Exceptional Algebra and G_2-modular forms.

SWANTAG

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The Cayley Octonions.

The Cayley Octonions $\mathbb{O} = \mathbb{Q}\text{-span}\{1, e_1, e_2, \dots\}$ with unit 1 and multiplication

$$e_i^2 = (e_i e_{i+1})e_{i+3} = e_i(e_{i+1}e_{i+3}) = -1 \quad \text{and} \quad e_i = e_{i+7}.$$

Intepret the identity $(e_i e_{i+1})e_{i+3} = e_i(e_{i+1}e_{i+3}) = -1$ as saying

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$$

where $e_i = \mathbf{i}$, $e_{i+1} = \mathbf{j}$, and $e_{i+3} = \mathbf{k}$.

If $a_0, a_1, \dots, a_7 \in \mathbb{Q}$, $\alpha = a_0 + a_1 e_1 + \cdots + a_7 e_7$, define

$$N(\alpha) = a_0^2 + a_1^2 + \cdots + a_7^2 \quad \text{and} \quad \mathrm{tr}(\alpha) = 2a_0.$$

Every $\alpha \in \mathbb{O} - \mathbb{Q}$ has a quadratic minimal polynomial

$$X^2 - \mathrm{tr}(\alpha)X + N(\alpha) \in \mathbb{Q}[X].$$

$\mathbb{O}$ is structurally similar to a quadratic imaginary extension $L/\mathbb{Q}$.

We pursue this analogy with a view to studying arithmetic in $\mathbb{O}$.

Specifically, we'll see how modular forms can be used to count solutions to diophantine equations in $\mathbb{O}$.

The Coxeter Order R.

Let R denote the $\mathbb{Z}$-lattice in $\mathbb{O}$ spanned by 1, the e_i's and the elements

$$\frac{1}{2}(1+e_1+e_2+e_4), \quad \frac{1}{2}(1+e_1+e_5+e_6), \quad \frac{1}{2}(1+e_1+e_3+e_7), \quad \frac{1}{2}(e_1+e_2+e_3+e_4).$$

Then R is one of seven maximal orders in $\mathbb{O}$ containing

$$\mathbb{Z}\text{-span}\{1, e_1, \ldots, e_7\}.$$

In particular, R is a subring of $\mathbb{O}$ consisting of integral elements.

Remark

Consider $\mathbb{Q}(\sqrt{-3})/\mathbb{Q}$. Then $\mathbb{Z}[\frac{-1+\sqrt{-3}}{2}]$ is the unique maximal order in $\mathbb{Q}(\sqrt{-3})$.

$\mathbb{Z}[\frac{-1+\sqrt{-3}}{2}]$ admits a Euclidean division algorithm, unlike $\mathbb{Z}[\sqrt{-3}]$.

Question (Diophantine Equations over R.)

Let $D < 0$ be a fundamental discriminant and $f_D(X) = X^2 - DX + \frac{D^2-D}{4} \in \mathbb{Z}[X]$. How many solutions does $f(X)$ have in R?

Quadratic Octonion Equations I

Let $D < 0$ be a fundamental discriminant and $f_D(X) = X^2 - DX + \frac{D^2 - D}{4}$

$$N_D = \#\{\alpha \in R \colon f_D(\alpha) = 0\}.$$

Proposition (Elkies-Gross)

There exists a modular form g of weight $7/2$ such that if

$$g(z) = \sum_{n \geq 0} a_n e^{2\pi i n z}, \quad \textit{denotes the Fourier expansion of } g$$

then $a_{|D|} = N_D$ for all fundamental discriminants $D < 0$.

Let $g(z) = \sum a_n e^{2\pi i n z}$ be the θ-series of the lattice

$$\Lambda = \{\lambda \in \mathbb{Z} + 2R \colon \mathrm{Tr}(\lambda) = 0\}.$$

So $a_n = \#\{\lambda \in \Lambda \colon N(\lambda) = n\}$ and if $D < 0$ is a fundamental discriminant then

$$\{\alpha \in R \colon f_D(\alpha) = 0\} \xrightarrow{\sim} \{\lambda \in \Lambda \colon N(\lambda) = -D\}, \qquad \alpha \mapsto 2\alpha - D.$$

Quadratic Octonion Equations II

Let $g(z) = \sum a_n q^n$ be the θ-series of the lattice Λ. So $g(z)$ is a modular form and

$$a_{|D|} = N_D$$

for all fundamental discriminants $D < 0$. Using "known facts" about the space of modular forms of weight $7/2$ one has

$$a_{|D|} = -252 \cdot L(\varepsilon_D, -2)$$

where $L(\varepsilon_D, s) = \sum_{n \geq 0} \frac{\varepsilon_D(n)}{n^s}$ is a Dirichlet L-function associated to $\mathbb{Q}(\sqrt{D})$.

The computation of N_D is reduced to a tractable problem in analytic number theory about estimating the value $L(\varepsilon_D, -2)$.

$$\Rightarrow 3 \cdot |D|^{5/2} \leq N_D \leq 5 \cdot |D|^{5/2}.$$

There are exact value expressions for $L(\varepsilon_D, -2)$ which can be used to compute

D	-3	-4	-7	-8	-11	-15	-19	-20	-23	-24
N_D	56	126	576	756	1512	4032	5544	7560	12096	11592

Modular Forms for SL_2

Let $G = \mathrm{SL}_2(\mathbb{R})$, $K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$, and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Definition

Suppose $k \in \mathbb{Z}_{\geq 0}$, a modular form of weight k is a function

$$\phi \colon \Gamma \backslash G \to \mathbb{C}$$

such that:

(a) If $\theta \in \mathbb{R}$ and $\overline{g} \in \Gamma \backslash G$ then $\phi(\overline{g} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}) = e^{-i\theta k} \phi(\overline{g})$.

(b) The function $G \to G \times \mathbb{C}$, $g \mapsto (g, \phi(\overline{g}))$ descends to a Γ-invariant holomorphic section of moderate growth of the line bundle

$$G \times_K \mathbb{C} \to G/K$$

where $G \times_K \mathbb{C} = G \times \mathbb{C} / \langle (g, z) \simeq (g \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, e^{-i\theta k} z) \colon \theta \in \mathbb{R} \rangle$. Note that $G/K \simeq \{z \colon \mathrm{im}(z) > 0\}$ is a complex manifold.

This definition directly generalizes to semi-simple real Lie groups provided G/K has Γ-invariant complex structure.

The Split Octonions Θ

Let $\mathbb{H} = \mathbb{R}\langle 1, \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$ denote the ring with product

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Write $(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})^* = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$. The split octonion algebra is $\Theta := \mathbb{H} \oplus \mathbb{H}$,

$$\text{with product } (q_1, q_2) \cdot (p_1, p_2) = (q_1 p_1 + p_2^* q_2, p_2 q_1 + q_2 p_1^*) \tag{0.1}$$

where $q_1, q_2, p_1, p_2 \in \mathbb{H}$.

Write $\mathbb{H}^1 = \{q \in \mathbb{H} \colon qq^* = 1\}$, we obtain an action

$$\rho \colon \mathbb{H}^1 \times \mathbb{H}^1 \to \mathrm{GL}(\Theta), \quad \rho(g, h)(q, p) = (gqg^{-1}, hqg^{-1}).$$

preserving the product (0.1).

Definition (Definition of G_2)

Define

$$\mathrm{G}_2(\mathbb{R}) = \{g \in \mathrm{GL}(\Theta) \colon \text{if } x, y \in \Theta \text{ then } g \cdot (xy) = g(x) \cdot g(y)\}.$$

Let $K \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)/\mu_2$ denote the image of ρ in $\mathrm{G}_2(\mathbb{R})$.

Modular Forms for G_2

Let $G = \mathrm{G}_2(\mathbb{R})$, $K = \mathrm{SU}(2) \times \mathrm{SU}(2)/\mu_2$, and $\Gamma = \mathrm{G}_2(\mathbb{Z})$.

Definition

Suppose $\ell \in \mathbb{Z}_{\geq 0}$, a G_2-modular form of weight ℓ is a function

$$\phi \colon \Gamma \backslash G \to \mathrm{Sym}^{2\ell}(\mathbb{C}^2) \quad \text{such that}$$

(a) If $(k_1, k_2) \in K$ and $\overline{g} \in \Gamma \backslash G$ then $\phi(\overline{g} \cdot (k_1, k_2)) = k_1^{-1}\phi(\overline{g})$.

(b) The function $G \to G \times \mathrm{Sym}^{2\ell}(\mathbb{C}^2)$, $g \mapsto (g, \phi(\overline{g}))$ descends to a Γ-invariant smooth section of moderate growth of the vector bundle

$$G \times_K \mathrm{Sym}^{2\ell}(\mathbb{C}^2) \to G/K$$

where $G \times_K \mathrm{Sym}^{2\ell}(\mathbb{C}^2) = G \times \mathrm{Sym}^{2\ell}(\mathbb{C}^2)/\langle (g, v) \simeq (gk, k \cdot v) \colon k \in K \rangle$ satisfying an analogue of the Cauchy Riemann Equations.

A commutative associative ring A is called cubic if $A \simeq \mathbb{Z}^3$ as a $\mathbb{Z}$-module.

Gan-Gross-Savin have defined Fourier coefficients $c_{\varphi,A}$ for a G_2-modular form φ such that the Fourier coefficients of φ are indexed by isomorphisms classes of cubic rings.

Hermitian Matrices over $\mathbb{O}$

Consider the set of Hermitian 3-by-3 matrices over $\mathbb{O}$,

$$\mathcal{H}_3(\mathbb{O}) = \left\{ \begin{pmatrix} \xi_1 & c_3 & \bar{c}_2 \\ \bar{c}_3 & \xi_2 & c_1 \\ c_2 & \bar{c}_1 & \xi_3 \end{pmatrix} : c_1, c_2, c_3 \in \mathbb{O}, \xi_1, \xi_2, \xi_3 \in \mathbb{Q} \right\}.$$

If $x, y \in \mathcal{H}_3(\mathbb{O})$, define $x \circ_I y := \frac{1}{2}(x \cdot y + y \cdot x)$, where $x \cdot y$ denotes the matrix product of x and y. Then

$$(x \circ_I x) \circ_I (x \circ_I y) = x \circ_I ((x \circ_I x) \circ_I y) \qquad \forall x, y \in \mathcal{H}_3(\mathbb{O}).$$

An element $\alpha \in \mathcal{H}(\mathbb{O})$ satisfies a Cayley-Hamilton equation

$$X^3 - \mathrm{tr}(\alpha)X^2 - (\frac{1}{2}\mathrm{tr}(\alpha^2) - \frac{1}{2}\mathrm{tr}(\alpha)^2)X - \det(\alpha) \cdot I$$

where $\mathrm{tr} \begin{pmatrix} \xi_1 & c_3 & \bar{c}_2 \\ \bar{c}_3 & \xi_2 & c_1 \\ c_2 & \bar{c}_1 & \xi_3 \end{pmatrix} = \xi_1 + \xi_2 + \xi_3$ and

$$\det \begin{pmatrix} \xi_1 & c_3 & \bar{c}_2 \\ \bar{c}_3 & \xi_2 & c_1 \\ c_2 & \bar{c}_1 & \xi_3 \end{pmatrix} = \xi_1\xi_2\xi_3 + \mathrm{tr}((c_1 c_2)c_3) - \xi_1 N(c_1) - \xi_2 N(c_2) - \xi_3 N(c_3)$$

is a cubic norm on $\mathcal{H}_3(\mathbb{O})$.

The Integral Structure $\mathcal{H}_3(R)$

Consider the lattice

$$\mathcal{H}_3(R) = \left\{ \begin{pmatrix} \xi_1 & c_3 & \overline{c}_2 \\ \overline{c}_3 & \xi_2 & c_1 \\ c_2 & \overline{c}_1 & \xi_3 \end{pmatrix} : c_1, c_2, c_3 \in R, \xi_1, \xi_2, \xi_3 \in \mathbb{Z} \right\},$$

The cubic norm $\det\colon \mathcal{H}_3(\mathbb{O}) \to \mathbb{Q}$ is integer valued on $\mathcal{H}_3(R)$. Let A be the ring of integers in a cubic extension of $\mathbb{Q}$ with norm $\mathbb{N}$. There is an analogy between

$$(\mathcal{H}_3(R), \det, I) \quad \text{and} \quad (A, \mathbb{N}, 1).$$

Define

$$N_A = \#\left\{\text{ring homorphisms } f\colon A \hookrightarrow \mathcal{H}_3(R)\right\}.$$

Theorem (Gan-Gross-Savin)

There exists a G_2-modular form θ of weight 4 such that if A is the ring of integers in a cubic extension $L/\mathbb{Q}$, and c_A is the Fourier coefficient of θ indexed by A, then

$$N_A = c_{\theta,A}.$$

In particular, the quantity c_A is an integer.

Thank You!

THANK YOU SO MUCH!!!!

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