

Characterizing Cuspidal G_2 -Modular Forms by the Growth of their Fourier Coefficients

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- 1 Characterizing Siegel Cusp Forms
- 2 Characterizing Quaternionic Cusp Forms on G_2
- 3 Power Saving over the Hecke Bound

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- Let $n \geq 1$, and Γ be a congruence subgroup of $\mathrm{Sp}_{2n}(\mathbb{Q})$.
- $F \in M_\ell^n(\Gamma)$ a Siegel modular form of weight ℓ , genus n , and level Γ .
- Then F Fourier expands at the cusp ∞ into a series

$$F(Z) = \sum_{T \in \Lambda_n^{\geq 0}} a_F(T) \exp(2\pi i \mathrm{tr}(TZ)).$$

- Here Z is in the Siegel upper half space of degree n ,
- $\Lambda_n^{\geq 0} =$ positive semidefinite symmetric matrices in $M_n(\mathbb{Q})$,
- $a_F(T)$ is the Fourier coefficient of F indexed by T .

The Hecke Bound

If $F \in M_\ell^n(\Gamma)$ is cuspidal then

$$a_F(T) \ll_F \det(T)^{\ell/2}$$

whenever $\det(T) > 0$.

Characterizing Cusp Forms by the Growth of their Fourier Coefficients

Das shows that cuspidality is detected by the sizes of Fourier coefficients.

Theorem (Das 2026)

Suppose $\ell \in \frac{1}{2}\mathbb{Z}_{\geq n}$ and $F \in M_\ell^n(\Gamma)$. If there exists $\epsilon > 0$ such that

$$a_F(T) \ll_F \det(T)^{\ell-(n+1)/2-\epsilon}$$

for all T satisfying $\det(T) > 0$, then F is cuspidal.

The above result has many antecedents for holomorphic forms (e.g. Linowitz 2014, Kohnen-Martin 2014, Böcherer-Das 2014, Mizuno 2015)

Remark

Das (2026) implies that if $\ell > n + 1$, then the Hecke bound

$$a_F(T) \ll \det(T)^{\ell/2}, \quad \det(T) > 0$$

characterizes the subspace of cusp forms.

Today, we study a characterization of a class of non-holomorphic, **quaternionic** cusp forms on G_2 by the sizes of their Fourier coefficients.

Outline

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- Let G_2 be the split real Lie group with Dynkin diagram 
- Fix a maximal compact subgroup of G_2 ,

$$K \simeq (\mathrm{SU}_2^{\mathrm{short}} \times \mathrm{SU}_2^{\mathrm{long}})/\{\pm 1\}.$$

- Let $\mathrm{SU}_2^{\mathrm{long}}$ act trivially on the representation

$$\mathrm{Sym}^{2\ell} \mathbb{C}^2$$

and $\mathrm{SU}_2^{\mathrm{short}}$ act on $\mathbb{C}^2$ by its standard representation.

Definition

The space $M_\ell(G_2)$ (resp. $S_\ell(G_2)$) of weight $\ell \in \mathbb{Z}_{\geq 1}$ quaternionic modular (resp. cusp) forms of level one consists of functions

$$\varphi: G_2 \rightarrow \mathrm{Sym}^{2\ell} \mathbb{C}^2$$

such that φ is smooth, of moderate growth, $Z(\mathfrak{g})$ -finite, and satisfies:

- φ is left invariant under $G_2(\mathbb{Z})$,
- If $k \in K$ and $g \in G_2$ then $\varphi(gk) = k^{-1} \cdot \varphi(g)$,
- The function φ satisfies a specific differential equation $D_\ell^\infty \varphi \equiv 0$.

The Fourier Expansion of Modular Forms on G_2

Write $P = \mathrm{GL}_2 \ltimes N$ for the short root parabolic subgroup in G_2

- Let Z denote the center of the Heisenberg group N .
- For a unipotent subgroup $W \leq G_2$, let $\varphi_W(g) = \int_{W(\mathbb{Z}) \backslash W} \varphi(wg) dw$.

Theorem (Wallach 2003, Gan-Gross-Savin 2002, Pollack 2020)

A modular form $\varphi \in M_\ell(G_2)$ has a Fourier expansion

$$\varphi_Z(g) = \varphi_N(g) + \sum_{\mu \neq 0} c_\varphi(\mu) \mathcal{W}_{\mu, \ell}(g).$$

Here the **Fourier Coefficient** $c_\varphi(\mu) \in \mathbb{C}$ is indexed by an element μ in

$$\mathrm{Sym}^3(\mathbb{Z}^2) = \{ax^3 + bx^2y + cxy^2 + dy^3 : a, b, c, d \in \mathbb{Z}\}$$

such that $\mathrm{disc}(\mu) = b^2c^2 + 18abcd - 4ac^3 - 4db^3 - 27a^2d^2 \geq 0$.

$$\mathcal{W}_{\mu, \ell} : G_2 \rightarrow \mathrm{Sym}^{2\ell} \mathbb{C}^2$$

is a generalized Whittaker function, which is independent of φ .

Characterizing Cusp Forms on G_2 by the Growth of their Fourier Coefficients

By Gan-Gross-Savin (2002), if $\varphi \in S_\ell(G_2)$ and $\ell \geq 2$, then

$$c_\varphi(\mu) \ll_\varphi \text{disc}(\mu)^{(\ell+1)/2}. \quad (\text{Quaternionic Hecke bound})$$

Main Theorem (M. in preparation)

(a) If $\ell \geq 2$, $\varphi \in M_\ell(G_2)$, and there exists $\epsilon > 0$ such that

$$c_\varphi(\mu) \ll_\varphi \text{disc}(\mu)^{(2\ell-1)/2-\epsilon}$$

for all $\mu \in \text{Sym}^3 \mathbb{Z}^2$ with $\text{disc}(\mu) > 0$, then φ is cuspidal.

(b) There exists $\epsilon \in (0, 1)$ such that if $\varphi \in S_\ell(G_2)$ and $\mu \in \text{Sym}^3 \mathbb{Z}^2$ satisfies $\text{disc}(\mu) > 0$ then

$$c_\varphi(\mu) \ll_\varphi \text{disc}(\mu)^{(\ell+1)/2-\epsilon}$$

Corollary

Let $\ell \geq 2$. Then $\varphi \in M_\ell(G_2)$ is cuspidal $\iff$ there exists $\epsilon > 0$ such that

$$c_\varphi(\mu) \ll_\varphi \text{disc}(\mu)^{(\ell+1)/2-\epsilon}, \quad \text{disc}(\mu) > 0.$$

- Define $GL_2 \ltimes U$ to be the long root parabolic in G_2 .
- Let Z_U be the center of U so $U/Z_U \simeq \mathbb{R}^2 \ltimes \mathbb{R}$ is a Heisenberg group.
- For $B \in \mathbb{Q}_{>0}$ define $\psi_B: \mathbb{R} \rightarrow \mathbb{C}^\times$, $\psi_B(x) = e^{2\pi i Bx}$
- ω_B the associated Weil representation of $Mp_2 \ltimes (U/Z_U)$
- For $B > 0$ define the Gaussian $\phi_B(x) = e^{-6\pi Bx^2}$ for $x \in \mathbb{R}$.

Theorem (after Pollack 2024)

There exists a non-zero functional $\mathcal{L}: \text{Sym}^{2\ell} \mathbb{C}^2 \rightarrow \mathbb{C}$ such that if $\varphi \in M_\ell(G_2)$ and $a, B \in \mathbb{Q}_{>0}$ then

$$\mathcal{F}_{B,a}(\varphi) = \int_{(U'(\mathbb{Z})Z_U) \backslash U} (\mathcal{L} \circ \varphi_{Z_U})(u\bar{h}) \sum_{\xi \in a\mathbb{Z}} \omega_B(uh) \phi_B(\xi) du$$

is the automorphic form associated to a **holomorphic modular form** of weight $\ell + 1/2$. Here $U'(\mathbb{Z}) \leq U(\mathbb{Z})$ is a finite index subgroup, φ_{Z_U} is the constant term of φ along Z_U , and $\bar{h} \in SL_2$ is the image of $h \in Mp_2$.

Detecting Cuspidality on G_2 by the Growth of Fourier Coefficients

There is an established framework for showing the Hecke bound implies cuspidality using Fourier-Jacobi coefficients (e.g. Kohnen-Martin (2014))

- Let $\varphi \in M_\ell(G_2)$ and assume there exists $\epsilon > 0$ such that

$$c_\varphi(\mu) \ll_\varphi \text{disc}(\mu)^{(2\ell-1)/2-\epsilon}, \quad \text{disc}(\mu) > 0.$$

- Applying the half integral weight elliptic case of Das' 2026 theorem, one proves that the Fourier-Jacobi coefficients $\mathcal{F}_{B,a}(\varphi)$ are cuspidal.

Proposition (M. in preparation)

If $\ell \geq 2$ and $\mathcal{F}_{B,a}(\varphi)$ is cuspidal for all $B, a \in \mathbb{Q}_{>0}$ then φ is cuspidal.

- Assume each Fourier-Jacobi coefficient $\mathcal{F}_{B,a}(\varphi)$ is cuspidal.
- Then the scalar components of the constant term φ_U are Maass eigenforms of weight $k \in [-\ell, \ell]$ and eigenvalue $\lambda = (1 - \ell^2)/4$.
- Since $\ell \geq 2$, square integrability of the scalar components of φ_U contradicts $\lambda \geq \frac{|k|}{2} \left(1 - \frac{|k|}{2}\right)$. So to prove $\varphi_U = 0$, it suffices to prove that the scalar components of φ_U are square integrable.

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- For $F \in S_\ell(\mathrm{Sp}_{2n}(\mathbb{Z}))$, define the Rankin convolution

$$R(F, F, s) = \int_{\mathrm{Sp}(2n, \mathbb{Z}) \backslash \mathbb{H}_n} |F(Z)|^2 G(Z, s) \det(Y)^{\ell - (n+1)} dX dY$$

where $G(Z, s) = (\det Y)^s \sum_{\{C, D\}} |\det(CZ + D)|^{-2s}$.

- Up to explicit archimedean factors, Parseval's theorem gives

$$R(F, F, s) = (*) \sum_{T \in \Lambda_n^+ / \mathrm{GL}(n, \mathbb{Z})} \frac{|a_F(T)|^2}{\varepsilon(T) (\det T)^{s + \ell - (n+1)/2}}$$

where $\varepsilon(T) = \{\gamma \in \mathrm{GL}_2(\mathbb{Z}) : T \cdot \gamma = T\}$.

- Analytic properties of $G(Z, s)$ (Kalinin 1977) give analytic continuation and functional equation for the Dirichlet series.
- A Tauberian theorem (Sato-Shintani 1974) gives that for any $\epsilon > 0$,

$$|a_F(T)| \ll_{F, \epsilon} \det(T)^{\ell/2 - 1/\alpha_n + \epsilon}$$

where $\alpha_n = 2n + 2 + 4\lfloor n/2 \rfloor + 2/(n+1)$ (Böcherer-Raghavan 1988).

Adapting the Power Saving Hecke Bound for Modular Forms on G_2

- Let $\varphi \in S_\ell(G_2)$ and $E(g, s)$ be a spherical Eisenstein series on G_2 .
- Since N is non-abelian, Parseval's theorem doesn't directly express

$$\int_{G_2(\mathbb{Z}) \backslash G_2} \|\varphi(g)\|^2 E(g, s) dg$$

as a Dirichlet series involving the Fourier coefficients $c_\varphi(\mu)$.

- Instead, we replace φ with φ_Z , and consider the modified integral

$$R(\varphi, \varphi) = \int_{P(\mathbb{Z}) \backslash P} \|\varphi_Z(p)\|^2 dp.$$

Here $P = \mathrm{GL}_2 \ltimes N$ is the parabolic attached to short root in .

- Up to explicit archimedean factors, Parseval's theorem gives

$$R(\varphi, \varphi) = (*) \sum_{[\mu] \in \mathrm{Sym}^3 \mathbb{Z}^2 / \mathrm{GL}(2, \mathbb{Z})} \frac{|c_\varphi(\mu)|^2}{\eta(\mu) \mathrm{disc}(\mu)^{\ell+1}}$$

where $\eta(\mu) = \#\{\gamma \in \mathrm{GL}_2(\mathbb{Z}) : \mu \cdot \gamma = \mu\}$.

- However, since $R(\varphi, \varphi)$ is not directly related to an Eisenstein series, this second moment sum is less accessible by Tauberian methods.

- Let $H = \mathrm{SL}_2 \ltimes N \leq P$ and $m_T = \begin{pmatrix} \sqrt{T} & \\ & \sqrt{T} \end{pmatrix} \in \mathrm{GL}_2 \leq P$.
- If $\varphi \in S_\ell(G_2)$ then $R(\varphi, \varphi) = \int_0^\infty \mathcal{P}_T(\varphi) d^\times T$ where

$$\mathcal{P}_T(\varphi) = \int_{H(\mathbb{Z}) \backslash H} \|\varphi_Z(hm_T)\|^2 dh, \quad T > 0$$

- Setting $T = X^{-1/2}$, Parseval's theorem gives

$$\sum_{\substack{[\mu] \in \mathrm{Sym}^3 \mathbb{Z}^2 / \mathrm{SL}(2, \mathbb{Z}) \\ X \leq \mathrm{disc}(\mu) \leq 2X}} |c_\varphi(\mu)|^2 \ll_\ell X^{\ell+1} \cdot \mathcal{P}_T(\varphi)$$

- One can bound the period $\mathcal{P}_T(\varphi)$ by applying a framework of Venkatesh based on equidistribution and quantitative mixing.

Proposition (after Venkatesh 2010)

There exists $\epsilon \in (0, 1)$ such that if $\varphi \in S_\ell(G_2)$ and $0 < T \leq 1$, then

$$\mathcal{P}_T(\varphi) \ll_\varphi T^\epsilon.$$

- Let $X := G_2(\mathbb{Z}) \backslash G_2$, $H = \mathrm{SL}_2 \ltimes N$, and $m_T = \begin{pmatrix} \sqrt{T} & \\ & \sqrt{T} \end{pmatrix}$. Define

$$Y_T := G_2(\mathbb{Z}) H m_T \subseteq X$$

- Then for a smooth function φ on X of sufficiently rapid decay

$$\mathcal{P}_T(\varphi) = \|\varphi_Z\|_{L^2(v_T)}^2.$$

Here $v_T(f) = \int_{H(\mathbb{Z}) \backslash H} f(hm_T) dh$ for any integrable function f on Y_T .

Equidistribution (after Kleinbock-Margulis 1999)

There exists a Haar measure μ_X on X and $\kappa_1 > 0$ such that if $T \in (0, 1]$ and $f \in C^\infty(X)$ is of sufficiently rapid decay, then $|v_T(f) - \mu_X(f)| \ll_f T^{\kappa_1}$.

Mixing (Kleinbock-Margulis 1999)

Let E be a basis of $\mathrm{Lie}(Z)$. There exists $\kappa_2 > 0$ such that if $f \in C^\infty(X)$ is of sufficiently rapid decay and $\mu_X(f) = 0$, then

$$\langle \exp(uE) \cdot f, f \rangle_{L^2(\mu_X)} \ll_f (1 + |u|)^{-\kappa_2}, \quad (u \in \mathbb{R}).$$

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LLM Usage Disclosure

Consumer LLM tools were used to help develop and check mathematical arguments presented in these slides. The author takes full responsibility for the accuracy of the results, and the correctness of the attributions.

Thank you