathbf{Q})$ is non-zero and $s \in \mathbf{A}$. One checks that for $m = \exp(sb_1 \wedge b_{-2})$ and $[T_1, T_2] = [0, T]$, the hypotheses of Lemma 8.2 are satisfied, and thus the term $\varphi_{[0,T]}(\exp(sb_1 \wedge b_{-2})g_\infty)$ appearing in the integrand of (6.2) is independent of s. It follows that

$$\mathcal{F}_T^N(\varphi; nb_2)(g_\infty) = \varphi_{[0,T]}(g_\infty) \int_{\mathbf{Q}\backslash\mathbf{A}} \psi^{-1}(ns)\, ds = \varphi_{[0,T]}(g_\infty) \cdot 0 = 0.$$

$\square$

The lemma above shows that the degenerate Fourier-Jacobi coefficient $\mathcal{F}(\varphi; nb_2)$ factors across the constant term φ_{N_P}. In Subsection 5.1, we saw that φ_{N_P} is closely related to a holomorphic modular form on the group $M_P^{\mathrm{der}} = \mathrm{SL}_2^3$. As such, it is natural to consider Fourier coefficients of φ_{N_P} along the unipotent radical of a Borel subgroup of M_P^{der}. Using the isomorphism of [Joh+24, Theorem A.8], the unipotent radical of such a subgroup has Lie algebra

$$e_2 \otimes E = \mathbf{Q}\text{-span}\{b_{-3} \wedge b_{-4}, b_2 \wedge b_{-1}, b_3 \wedge b_{-4}\}.$$

Given a triple of $C = (a, b, c) \in \mathbf{Q}^3$, we let η_C denote the character of $[\exp(e_2 \otimes E)]$ corresponding to the linear functional

$$xb_2 \wedge b_{-1} + yb_{-3} \wedge b_{-4} + xb_3 \wedge b_{-4} \mapsto bz - ay - cx.$$

Proposition 8.3 shows that the Fourier coefficient $(\varphi_{N_P})_{\exp(e_2 \otimes E), C} \colon G(\mathbf{A}) \to \mathbf{V}_\ell$, defined by

$$(\varphi_{N_P})_{\exp(e_2 \otimes E), C}(g) = \int_{[\exp(e_2 \otimes E)]} \varphi_{N_P}(ug)\eta_C^{-1}(u)\, du, \tag{6.15}$$

is equal to an integral transform of a Fourier coefficient of φ along N_P. More precisely, we have the following corollary to Proposition 8.3.

Corollary 6.5. *Let $\varphi \in M_\ell$ be a weight ℓ modular form of level 1 of $G = \mathrm{Spin}(V)$. As in (5.6), write Φ for the holomorphic modular form on M_P^{der} associated to the constant term φ_{N_P}. Folllowing the notation in (4.21), write the Fourier expansion of the classical modular form associated to Φ as*

$$f_\Phi(z_1, z_2, z_3) = \sum_{C=(a,b,c) \in \mathbf{Z}^3_{\geq 0}} a_\varphi(C) e^{2\pi i(az_1 + bz_2 + cz_3)}. \tag{6.16}$$

Then there exists a non-zero scalars $B \in \mathbf{C}$ such that if $C \in \mathbf{Z}^3$, then

$$a_\varphi(C) = B \cdot \int_{S_C(\mathbf{A}_f)\backslash \exp(e_2 \otimes E)(\mathbf{A}_f)} a_{[ab_3 + cb_{-3}, -bb_{-4}]}(\varphi, u)\, du. \tag{6.17}$$

Here S_C denotes the stabilizer of the character $\varepsilon_{[ab_3 + cb_{-3}, -bb_{-4}]}$ in $\exp(e_2 \otimes E)$.

Proof. This is a restatement of the equality (8.9) in the case $G_J = G^{\mathrm{ad}}$ and $g_f = 1$. $\square$

6.5. **Hard Analysis of $\mathcal{F}(\varphi; y)$: The case of non-isotropic y.** Our goal in this subsection is to refine the result of Lemma 6.2 in the special case when $\varphi \in M_\ell(1)$ is a modular form on G of level 1. The main result is Proposition 6.7, which shows that for such a φ, the non-degenerate Fourier-Jacobi coefficients of Lemma 6.2 give rise to holomorphic modular forms in the sense of Subsection 4.1.

We adopt the notation of Lemma 6.2, so $n \in \mathbf{Z}_{\geq 1}$, $\alpha \in 2\mathbf{Z}\backslash\{0\}$, and $ny_\alpha = nb_3 + n\alpha b_{-3}/2$. By Proposition 6.3, if $\varphi \in M_\ell(1)$ then $\mathcal{F}(\varphi; ny_\alpha)(g_\infty) = 0$ whenever $\alpha < 0$ and $g_\infty \in G(\mathbf{R})$. Hence, we may assume $\alpha > 0$, in which case $\mathcal{F}(\varphi; y)$ Fourier expands along $N_{Q'}$ as

$$\mathcal{F}(\varphi; ny_\alpha)(g) = \sum_{S \in V'_{1,2}(\mathbf{Q})} \mathcal{F}_S^{Q'}(\varphi; ny_\alpha)(g). \tag{6.18}$$

In fact, if $S \in V'_{1,2}$ is a vector of negative norm, then the line of reasoning applied during the proof of Proposition 6.3 implies that $\mathcal{F}_S^{Q'}(\varphi; ny_\alpha)$ vanishes identically on $G(\mathbf{R})$. Thus, if $g = g_\infty \in G(\mathbf{R})$, then (6.18) takes the form

$$\mathcal{F}(\varphi; ny_\alpha)(g_\infty) = \sum_{S \in V'_{1,2}(\mathbf{Q}) \colon (S,S) \geq 0} \mathcal{F}_S^{Q'}(\varphi; ny_\alpha)(g_\infty). \tag{6.19}$$

The summation indices appearing in (6.19) resemble the indices which appear in the Fourier expansion of holomorphic modular forms on M' (see (4.6)). However, since $K_\infty \cap M'$ is not necessarily a maximal compact subgroup of M', it is necessary to introduce a modified definition of $\mathcal{F}(\varphi; ny_\alpha)$ in order to obtain holomorphic modular forms on M' from $\mathcal{F}(\varphi; ny_\alpha)$. For this we recall the element $g_y \in G(\mathbf{R})$ and the representation $\mathbf{V}_\ell^\alpha$ introduced in Subsection 4.1. Define

$$\widetilde{\mathcal{F}}(\varphi; ny_\alpha) \colon G(\mathbf{A}) \to \mathbf{V}_\ell^\alpha, \qquad g \mapsto \mathcal{F}(\varphi; ny_\alpha)(gg_{y_\alpha}).$$

Then $\widetilde{\mathcal{F}}(\varphi; ny_\alpha)(gk) = k^{-1} \cdot \widetilde{\mathcal{F}}(\varphi; ny_\alpha)(g)$ for all $g \in G(\mathbf{A})$ and $k \in K_{y_\alpha}$. Furthermore, Lemma 6.2 and Theorem 5.3 imply that

$$\widetilde{\mathcal{F}}(\varphi; ny_\alpha)(g_\infty) = \sum_{S \in V'_{1,2}(\mathbf{Q}) \colon (S,S) \geq 0} \widetilde{\mathcal{F}}_S^{Q'}(\varphi; ny_\alpha)(g_\infty),$$

where for $S \in V'_{1,2}(\mathbf{Q})$ satisfying $(S, S) \geq 0$,

$$\widetilde{\mathcal{F}}_S^{Q'}(\varphi; ny_\alpha)(g_\infty) = \int_{X(\mathbf{A}_f)} a_{[ny_\alpha, S]}(\varphi, x_f)\, dx_f \cdot \int_{X(\mathbf{R})} \mathcal{W}_{[2\pi ny_\alpha, 2\pi S]}(x_\infty g_\infty g_{y_\alpha})\, dx_\infty. \tag{6.20}$$

Here $g_\infty \in G(\mathbf{R})$ and $X = \{\exp(sb_1 \wedge b_{-2}) \colon s \in \mathbb{G}_a\}$. Given $g_\infty \in M_{Q'}(\mathbf{R})$, let $\overline{g}_\infty$ to be the image of g_∞ in $\mathrm{SO}(V'_{2,3})$. We express $\overline{g}_\infty$ as an ordered pair $\overline{g}_\infty = (t, u)$ with $t \in \mathbf{R}_{>0}$ and $u \in \mathrm{SO}(V'_{1,2})(\mathbf{R})^0$. The coordinate $t \in \mathbf{R}$ is normalized so that $(t, 1) \cdot b_2 = tb_2$. In [Joh+24, Proposition 7.3], the archimedean integral in (6.20) is evaluated when $\alpha = 2$ and $n = 1$. The computation in (loc. cit.) generalizes to the case of general α and general n as follows.

Lemma 6.6. *Suppose $\alpha \in 2\mathbf{Z}_{>0}$, $n \in \mathbf{Z}_{\geq 1}$, and $S \in V'_{1,2}(\mathbf{Q})$ is such that $[ny_\alpha, S] \succeq 0$. Let $g_\infty \in M_{Q'}(\mathbf{R})^0$ and write $\overline{g}_\infty = (t, u)$ with $t \in \mathbf{R}_{>0}$ and $u \in \mathrm{SO}(V'_{1,2})(\mathbf{R})^0$. Then*

$$\int_{X(\mathbf{R})} \mathcal{W}_{[2\pi ny_\alpha, 2\pi S]}(x_\infty g_\infty g_{y_\alpha})\, dx_\infty = \frac{t^\ell e^{-2\sqrt{2}\pi(n\alpha^{1/2} - t(S, u \cdot v_2))}}{2\sqrt{2}n\alpha^{1/2}} \sum_{-\ell \leq v \leq \ell} i^v \frac{x^{\ell+v} y^{\ell-v}}{(\ell+v)!(\ell-v)!}$$

Proof. The result is proven by applying the same computational techniques as in [Joh+24, Proposition 7.3]. For a detailed account, the reader may consult [McG25, Lemma 4.0.9]. ☐

Lemma 6.6 is applied to establish the holomorphy statement in the following result.

Proposition 6.7. *Let $\alpha \in 2\mathbf{Z}_{>0}$, $n \in \mathbf{Z}_{>0}$, and suppose $\varphi \in M_\ell(1)$. Then the function*

$$\xi^\varphi(ny_\alpha)\colon M'(\mathbf{R}) \to \mathbf{C}, \qquad g_\infty \mapsto \overline{\left\{\widetilde{\mathcal{F}}(\varphi; ny_\alpha)(g_\infty), (-ix+y)^{2\ell}\right\}}_{K_\infty}$$

is the automorphic form associated to a weight ℓ holomorphic modular form on M'.

Proof. The result is proven by an argument parallel to the one given in the proof [Joh+24, Corollary 7.6]. Again, the reader may consult [McG25, Proposition 4.0.12] for full details. Since it will be relevant shortly, we note that as a consequence of the argument in (loc. cit.), the Fourier expansion of $\xi^\varphi(ny_\alpha)$ along the $N_{Q'}$ takes the form

$$\xi^\varphi(ny_\alpha)(g_\infty) = \sum_{S \in V'_{1,2}(\mathbf{Q})_{\geq 0}} \xi^\varphi(ny_\alpha)_S(g_\infty)$$

where

$$j_{y_\alpha}(h, -i\sqrt{2}v_2)^\ell \xi^\varphi(ny_\alpha)_S(g_\infty) = \eta_S \cdot e^{2\pi i(S,Z)}.$$

Here $j_{y_\alpha}(\cdot, -i\sqrt{2}v_2)$ is the automorphy factor of subsection 4.1 and

$$\eta_S = \frac{e^{-2\sqrt{2}\pi n\alpha^{1/2}}}{2\sqrt{2}n\alpha^{1/2}} \cdot \overline{\int_{X(\mathbf{A}_f)} a_{[ny_\alpha,S]}(\varphi, x_f)\,dx_f \cdot \left\{\sum_{-\ell\leq v\leq \ell} i^v \frac{x^{\ell+v}y^{\ell-v}}{(\ell+v)!(\ell-v)!}, (-ix+y)^{2\ell}\right\}}_{K_\infty}.$$

□

By examining the explicit formula for η_S given in the proof above, we obtain the following corollary to Proposition 6.7.

Corollary 6.8. *Given $\varphi \in M_\ell(1)$ and $S \in V'_{1,2}(Q)$ such that $(S,S) \geq 0$, define*

$$A_{\xi^\varphi(ny_\alpha)}[S] = \overline{\int_{\mathbf{A}_f} a_{[ny_\alpha,S]}(\varphi, \exp(s_f b_1 \wedge b_{-2}))\,ds_f}. \tag{6.21}$$

Then the numbers $\{A[S]\colon S \in V'_{1,2}(\mathbf{Q})\colon (S,S) \geq 0\}$ are the Fourier coefficients of a holomorphic modular form on $\mathfrak{h}_{y_\alpha}$ of level $M'(\mathbf{Z})$. Hence, $A_{\xi^\varphi(ny_\alpha)}[S] \neq 0$ implies $S \in V'_{1,2}(\mathbf{Z})^\vee_{\geq 0}$.

6.6. **The Primitivity Theorem.** In this subsection, we apply the results of Subsections 6.4 and 6.5 to establish the main result of this section, Theorem 1.1. Before giving the proof of Theorem 1.1 we require the following lemma.

Lemma 6.9. *Assume $\ell \in \mathbf{Z}_{>0}$ and suppose $\varphi \in M_\ell(1)$. Recall the orthogonal parabolic subgroup $R = M_R N_R$ defined in Subsection 3.5. If $\varphi \equiv \varphi_{N_R}$ then $\varphi \equiv 0$.*

Proof. Assume $\varphi \equiv \varphi_{N_R}$. The weight ℓ is positive, and hence it suffices to show that φ is constant. Moreover, since $G(\mathbf{A}) = G(\mathbf{Q})G(\mathbf{R})G(\widehat{\mathbf{Z}})$, it is enough to prove that $\varphi|_{G(\mathbf{R})}$ is constant. Let X denote the subgroup of $G(\mathbf{R})$ generated by $G(\mathbf{Z})$ and $N_R(\mathbf{R})$. Then $\varphi(x) = \varphi(1)$ for all $x \in X$, and so we are reduced to proving that $X = G(\mathbf{R})$.

As G is semi-simple and split, G is the special fiber of a finite Chevalley group $\underline{G}$ defined over $\mathbf{Z}$ [Ste68, Theorem 6]. So, $G(\mathbf{Z})$ contains representatives for each of the Weyl reflections in G. Since the root system of G is simply laced, the Weyl group of G acts transitively on the set of roots of G [Spr98, Lemma 10.2.2(ii)]. Hence, X contain the root subgroup $U_\alpha(\mathbf{R})$ for every root α of G. Since G is simply connected, $G(\mathbf{R})$ is generated by the root subgroup $U_\alpha(\mathbf{R})$ as α runs over the set of roots (see for example [Spr98, §9.4]). Hence, $X = G(\mathbf{R})$. □

We are ready to give the proof of our first main result.

Proof of Theorem 1.1. Suppose for a contradiction that $\varphi \not\equiv 0$. Then $\varphi|_{G(\mathbf{R})} \not\equiv 0$, and applying Lemma 6.9 in tandem with (6.11), there exists a non-zero vector $y \in V_{3,3}(\mathbf{Z})$ such that $(y, y) \geq 0$ and $\mathcal{F}(\varphi; y)|_{G(\mathbf{R})}$ is not identically zero. The proof partitions into the case when y is non-isotropic, and the case when y is isotropic.

First we suppose y is non-isotropic. Then there exists $\alpha \in 2\mathbf{Z}_{>0}$ and $n \in \mathbf{Z}_{\geq 1}$ such that $\mathcal{F}(\varphi; ny_\alpha)|_{G(\mathbf{R})} \not\equiv 0$. Applying (6.20) and (6.21), there exists $S \in V'_{1,2}(\mathbf{Q})$ such that

$$A_{\xi^\varphi(ny_\alpha)}[S] \neq 0$$

By Corollary 6.8 and Theorem 4.5, we may assume S is primitive. Then (6.21) gives that

$$\overline{A_{\xi^\varphi(ny_\alpha)}[S]} = \int_{\mathbf{A}_f/\widehat{\mathbf{Z}}} a_{[ny_\alpha, S]}(\exp(sb_1 \wedge b_{-2})) = \sum_{s \in \mathbf{Q}/\mathbf{Z}} \Lambda_\varphi[ny_\alpha, S + sny_\alpha]. \tag{6.22}$$

If $s \in \mathbf{Q}_{[0,1)}$ and $\Lambda_\varphi[ny_\alpha, S+sny_\alpha] \neq 0$, then $S + sny_\alpha \in V_{2,2}(\mathbf{Z})$. Fix $n', m', r' \in \mathbf{Z}$ such that $\gcd(n', m', r') = 1$ and

$$S = -n'b_4 - m'b_{-4} - \frac{r'}{\alpha} y_\alpha^\vee.$$

Then $S + sny_\alpha \in V_{2,2}(\mathbf{Z})$ implies $\dfrac{-r'}{\alpha} + sn \in \mathbf{Z}$, and thus $n\alpha s \in \mathbf{Z}$. Hence,

$$\overline{A_{\xi^\varphi(ny_\alpha)}[S]} = \sum\nolimits_{s=0}^{n\alpha-1} \Lambda_\varphi \left[ny_\alpha, -n'b_4 - m'b_{-4} + \frac{s - r'}{\alpha} b_3 + \frac{s + r'}{2} b_{-3}\right].$$

Let $s \in \mathbf{Z}_{[0,n\alpha-1]}$ be such that $(s+r')/2$ and $(s-r')/\alpha$ are integral. Then, given $d \in \mathbf{Z}_{\geq 1}$ such that d divides n', m', $(s + r')/2$, and $(s - r')/\alpha$, d also divides r'. Hence, $d = 1$ and (6.22) expresses $\overline{A_{\xi^\varphi(ny_\alpha)}[S]}$ as a sum of primitive Fourier coefficients of φ, which is a contradiction.

Next we consider the case when y is isotropic. Hence, there exists an isotropic vector $y \in V_{3,3}(\mathbf{Z})$ such that $\mathcal{F}(\varphi; y)|_{G(\mathbf{R})} \not\equiv 0$. Then we may suppose $y = nb_2$ for some $n \in \mathbf{Z}\backslash\{0\}$. By Corollary 6.4, the constant term φ_{N_P} satisfies $\varphi_{N_P}|_{G(\mathbf{R})} \not\equiv 0$. Hence, there exists $C \in \mathbf{Q}^3$ such that the Fourier coefficient $(\varphi_{N_P})_{\exp(e_2 \otimes E), C}$ of (6.15) is not identically zero on $G(\mathbf{R})$. Hence, we may may assume the coefficient $a_\varphi(C)$ of (6.16) is non-zero, and by Lemma 4.11, we may take $C = (a, b, c) \in \mathbf{Z}^3$ to be such that $\gcd(a, b, c) = 1$.

Let $x \in E$ be such that $(x, C)_E \neq 0$, then by Corollary 6.5,

$$a_\varphi(C) = B_C \cdot \sum_{s \in \mathbf{Q}/\mathbf{Z}} \Lambda_\varphi[ab_3 + cb_{-3} + s(C, x)_E b_{-4}, -bb_{-4}]. \tag{6.23}$$

However, since $\gcd(a, b, c) = 1$, if $s \in \mathbf{Q}/\mathbf{Z}$, then by assumption,

$$\Lambda_\varphi[ab_3 + cb_{-3} + s(C, x)_E b_{-4}, -bb_{-4}] = 0.$$

So, (6.23) contradicts $a_\varphi(C) \neq 0$. Hence $\mathcal{F}(\varphi; y)|_{G(\mathbf{R})} \equiv 0$, which completes the proof. $\square$

Definition 6.10. We say that a pair $[T_1, T_2] \in V_{2,2}(\mathbf{Z})^{\oplus 2}$ is slice primitive if

$$\mathbf{Q}\text{-span}\{T_1, T_2\} \cap V_{2,2}(\mathbf{Z}) = \mathbf{Z}\text{-span}\{T_1, T_2\}.$$

Corollary 6.11. *Let $\ell > 0$ and suppose φ is a weight ℓ cuspidal quaternionic modular form of level one. Write the Fourier expansion of φ_Z as*

$$\varphi(g_\infty) = \sum_{T_1, T_2 \in V_{2,2}(\mathbf{Z}) \colon [T_1, T_2] \succ 0} \Lambda_\varphi[T_1, T_2] \mathcal{W}_\ell(g_\infty), \tag{6.24}$$

and assume $\Lambda_\varphi[T_1,T_2]=0$ for all slice primitive vectors $[T_1,T_2]\in V_{2,2}(\mathbf{Z})^{\oplus 2}$. Then $\varphi\equiv 0$.

Proof. Assume $\Lambda_\varphi[T_1,T_2]=0$ for all slice primitive vectors $[T_1,T_2]\in V_{2,2}(\mathbf{Z})^{\oplus 2}$. By Theorem 1.1, it suffices to show that if $[T_1,T_2]\in V_{2,2}(\mathbf{Z})^{\oplus 2}$ is primitive, then $\Lambda_\varphi[T_1,T_2]=0$.

Assume $[T_1,T_2]\succ 0$ is primitive. Applying (3.2), we may write

$$[T_1,T_2]=\left[\begin{pmatrix} a & b\\ c & d\end{pmatrix},\begin{pmatrix} e & f\\ g & h\end{pmatrix}\right]$$

where $a,\dots,h\in\mathbf{Z}$ are jointly coprime. Since φ is of level 1, [Bha04, Appendix Ch. II] implies that we may assume

$$[T_1,T_2]=[y_\alpha,-nb_4-mb_{-4}+rb_{-3}]\,. \tag{6.25}$$

Moreover, since $[T_1,T_2]\succ 0$, Proposition 5.6 implies $\alpha>0$.

Consider the Fourier expansion of the holomorphic modular form $\xi^\varphi(y_\alpha)$ of Corollary 6.8. By (6.22), if $n',m',r'\in\mathbf{Z}$ and $S=-n'b_4-m'b_{-4}-\dfrac{r'}{\alpha}y_\alpha^\vee$ then $\overline{A_{\xi^\varphi(y_\alpha)}[S]}$ equals

$$\sum_{s=0}^{\alpha-1}\Lambda_\varphi\left[y_\alpha,-n'b_4-m'b_{-4}+\frac{s-r'}{\alpha}b_3+\frac{s+r'}{2}b_{-3}\right]=\Lambda\left[y_\alpha,-n'b_4-m'b_{-4}+r'b_{-3}\right]. \tag{6.26}$$

Clearly, if S is primitive, then $[y_\alpha,-n'b_4-m'b_{-4}+r'b_{-3}]$ is slice primitive. Thus, $\overline{A_{\xi^\varphi(y_\alpha)}[S]}=0$ for all primitive S, and Corollary 6.8, together with Theorem 4.5, implies $\xi^\varphi(y_\alpha)\equiv 0$. Hence, by (6.26), $\Lambda\left[y_\alpha,-n'b_4-m'b_{-4}+r'b_{-3}\right]=0$ for all $n',m',r'\in\mathbf{Z}$. Thus, (6.25) implies $\Lambda[T_1,T_2]=0$ as required. □

As a corollary to Theorem 1.1 and Corollary 6.11, we obtain the following.

Corollary 6.12. *Let $\varphi_1,\varphi_2\in M_\ell(1)$ be level one quaternionic modular forms on G.*

(i) If $\Lambda_{\varphi_1}[B]=\Lambda_{\varphi_2}[B]$ for all primitive $B\in V_{2,2}(\mathbf{Z})^{\oplus 2}$. Then $\varphi_1=\varphi_2$.

(ii) If φ_1 and φ_2 are cuspidal and

$$\Lambda_{\varphi_1}[B]=\Lambda_{\varphi_2}[B],$$

for all slice primitive $B\in V_{2,2}(\mathbf{Z})^{\oplus 2}$, then $\varphi_1=\varphi_2$.

7. An Application to The Quaternionic Maass Spezialschar

7.1. **Quaternionic Modular Forms on SO_8.** Let V denote the 8-dimensional split quadratic space of Subsection 3.2. In this subsection, we review the definition of quaternionic modular forms on $\mathrm{SO}(V)$ following [Pol21]. We also explain some notation concerning the Fourier coefficients of modular forms on $\mathrm{SO}(V)$.

Write $\pi\colon G\to\mathrm{SO}(V)$ for the projection homomorphism defined in Subsection 3.3. The representation $\mathbf{V}$ of Subsection 3.7 occurs in the Lie algebra of the maximal compact subgroup $K_\infty\le\mathrm{Spin}(V)(\mathbf{R})$. Therefore, $\mathbf{V}$ descends along π to give a representation of the identity component of the maximal compact subgroup $K_{\mathrm{SO}_8}\le\mathrm{SO}(V)(\mathbf{R})$, which is the image of K_∞ under π. Similarly, the differential operator D_ℓ of Subsection 5.1 descends to give a $K^0_{\mathrm{SO}_8}$ invariant differential operator on smooth functions $\mathrm{SO}(V)(\mathbf{R})\to\mathbf{V}_\ell$. Let $\mathfrak{so}(V)=\mathrm{Lie}(\mathrm{SO}(V))\otimes_{\mathbf{Q}}\mathbf{C}$ and write $Z(\mathfrak{so}(V))$ for the center of the universal enveloping algebra of $\mathfrak{so}(V)$.

Definition 7.1. A weight $\ell>0$ quaternionic modular form on $\mathrm{SO}(V)$ of level one is a smooth moderator growth function $\varphi\colon G(\mathbf{A})\to\mathbf{V}_\ell$ such that

(i) If $g \in \mathrm{SO}(V)(\mathbf{A})$ and $\gamma \in \mathrm{SO}(V)(\mathbf{Q})$ then $\varphi(\gamma g) = \varphi(g)$,
(ii) If $g \in \mathrm{SO}(V)(\mathbf{A})$ and $k \in \mathrm{SO}(V)(\widehat{\mathbf{Z}})$ then $\varphi(gk) = \varphi(g)$,
(iii) If $g \in \mathrm{SO}(V)(\mathbf{A})$ and $k_\infty \in K^0_{\mathrm{SO}_8}$ then $\varphi(gk_\infty) = k_\infty^{-1}\varphi(g)$,
(iv) $D_\ell \varphi \equiv 0$, and
(v) φ is $Z(\mathfrak{so}(V))$-finite.

Clearly, if φ is a quaternionic modular form on $\mathrm{SO}(V)$ of level one, then the pull-back of φ by π is a level one quaternionic modular form $\mathrm{Spin}(V)$ in the sense of Subsection 5.1.

Lemma 7.2. *Suppose φ_1 and φ_2 are quaternionic modular forms on $\mathrm{SO}(V)$ of level one. For $i = 1, 2$, let $\pi^*\varphi_i$ denote the pull-back of φ_i to $\mathrm{Spin}(V)$. If $\pi^*\varphi_1 = \pi^*\varphi_2$ then $\varphi_1 = \varphi_2$.*

Proof. Let $V(\mathbf{Z}) = \mathbf{Z}\text{-span}\{b_{\pm i}\colon i = 1, 2, 3, 4\}$.The quotient $\mathrm{SO}(V)(\mathbf{Q})\backslash \mathrm{SO}(V)(\mathbf{A}_f)/\mathrm{SO}(V)(\widehat{\mathbf{Z}})$ classifies isomorphism classes of lattices in the genus of $V(\mathbf{Z})$. Thus, by the Hasse principle

$$|\mathrm{SO}(V)(\mathbf{Q})\backslash \mathrm{SO}(V)(\mathbf{A}_f)/\mathrm{SO}(V)(\widehat{\mathbf{Z}})| = 1,$$

and if $\Gamma = \mathrm{SO}(V)(\widehat{\mathbf{Z}}) \cap \mathrm{SO}(V)(\mathbf{Q})$, then $\mathrm{SO}(V)(\mathbf{Q})\backslash \mathrm{SO}(V)(\mathbf{A})/G(\widehat{\mathbf{Z}}) = \Gamma\backslash \mathrm{SO}(V)(\mathbf{R})$. Since Γ contains elements in the non-identity component of $\mathrm{SO}(V)(\mathbf{R})$, it follows that φ_i is determined by its restriction to $\mathrm{SO}(V)(\mathbf{R})^0$. As $\mathrm{Spin}(V)(\mathbf{R})$ surjects onto $\mathrm{SO}(V)(\mathbf{R})^0$, the statement follows. ☐

Suppose φ is a level one quaternionic modular form on $\mathrm{SO}(V)$ and $B \in V_{2,2}(\mathbf{Z})^{\oplus 2}$. Define the Fourier coefficient $\Lambda_\varphi[B]$ via the equality $\Lambda_\varphi[B] = \Lambda_{\pi^*\varphi}[B]$. Then we obtain the following Corollary to Lemma 7.2 and Corollary 6.11.

Corollary 7.3. *Suppose φ_1 and φ_2 are cuspidal level one quaternionic modular forms on $\mathrm{SO}(V)$. Assume $\Lambda_{\varphi_1}[B] = \Lambda_{\varphi_2}[B]$ for all slice primitive $B \in V_{2,2}(\mathbf{Z})^{\oplus 2}$. Then $\varphi_1 = \varphi_2$.*

7.2. **The Quaternionic Saito-Kurokawa Lifting.** In this section we define the quaternionic Saito-Kurokawa subspace SK_ℓ of modular forms on SO_8. Suppose $B = [T_1, T_2] \in V^{\oplus 2}$ and define a 2-by-2 matrix with entries in $\mathbf{Q}$ via the formula

$$T(B) = \frac{1}{2}\begin{pmatrix} (T_1, T_1) & (T_1, T_2) \\ (T_2, T_1) & (T_2, T_2) \end{pmatrix}. \tag{7.1}$$

Theorem 7.4. [Pol21, Theorem 4.1.1] *Suppose $\xi\colon \mathrm{Sp}_4(\mathbf{Q})\backslash \mathrm{Sp}_4(\mathbf{A}) \to \mathbf{C}$ is the automorphic function associated to a level one, cuspidal, holomorphic modular form of even weight $\ell \geq 16$. Write the classical Fourier expansion of ξ in the form $F_\xi(Z) = \sum_{T>0} B_\xi[T]\exp(2\pi i \operatorname{tr}(TZ))$. For all $B \in V_{2,2}(\mathbf{Z})^{\oplus 2}$ satisfying $B \succ 0$, define*

$$\Lambda_{\theta^*(F_\xi)}(B) = \sum_{r \,\in\, \mathrm{GL}_2(\mathbf{Z})\backslash \mathrm{M}_2(\mathbf{Z})^{\det \neq 0} \;\; s.t. \;\; Br^{-1} \,\in\, V_{2,2}(\mathbf{Z})^{\oplus 2}} |\det(r)|^{\ell-1}\overline{B_\xi[{}^t r^{-1} T(B) r^{-1}]}. \tag{7.2}$$

Here Br^{-1} is computed as a matrix product where $B = [T_1, T_2]$ is a 1×2 matrix with entries in $V_{2,2}(\mathbf{Z})$. Then the numbers $\Lambda_{\theta^*(F_\xi)}(B)$ are the Fourier coefficients of a non-zero, weight ℓ, level one, cuspidal quaternionic modular form, $\theta^*(F_\xi)$, on SO_8.

Definition 7.5. Suppose $\ell \geq 16$ is even and let $S_\ell(\mathrm{Sp}_4(\mathbf{Z}))$ denote the space of cuspidal holomorphic Siegel modular forms of weight ℓ, genus 2, and level 1. Define the quaternionic Saito-Kurokawa subspace of weight ℓ as $\mathrm{SK}_\ell = \{\theta^*(F)\colon F \in S_\ell(\mathrm{Sp}_4(\mathbf{Z}))\}$.

7.3. **The Quaternionic Maass Spezialschar.** The purpose of this subsection is to present a characterization of SK_ℓ which is analogous to the characterization of the classical Saito-Kurokawa subspace given by [Zag81, Theorem 1(iii)].

Definition 7.6. The weight ℓ quaternionic Maass Spezialschar MS_ℓ is the subspace of level one, weight ℓ quaternionic modular forms φ on $\mathrm{SO}(V)$ such that if $B_1, B_2 \in V_{2,2}(\mathbf{Z})^{\oplus 2}$ are slice primitive and $T(B_1) = T(B_2)$ (see (7.1)), then $\Lambda_\varphi[B_1] = \Lambda_\varphi[B_2]$.

Definition 7.6 should be compared to [Joh+24, Definition 5.8]. The reader will note that the condition in Definition 7.6 is equivalent to condition (i) in (loc. cit.). The next Lemma shows that condition (ii) of (loc.cit) is redundant, which explains its absence in Definition 7.6.

Lemma 7.7. *Suppose $\varphi \in \mathrm{MS}_\ell$. Given $B \in V_{2,2}(\mathbf{Z})^{\oplus 2}$, choose a slice primitive $\breve{B} \in V_{2,2}(\mathbf{Z})^{\oplus 2}$ such that $T(B) = T(\breve{B})$, such a choice of $\breve{B}$ exists by [Joh+24, Lemma 5.7]. Define*

$$\Lambda_\varphi^{\mathrm{prim}}[B] = \Lambda_\varphi[\breve{B}].$$

Then, for all $B \in V_{2,2}(\mathbf{Z})^{\oplus 2}$,

$$\Lambda_\varphi[B] = \sum_{r \in \mathrm{GL}_2(\mathbf{Z})\backslash \mathrm{M}_2(\mathbf{Z})^{\det \neq 0}:\ Br^{-1} \in V_{2,2}(\mathbf{Z})^{\oplus 2}} |\det(r)|^{\ell-1} \Lambda_\varphi^{\mathrm{prim}}[Br^{-1}]. \tag{7.3}$$

In other words, Definition 7.6 is equivalent to [Joh+24, Definition 5.8].

Proof. Suppose $\varphi \in \mathrm{MS}_\ell$ and let F_ξ be the holomorphic Siegel modular form that is associated to φ by [Joh+24, Corollary 7.7]. Define $\varphi' = \theta^*(F_\xi)$. By [Joh+24, Lemma 5.10], the Fourier coefficients $\Lambda_{\varphi'}[B]$ satisfy (7.3). Hence, it suffice to prove that $\varphi = \varphi'$. Applying Corollary 7.3, it is enough to show $\Lambda_\varphi[B] = \Lambda_{\varphi'}[B]$ for all slice primitive $B \in V_{2,2}(\mathbf{Z})^{\oplus 2}$.

Suppose

$$T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

with $a, b, c \in \mathbf{Z}$. On the one hand, [Joh+24, Corollary 7.7] implies $\overline{B_\xi\,[T]} = \Lambda_\varphi[T_1, T_2]$ where $T_1 = b_3 + bb_4 + ab_{-3}$ and $T_2 = -cb_4 - b_{-4}$. Since $[T_1, T_2]$ is slice primitive and $\varphi \in \mathrm{MS}_\ell$, this implies that $\Lambda_\varphi[B] = \overline{B_\xi\,[T]}$ for all slice primitive $B \in V_{2,2}(\mathbf{Z})^{\oplus 2}$ satisfying $T(B) = T$.

On the other hand, when B is slice primitive, the summation (7.2) consists of a single term, and so the equality $\varphi' = \theta^*(F_\xi)$ implies $\Lambda_{\varphi'}[B] = \overline{B_\xi[T]}$ for all slice primitive $B \in V_{2,2}(\mathbf{Z})^{\oplus 2}$ satisfying $T(B) = T$. Since T was arbitrary, we conclude that $\Lambda_\varphi[B] = \overline{B_\xi[T(B)]} = \Lambda_{\varphi'}[B]$ for all slice primitive $B \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$, completing the proof. □

Proof of Theorem 1.5. In light of the equivalence between Definition 7.6 and [Joh+24, Definition 5.8], Theorem 1.5 follows as corollary to [Joh+24, Theorem 1.3]. □

8. The Hecke Bound Characterization of Cusp Form on G

The purpose of this section is to prove Theorem 1.7 and Theorem 1.8. We work in the general setting a group G_J, associated to cubic norm structure J. We have that $G_J = G^{\mathrm{ad}}$ when $J = E$ is the algebra of diagonal 3×3 matrices. Hence, the results we obtain for G_J will also apply to level one forms on the group G. Throughout, any unexplained notation has the same meaning as in [Pol20]. Similarly, we refer the reader to (loc. cit.) for the precise definitions of quaternionic modular forms on G_J.

8.1. **Non-Vanishing of Rank 3 Fourier Coefficients.** The purpose of this subsection is to explain the proof of the following proposition. We refer the reader to [Pol18, Definition 4.3.2] for the definition of *rank* as it pertains to elements of $W_E \simeq V_{2,2}^{\oplus 2}$.

Proposition 8.1. *Suppose $\varphi \in M_\ell(1)$ is non-zero and non-cuspidal. Then there exists a rank 3 primitive element $B \in V_{2,2}(\mathbf{Z})^{\oplus 2}$ such that $\Lambda_\varphi[B] \neq 0$.*

Proof of Proposition 8.1 assuming Theorem 1.8. Assume φ is non-cuspidal. Then Theorem 1.8 implies that $\varphi_{N_J} \neq 0$. Hence, the semi-classical holomorphic modular form Φ of (5.6) is non-zero. As in (4.21), write $a_\varphi(n_1, n_2, n_3)$ for the Fourier coefficients of f_Φ, the classical holomorphic modular form on $\mathfrak{h}_{M_P^{\mathrm{der}}}$ corresponding to Φ. By Lemma 4.11, there exists a triple of coprime positive integers $C = (n_1, n_2, n_3)$ such that $a_\varphi(C) \neq 0$. Now by (6.23), there exists $x \in E$ and $s \in \mathbf{Q}/\mathbf{Z}$ such that

$$\Lambda_\varphi[ab_3 + cb_{-3} + s(C,x)b_{-4}, -bb_{-4}] \neq 0.$$

Since $\gcd(a,b,c) = 1$, the element $B = [ab_3 + cb_{-3} + s(C,x)b_{-4}, -bb_{-4}]$ is primitive. Moreover, in the coordinates given by the isomorphism $V_{2,2}(\mathbf{Z})^{\oplus 2} \simeq W_E$, $B = (0,0,C,s(C,x))$. Hence $Q(B) = 0$ and $B^\flat = (0,0,0,N_E(C))$ where the operation $B \mapsto B^\flat$ is described in [Pol18, §4.3 pg. 20]. Therefore, since a, b, and c are positive, $B^\flat \neq 0$ and so B has rank 3 as required. □

8.2. **The Degenerate Coefficients of a Quaternionic Modular Form.** In this subsection, we assume J is a cubic norm structure such that trace pairing $(\cdot,\cdot)_J$ is positive definite. Let $K_{J,\infty}$ be the maximal compact subgroup of $G_J(\mathbf{R})$ from [Pol20, §5]. The Heisenberg parabolic subgroup of G_J [Pol20, §4.3.2] is denoted $P_J = N_J H_J$. Here H_J is the Levi subgroup of P_J defined in [Pol20, §2.2] . So $\mathrm{Lie}(N_J) = e \otimes W_J + \mathbf{Q}E_{13}$ and

$$W_J \simeq \mathrm{Lie}(N_J^{\mathrm{ab}}), \qquad w \mapsto e \otimes w,$$

where $V_2 = \langle e, f\rangle$ is the standard representation of SL_2. Write $\langle \cdot,\cdot\rangle_{W_J}$ for the usual symplectic form on W_J. We refer the reader to [Pol20] for any unexplained notation.

Given $w \in W_J$, define $\varepsilon_w \in \mathrm{Hom}(N_J(\mathbf{Q})\backslash N_J(\mathbf{A}), \mathbf{C}^\times)$ by

$$\varepsilon_w(\exp(e \otimes w')) = \psi(\langle w, w'\rangle_{W_J}).$$

If $\varphi\colon G_J(\mathbf{A}) \to \mathbf{V}_\ell$ is an automorphic function, the Fourier coefficient φ_w is defined as

$$\varphi_w\colon G_J(\mathbf{A}) \to \mathbf{V}_\ell, \quad \varphi_w(g) = \int_{[N_J]} \varphi(ng)\varepsilon_w(n)^{-1}\,dn.$$

Regarding the Fourier coefficients φ_w, we have the following.

Lemma 8.2. *Let $w \in W_J$ be non-zero and $g = g_f g_\infty \in G_J(\mathbf{A})$ be such that $g_\infty \in N_J(\mathbf{R})H_J(\mathbf{R})^\pm K_{J,\infty}^0$. Assume $m \in H_J(\mathbf{A}_f) \times H_J(\mathbf{R})^\pm$ is such that there exists $m_\mathbf{Q} \in H_J(\mathbf{Q})$, $m_\infty \in H_J(\mathbf{R})^\pm$, and $m_f \in H_J(\mathbf{A}_f)$ satisfying (i) $m = m_\mathbf{Q} m_\infty m_f$, (ii) $m_\mathbf{Q}$ and m_∞ stabilize ε_w, (iii) m_∞ is unipotent, and (iv) $\varphi(m_f g) = \varphi(g)$. Then $\varphi_w(mg) = \varphi_w(g)$.*

Proof. Let $n_g \in N_J(\mathbf{R})$, $m_g \in H_J(\mathbf{R})^\pm$, and $k_g \in K_{J,\infty}^0$ be such that $g_\infty = n_g m_g k_g$. By [Pol20, Theorem 1.2.1], there exists a unique collection of locally constant functions

$$\{a_w\colon G_J(\mathbf{A}_f) \to \mathbf{C}\colon w \in W_J\}$$

such that $\varphi_w(mg) = a_w(m_f g_f)\mathcal{W}_{J,2\pi w}(m_\infty g_\infty)$. Here $\mathcal{W}_{J,2\pi i w}$ is the generalized Whittaker function of (loc. cit.). Let ε_w^∞ be the archimedean component of ε_w. By the equivariance properties of $\mathcal{W}_{J,2\pi w}$,

$$\varphi_w(mg) = a_w(m_f g_f)\varepsilon_w^\infty(m_\infty n_g m_\infty^{-1})k_g^{-1}\cdot \mathcal{W}_{J,2\pi w}(m_\infty m_g). \tag{8.1}$$

Inspecting the formula in (loc. cit.), hypothesis (ii) and (iii) imply that $\mathcal{W}_{J,2\pi w}(m_\infty m_g) = \mathcal{W}_{J,2\pi w}(m_g)$. Moreover, (ii) implies that $\varepsilon_w^\infty(m_\infty n_g m_\infty^{-1}) = \varepsilon_w^\infty(n_g)$, and so (8.1) simplifies to $\varphi_w(mg) = \varphi_w(m_f g)$. Applying hypothesis (iv) completes the proof. □

The next proposition involves the 3-step parabolic subgroup $Q \le G_J$ associated to the element $h_Q = E_{11}+E_{22}-2E_{33}$. Since $\mathrm{ad}(h_Q)$ acts on $\mathrm{Lie}(G_J)$ with eigenvalues $-3,-2,-1,0,1,2,3$, Q admits a Levi decomposition $Q = M_Q N_Q$ where M_Q is the zero eigenspace of $\mathrm{ad}(h_Q)$. The Lie algebra of the unipotent radical N_Q decomposes into $\mathrm{ad}(h_Q)$ eigenspaces as

$$\mathrm{Lie}(N_Q) = \mathrm{Lie}(N_Q)^{[1]} \oplus \mathrm{Lie}(N_Q)^{[2]} \oplus \mathrm{Lie}(N_Q)^{[3]}.$$

Here $\mathrm{Lie}(N_Q)^{[1]} = (v_1 \otimes J) \oplus (v_2 \otimes J)$, $\mathrm{Lie}(N_Q)^{[2]} = \delta_3 \otimes J^\vee$, and $\mathrm{Lie}(N_Q)^{[3]} = \mathbf{Q}E_{13} \oplus \mathbf{Q}E_{23}$. Moreover,

$$\mathrm{Lie}([N_Q,N_Q]) = \mathrm{Lie}(N_Q)^{[1]} \oplus \mathrm{Lie}(N_Q)^{[2]}.$$

Since $h_Q = E_{11}+E_{22}-2E_{33}$, M_Q^{der} is the principle SL_2 in G_J containing the root subgroup $\exp(\mathbf{Q}E_{12})$. Let $n_\alpha \in M_Q^{\mathrm{der}}(\mathbf{Z})$ denote a representative for the Weyl reflection in the hyperplane orthogonal to α. We view $n_\alpha \in G_J(\mathbf{A})$ via the diagonal embedding $G(\mathbf{Q}) \hookrightarrow G(\mathbf{A})$.

Given $C \in J^\vee\backslash\{0\}$, let $\eta_C\colon [N_Q] \to \mathbf{C}^\times$ be the character of N_Q associated to the linear functional $v_1 \otimes X + v_2 \otimes Y \mapsto (C,Y)_J$, and let

$$S_C = \{\exp(v_2 \otimes x) \mid x \in J \text{ such that } (x,C)_J = 0\}.$$

be the stabilizer of $w_C = (0,0,C,0)$ inside $\exp(v_2 \otimes J)$.

Proposition 8.3. *Suppose φ is a weight ℓ modular form on G_J and fix $C \in J^\vee$. Write $w_C = (0,0,C,0) \in W_J$ and let φ_{N_J} denote the constant term of φ along N_J. Define*

$$\varphi_{N_J,C}(g) = \int_{[\exp(v_2\otimes J)]} \varphi_{N_J}(ug)\eta_C^{-1}(u)du.$$

If $g \in G(\mathbf{A})$ is such that $g_\infty \in N_J(\mathbf{R})H_J(\mathbf{R})^\pm K_{J,\infty}^0$ then

$$\varphi_{N_J,C}(g) = \int_{S_C(\mathbf{A})\backslash \exp(v_2\otimes J)(\mathbf{A})} \varphi_{w_C}(un_\alpha g)du. \tag{8.2}$$

Proof. The result is proven by calculating

$$\varphi_{N_Q,\eta_C}(g) := \int_{[N_Q]} \varphi(cg)\eta_C(c)^{-1}\,dc$$

in two different ways.

To obtain the left hand side of (8.2), we calculate φ_{N_Q,η_C} by Fourier expanding φ along the center Z. Let $\varphi_{[N_Q,N_Q]}$ be the constant term of φ along $[N_Q,N_Q]$. Then $\varphi_{[N_Q,N_Q]}$ Fourier expands in characters of $[N_J^{\mathrm{ab}}]$ as $\varphi_{[N_Q,N_Q]}(g) = \sum_{d\in\mathbf{Q},c\in J^\vee}\varphi_{(0,0,c,d)}(g)$. Hence,

$$\begin{aligned}\varphi_{N_Q,\eta_C}(g) &= \int_{[\exp(v_2\otimes J)\backslash N_Q^{\mathrm{ab}}]}\int_{[\exp(v_2\otimes J)]}\varphi_{[N_Q,N_Q]}(cxg)\eta_C^{-1}(x)\,dx\,dc\\ &= \int_{[\exp(v_2\otimes J)]}\sum_{d\in\mathbf{Q},c\in J^\vee}\varphi_{(0,0,c,d)}(xg)\eta_C^{-1}(x)\int_{[\exp(v_2\otimes J)\backslash N_Q^{\mathrm{ab}}]}\varepsilon_{(0,0,c,d)}(c)\,dc\,dx\end{aligned}\tag{8.3}$$

The quotient $\exp(v_2\otimes J)\backslash N_Q^{\mathrm{ab}}$ is identified with the subgroup of N_Q^{ab} whose Lie algebra is spanned by $v_1\otimes J$. So the inner integral in (8.3) vanishes unless the summation index $c=0$. Since the subgroup $\exp(v_2\otimes J)$ stabilizes $\varepsilon_{(0,0,0,d)}$ for all $d\in\mathbf{Q}$, (8.3) simplifies to

$$\varphi_{N_Q,\eta_C}(g) = \int_{[\exp(v_2\otimes J)]}\varphi_{N_J}(xg)\eta_C^{-1}(x)dx + \int_{[\exp(v_2\otimes J)]}\sum_{d\neq 0}\varphi_{(0,0,0,d)}(xg)\eta_C^{-1}(x)dx\tag{8.4}$$

Applying Lemma 8.2, the function $x\mapsto\varphi_{(0,0,0,d)}(xg)$ is constant for all $d\neq 0$. Hence the second integral in (8.4) vanishes and we obtain $\varphi_{N_Q,\eta_C}(g) = \varphi_{N_J,C}(g)$.

To finish the proof, we obtain the right hand side of (8.2) by calculating φ_{N_Q,η_C} in a different way. This time we factor $\varphi_{[N_Q,N_Q]}$ across the central subgroup $Z' = n_\alpha Z n_\alpha^{-1}$ as

$$\varphi_{[N_Q,N_Q]}(g) = \sum_{c\in J^\vee,d\in\mathbf{Q}}\varphi_{(0,0,c,d)}(n_\alpha g).\tag{8.5}$$

Then applying (8.5), we obtain

$$\begin{aligned}\varphi_{N_Q,\eta_C}(g) &= \int_{[\exp(v_2\otimes J)]}\int_{[\exp(v_1\otimes J)]}\varphi_{[N_Q,N_Q]}(cxg)\eta_C^{-1}(c)\,dx\,dc\\ &= \int_{[\exp(v_1\otimes J)]}\sum_{d\in\mathbf{Q},c\in J^\vee}\varphi_{(0,0,c,d)}(n_\alpha xg)\int_{[\exp(v_2\otimes J)]}\varepsilon_{(0,0,c,d)}(n_\alpha c n_\alpha^{-1})\eta_C^{-1}(c)\,dc\,dx.\end{aligned}\tag{8.6}$$

The inner integral appearing in (8.6) vanishes unless $c=C$. Hence

$$\begin{aligned}\varphi_{N_Q,\eta_C}(g) &= \int_{[\exp(v_2\otimes J)]}\sum_{d\in\mathbf{Q}}\varphi_{(0,0,C,d)}(xn_\alpha g)\,dx\\ &= \int_{[\exp(v_2\otimes J)]}\sum_{u\in S_C(\mathbf{Q})\backslash\exp(v_2\otimes J)(\mathbf{Q})}\varphi_{w_C}(uxn_\alpha g)\,dx\\ &= \int_{S_C(\mathbf{Q})\backslash\exp(v_2\otimes J)(\mathbf{A})}\varphi_{w_C}(xn_\alpha g)\,dx.\end{aligned}\tag{8.7}$$

By Lemma 8.2, the function $x\mapsto\varphi_{w_C}(xn_\alpha g)$ is left $S_C(\mathbf{A})$ invariant. So, (8.7) simplifies to $\varphi_{N_Q,\eta_C}(g) = \int_{S_C(\mathbf{A})\backslash\exp(v_2\otimes J)(\mathbf{A})}\varphi_{w_C}(xn_\alpha g)dx$ and the proof is complete. □

Recall [Pol20, Proposition 11.1.1], which states that if $m\in H_J(\mathbf{A}_f)\times H_J(\mathbf{R})^\pm$, then

$$\varphi_{N_J}(m) = \nu(m)^\ell|\nu(m)|(\Phi(m)[x^{2\ell}] + \beta[x^\ell y^\ell] + \Phi'(m)[y^{2\ell}]).$$

Here $\beta\in\mathbf{C}$ is constant and Φ is a holomorphic modular form on $\mathcal{H}_J^+$.

For $C\in J^\vee\backslash\{0\}$, write $C\geq 0$ if $(C,1_J)_J$ and $(C^\#,1_J)_J)\geq 0$. Then, Φ Fourier expands as

$$\Phi(m_f m_\infty) = \Phi_{\exp(v_2\otimes J)}(m_f m_\infty) + \sum_{C\in J^\vee\backslash\{0\}:\,C\geq 0}A_C(m_f)j(m_\infty,i)^{-\ell}e^{2\pi i(m_\infty\cdot(i1_J),C)_J}.$$

Here notation is as follows:

(1) $\Phi_{\exp(v_2\otimes J)}$ is the constant term of Φ along $\exp(v_2\otimes J)$,

(2) $j(m_\infty, i)$ is the automorphy factor of [Pol20, §2.3], and

(3) $\{A_C\colon H_J(\mathbf{A}_f)\to\mathbf{C}\}$ is a family of locally constants functions.

Let $u_0\in\mathbf{V}_\ell$ be such that if $\{\cdot,\cdot\}_{K^0_{J,\infty}}$ is the $K^0_{J,\infty}$ invariant pairing on $\mathbf{V}_\ell$, then

$$\left\{[x^{\ell-v}y^{\ell+v}],u_0\right\}_{K^0_{J,\infty}}=\begin{cases}1, & \text{if } v=\ell,\\ 0, & \text{else.}\end{cases}$$

So by definition, if $m=m_fm_\infty\in H_J(\mathbf{A}_f)H_J(\mathbf{R})^{\pm}$,

$$\{\varphi_{N_J,C}(m_fm_\infty),u_0\}_{K^0_{J,\infty}}=\nu(m)^\ell|\nu(m)|A_C(m_f)j(m_\infty,i)^{-\ell}e^{2\pi i(m_\infty\cdot(i1_J),C)_J}. \tag{8.8}$$

Since $\varphi_{w_C}(g_fg_\infty)=a_{w_C}(g_f)\mathcal{W}_{J,2\pi w_C}(g_\infty)$, Proposition 8.3 implies that $\varphi_{N_J,C}(g)$ is factorizable for all $g_f\in G_J(\mathbf{A}_f)$ and $g_\infty\in N_J(\mathbf{R})H_J(\mathbf{R})^{\pm}K^0_{J,\infty}$. In particular, the functions $\nu^\ell|\nu|A_C$ extend to functions on $G_J(\mathbf{A}_f)$ by the formula

$$\nu^\ell|\nu|A_C(g_f)\doteq\int_{S_C(\mathbf{A}_f)\backslash\exp(v_2\otimes J)(\mathbf{A}_f)}a_{w_C}(un_\alpha g_f)du. \tag{8.9}$$

Here $\doteq$ means that the left-hand side is a constant multiple of the right hand side.

Lemma 8.4. *Suppose $C\in J^\vee\backslash\{0\}$ is such that $C\geq 0$. Then the implied constant in (8.9) is non- zero and indepedent of C.*

Proof. In [Pol20, §11], the author constructs Klingen Eisenstein series on G_J for which the Fourier coefficients A_C are non-trivial. Hence, the non-vanishing statement is a consequence of showing that the implied constant in (8.9) is independent from C.

Assume $C\geq 0$. Since the trace pairing $(\cdot,\cdot)_J$ on J is positive definite, S_C is contained in $\exp(v_2\otimes J)$ as subgroup of codimension 1. Hence, $\mathbf{G}_a\xrightarrow{\sim}S_C\backslash\exp(v_2\otimes J)$ via the map

$$t\mapsto x(t):=\exp(v_2\otimes tC/(C,C)). \tag{8.10}$$

If R is a topological ring equipped with a measure, then $S_C(R)\backslash\exp(v_2\otimes J)(R)$ inherits a measure via (8.10). With this normalization, the implied constant in (8.9) is equal to

$$e^{2\pi(1_J,C)_J}\int_{\mathbf{R}}\{\mathcal{W}_{J,2\pi w_C}(x(t)n_\alpha),u_0\}_{K_{J,\infty}}\,dt. \tag{8.11}$$

Since $n_\alpha\in K^0_{J,\infty}$, (8.11) simplifies to

$$e^{2\pi(1_J,C)_J}\left\{\int_{\mathbf{R}}\mathcal{W}_{J,2\pi w_C}(x(s))\,ds,n_\alpha\cdot u_0\right\}_{K_{J,\infty}}$$

Therefore, it suffices to show that

$$e^{(C,1_J)_J}\cdot\int_{\mathbf{R}}\mathcal{W}_{J,2\pi w_C}(x(s))\,ds \tag{8.12}$$

is independent of C. If $r_0(i)=(1,-i1_J,-1_J,i)_J\in W_J(\mathbf{C})$, then

$$\langle w_C,x(s)\cdot r_0(i)\rangle_{W_J},=s+i(C,1_J).$$

Hence,by [Pol20, Theorem 1.2.1],

$$\int_{\mathbf{R}} \mathcal{W}_{J,2\pi w_C}(x(s))\,ds = \frac{1}{2\pi} \sum_{-\ell \le v \le \ell} [x^{\ell-v}][y^{\ell+v}] \int_{\mathbf{R}} \left(\frac{|s+2\pi i(C,1_J)_J|}{s+2\pi i(C,1_J)_J} \right)^v K_v(|s+2\pi i(C,1_J)_J|)\,ds.$$

Applying [Pol24, Lemma A.4] and the fact that $(C,1_J)_J > 0$,

$$\int_{\mathbf{R}} \left(\frac{|s+2\pi i(C,1_J)_J|}{s+2\pi i(C,1_J)_J} \right)^v K_v(|s+2\pi i(C,1_J)_J|)\,ds = \frac{1}{2} \cdot i^v \cdot e^{-2\pi(C,1_J)_J}$$

Hence, (8.12) is independent of C as required. □

8.3. **The General Cuspidality Criterion.** We continue to assume J is a general cubic norm structure with a positive definite trace form. The proof of the following lemma is adapted from [Pol24, Lemma 13.6.].

Lemma 8.5. *Suppose $C \in J^\vee$ is non-zero and define*

$$\overline{\varphi}_C \colon G_J(\mathbf{A}_f) \to \mathbf{C}, \qquad \overline{\varphi}_C(g) = \int_{S_C(\mathbf{A}_f)\backslash \exp(v_2 \otimes J)(\mathbf{A}_f)} a_{w_C}(ug)\,du.$$

If $g \in G_J(\mathbf{A}_f)$, then

$$a_{w_C}(g) = \int_{\mathbf{A}_f} \overline{\varphi}_C(\exp(\alpha E_{12})g)\,d\alpha.$$

Proof. Using the measure defined by (8.10), $\overline{\varphi}_C(g) = \int_{\mathbf{A}_f} a_{w_C}(\exp(v_2 \otimes sx)g)ds$ where $x = C/(C,C)$. Then, if $\varepsilon^f_{w_C}$ is the finite adelic part of the character ε_{w_C}, $\alpha \in \mathbf{A}_f$, and $g \in G_J(\mathbf{A}_f)$,

$$\begin{aligned} \overline{\varphi}_C(\exp(\alpha E_{12})g) &= \int_{\mathbf{A}_f} a_{w_C}(\exp(sv_2 \otimes x)\exp(\alpha E_{12})g)\,ds \\ &= \int_{\mathbf{A}_f} \varepsilon^f_{w_C \cdot \exp(sv_2 \otimes x)}(\exp(\alpha E_{12}))a_{w_C}(\exp(sv_2 \otimes x)g)\,ds \\ &= \int_{\mathbf{A}_f} \psi^f((C,\alpha sx)_J)a_{w_C}(\exp(sv_2 \otimes x)g)ds. \end{aligned}$$

Now fix $g_f \in G_J(\mathbf{A}_f)$. Since $g' \mapsto a_{w_C}(g'g_f)$ is smooth on $G_J(\mathbf{A}_f)$, there exists $M_g \in \mathbf{Z}$ such that for $u \in \exp(M_g\widehat{\mathbf{Z}}E_{12})$, and $g' \in G_J(\mathbf{A}_f)$, $a_{w_C}(g'ug_f) = a_{w_C}(g'g_f)$. Then

$$\overline{\varphi}_C(\exp(\alpha E_{12})g) = \int_{\mathbf{A}_f/M_g\widehat{\mathbf{Z}}} \psi^f((C,\alpha sx)_J)a_{w_C}(\exp(sv_2 \otimes x)g) \int_{M_g\widehat{\mathbf{Z}}} \psi^f((C,\alpha s'x)_J)ds'ds.$$

So $\overline{\varphi}_C(\exp(\alpha E_{12})g) \neq 0$ implies $\alpha \in V_{C,g} := \{\alpha' \in \mathbf{A}_f \colon \psi^f((C,\alpha' M_g\widehat{\mathbf{Z}}x)) = 1\}$. Therefore,

$$\begin{aligned} \int_{\mathbf{A}_f} \overline{\varphi}_C(\exp(\alpha E_{12})g)\,d\alpha &= \int_{V_{C,g}} \int_{\mathbf{A}_f} \psi((C,\alpha sx)_J)a_{w_C}(\exp(sv_2 \otimes x)g)\,ds\,d\alpha \\ &= \int_{\mathbf{A}_f} a_{w_C}(\exp(sv_2 \otimes x)g) \int_{V_{C,g}} \psi((C,\alpha sx)_J)\,d\alpha\,ds \\ &= \sum_{s \in \mathbf{Q}/M_g\mathbf{Z}} a_{w_C}(\exp(sv_2 \otimes x)g) \int_{V_{C,g}} \psi((C,\alpha sx)_J)\,d\alpha. \end{aligned}$$

To complete the proof, it suffices to show that if $s \in \mathbf{Q}$ satisfies $\int_{V_{C,g}} \psi((C, \alpha s x)_J)\, d\alpha \neq 0$, then $s \in M_g\mathbf{Z}$. Let $\beta \in \mathbf{Q}$ be such that $\psi^f(a) = \psi^f_{\mathrm{std}}(\beta a)$ for all $a \in \mathbf{A}_f$. Here ψ^f_{std} is the standard additive character of $\mathbf{A}_f$ with $\ker(\psi^f_{\mathrm{std}}) = \widehat{\mathbf{Z}}$. So, if $\psi((C, \alpha s x)_J) = 1$ for all $\alpha \in V_{C,g}$, then $\beta(C, \alpha s x)_J \in \widehat{\mathbf{Z}}$ for all $\alpha \in V_{C,g}$. Since $1/(\beta(C, M_g x)_J) \in V_{C,g}$, it follows that $s \in M_g\mathbf{Z}$ as required. □

Before completing the proof of Theorem 1.8 we require one additional lemma.

Lemma 8.6. *Suppose $J = \mathbb{G}_a^3$ or $J = H_3(C)$ with C a composition algebra over $\mathbf{Q}$. If $w \in W_J(\mathbf{Q})$ is non-zero and* $\operatorname{rank}(w) < 4$ *then there exists $m \in H_J(\mathbf{Q})$ and $C \in J^\vee$ such that* $\operatorname{rank}(w) = \operatorname{rank}(C)$, *$\nu(m) = 1$, and $m \cdot w = w_C$.*

Proof. Applying [Pol20, Lemma 10.0.2], there exists an element $m \in H_J(\mathbf{R})$ such that $\nu(m) = 1$ and $m \cdot w = (0, 0, c, d)$ for some $C \in J^\vee$ and $d \in \mathbf{Q}$. In fact, the proof of (loc. cit.) is valid when the field $\mathbf{R}$ is replaced by $\mathbf{Q}$. Hence, we obtain $m \in H_J(\mathbf{Q})$ such that $\nu(m) = 1$ and $m \cdot w = (0, 0, C, d)$.

Since J contains a non-zero element y such that $y^\# \neq 0$, we may act on $m \cdot w$ through the element $n^\vee(y)$ of [Pol20, §2.2] to ensure that $C \neq 0$. Assuming $C \neq 0$ and $d \neq 0$, we may then arrange for the case when $d = 0$ by selecting $x \in J$ appropriately such that $(C, x) \neq 0$, and acting on $(0, 0, C, d)$ by the element $n(x)$ of (loc. cit.). In this way we construct $m \in H_J(\mathbf{Q})$ such that $\nu(m) = 1$ and

$$m \cdot w = w_C.$$

It remains to prove that $\operatorname{rank}(w_C) = \operatorname{rank}(C)$. This is immediate in the case when $\operatorname{rank}(w_C) = 3$, since $w_C^\flat = (0, 0, 0, N_J(C))$ and so $N_J(C) \neq 0$. For the case when $\operatorname{rank}(w_C) = 2$, $N_J(C) = 0$ since $w_C^\flat = 0$, and one can apply [Pol18, Lemma 4.3.4] to conclude that $C^\# \neq 0$. Finally, for the case when $\operatorname{rank}(w_C) = 1$, $\operatorname{rank}(C) = 1$ as a consequence of (loc. cit.). □

Proof of Theorem 1.8. Suppose $\varphi_{N_J} \equiv 0$. Then applying (8.8), if $C \in J^\vee$ then $\nu^\ell |\nu| A_C(g_f) = 0$ for all $g_f \in G(\mathbf{A}_f)$. Therefore, by (8.9), $\overline{\varphi}_C(g_f) = 0$ for all $C \in J^\vee$ and $g_f \in G(\mathbf{A}_f)$. Hence, by the result of Lemma 8.5, $a_{w_C}(g_f) = 0$ for all $C \in J^\vee$ and $g_f \in G(\mathbf{A}_f)$.

We conclude that $a_w(g_f) = 0$ whenever $w \in W_J$ is in the same $H_J(\mathbf{Q})$ orbit of an element of the form w_C for $C \in J^\vee$. Hence, by the result of Lemma 8.6, $a_w(g_f) = 0$ for all $g_f \in G(\mathbf{A}_f)$ and non-zero $w \in W_J$ satisfying $\operatorname{rank}(w) < 4$. In case of $J = \mathbb{G}_a^3$, one may now complete the proof by applying Corollary 5.7.

For the case when $J = H_3(C)$ it remains so that if $\varphi_{N_J,w} = 0$ for all $w \in W_J$ satisfying $\operatorname{rank}(w) < 4$, then φ cuspidal. This is achieved by a similar argument to the one given in Corollary 5.7. Namely, let U denote the intersection of the unipotent radicals of the maximal parabolic subgroups containing a fixed minimal parabolic subgroup P_0. In this case G_J has a rational root system of type F_4, and the description of the $\operatorname{Lie}(U)$ furnished by [Pol25, Remark 9.4.7] implies that the constant term of φ along U takes the form

$$\varphi_U(g) = \sum_{w = \left(*, \left(\begin{smallmatrix} 0&0&0\\ 0&*&*\\ 0&*&* \end{smallmatrix}\right), \left(\begin{smallmatrix} *&0&0\\ 0&0&0\\ 0&0&0 \end{smallmatrix}\right), 0\right) \in W_J} \varphi_{N_J,w}(g).$$

Since any element $w \in W_J$ of the form $w = \left(*, \left(\begin{smallmatrix} 0&0&0\\ 0&*&*\\ 0&*&* \end{smallmatrix}\right), \left(\begin{smallmatrix} *&0&0\\ 0&0&0\\ 0&0&0 \end{smallmatrix}\right), 0\right)$ satisfies $\operatorname{rank}(w) < 4$, $\varphi_U \equiv 0$ which implies that φ is cuspidal. □

8.4. **The Hecke Bound Implies Cuspidality.** We recall that if $\alpha \in 2\mathbf{Z}_{\geq 1}$, then

$$V_{2,2} = \mathbf{Q}y_\alpha + V'_{1,2}$$

where $V'_{1,2}$ denote the orthogonal complement of y_α in $V_{2,2}$. Before we establish the main result of this section, we require one preparatory lemma.

Lemma 8.7. *(i) Suppose $n, \alpha \in \mathbf{Z}$ and $T \in V_{2,2}$. Write S for the orthogonal projection of T onto the subspace $V'_{1,2}$. If $B = [ny_\alpha, T]$ then*

$$Q(B) = n^2\alpha(S, S).$$

(ii) Assume $B \in V_{2,2}(\mathbf{Z})^{\oplus 2}$ is a primitive element satisfying $\mathrm{rank}(B) = 3$ and $B \succeq 0$. Then there exists $\alpha \in 2\mathbf{Z}_{\geq 1}$, and $n, m, r \in \mathbf{Z}$ such that B is in the same $M_P^{\mathrm{der}}(\mathbf{Z})$-orbit as

$$[y_\alpha, -nb_4 - mb_{-4} + rb_{-3}]$$

Proof. Statement (i) is a direct computation using the formula

$$Q([T_1, T_2]) = (T_1, T_1)(T_2, T_2) - (T_1, T_2)^2.$$

For statement (ii), we apply the primitivity of B and [Bha04, Appendix Ch. II] to reduce to the case

$$B = [y_\alpha, -nb_4 - mb_{-4} + rb_{-3}]$$

where $\alpha, n, m, r \in \mathbf{Z}$. It remains to show that we may assume $\alpha > 0$.

Since $B \succeq 0$, Proposition 5.6(i) implies $\alpha \geq 0$. Hence, without loss of generality, we may assume $\alpha = 0$. Then via the isomorphism (3.10), B maps to an element of the form

$$w = (r, b, 0, 1)$$

where $b = (n, 0, m) \in E$. Since $\mathrm{rank}(B) = 3$,

$$w^\flat = (-r^2, rb, 2b^\#, r) \neq 0.$$

Therefore, either $r \neq 0$ or $b^\# \neq 0$. Since $Q(B) = r^2 + 4N_E(b) = 0$, $r \neq 0$ and $b^\# = 0$ gives a contradiction. Hence, we may assume $b^\# \neq 0$, in which case $T_2 = -nb_4 - mb_{-4} + rb_{-3}$ satisfies $(T_2, T_2) \neq 0$. Therefore, since $B \succeq 0$, Proposition 5.6 implies $(T_2, T_2) > 0$, which implies $mn > 0$. Therefore, since $\alpha = 0$, there exists $k \in \mathbf{Z}$ such that $\gcd(k, m, n) = 1$ and

$$(T_2 + ky_\alpha, T_2 + ky_\alpha) > 0.$$

Hence, B is in the $M_P^{\mathrm{der}}(\mathbf{Z})$ orbit of an element $[y_\alpha, T'_2]$ where T'_2 is a primitive element in $V_{2,2}(\mathbf{Z})$ of positive norm. Hence, there exists $h \in \mathrm{SO}(V_{2,2})(\mathbf{Z})$ and $\beta \in 2\mathbf{Z}_{\geq 1}$ such that $h \cdot T'_2 = y_\beta$. Therefore, B is in the $M_P^{\mathrm{der}}(\mathbf{Z})$ orbit of the element $[y_\beta, -h \cdot y_\alpha]$, as required. ∎

We are now ready to establish the main result of this section.

Theorem 8.8. *Suppose $\ell \geq 5$ and let $\varphi \in M_\ell(1)$ be a weight ℓ, level one, quaternionic modular form on $G = \mathrm{Spin}(V)$. Assume that if $B \in V_{2,2}(\mathbf{Z})^{\oplus 2}$ is primitive and satisfies $B \succ 0$, then*

$$\Lambda_\varphi[B] \ll_\varphi Q(B)^{\frac{\ell+1}{2}}.$$

Then φ is cuspidal.

Proof. By the result of Proposition 8.1, it suffices to show that if $B \in V_{2,2}(\mathbf{Z})^{\oplus 2}$ is primitive and $\mathrm{rank}(B) = 3$, then $\Lambda_\varphi[B] = 0$. Hence, we assume $B \succeq 0$ is primitive of rank 3. Then by Lemma 8.7(ii), there exists $\alpha \in \mathbf{Z}_{\geq 1}$, and $n, m, r \in \mathbf{Z}$ such that

$$\Lambda_\varphi[B] = \Lambda_\varphi[y_\alpha, -nb_4 - mb_{-4} + rb_{-3}].$$

By (6.26), if $S = -nb_4 - mb_{-4} - ry_\alpha^\vee/\alpha$,

$$\Lambda_\varphi[B] = \overline{A_{\xi^\varphi(y_\alpha)}[S]}.$$

Since S is the orthogonal projection of $T = -nb_4 - mb_{-4} + rb_{-3}$ onto the $V'_{1,2}$, Lemma 8.7(i) implies $(S, S) = 0$. Hence, to show $\Lambda_\varphi[B] = 0$, it suffices to prove that $\xi^\varphi(y_\alpha)$ is cuspidal.

With a view to to applying Theorem 4.6, assume $S' = -n'b_4 - m'b_{-4} - r'y_\alpha^\vee/\alpha \in V'_{1,2}(\mathbf{Z})^\vee_{\geq 0}$ satisfies $(S', S') > 0$. Then by (6.26),

$$\overline{A_{\xi^\varphi(y_\alpha)}[S']} = \Lambda_\varphi[B'],$$

where $B' = [y_\alpha, -n'b_4 - m'b_{-4} + r'b_{-3}]$. Without loss of generality, we may assume $B' \succeq 0$, and Lemma 8.7(ii) implies $Q(B') = \alpha(S', S') > 0$. Therefore, $B' \succ 0$, and so by assumption

$$\overline{A_{\xi^\varphi(y_\alpha)}[S']} = \Lambda_\varphi[B'] \ll_\varphi Q(B')^{\frac{\ell+1}{2}} \ll_\alpha (S', S')^{\frac{\ell+1}{2}}.$$

Hence, $\xi^\varphi(y_\alpha)$ satisfies the hypothesis of Theorem 4.6, and is thus cuspidal. □

Proof of Theorem 1.7. This is a direct consequence of Theorem 8.8 and Proposition 5.8. □

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