On the Fourier-Jacobi Expansion of Quaternionic Modular Forms on $\mathrm{Spin}(8)$

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Outline

Contents

Integral Binary Quadratic Forms

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$$T(x, y) * \begin{pmatrix} a & b \\ c & d \end{pmatrix} = T(ax + by, cx + dy), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

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- Equivalently, $(\mathrm{Sym}^2\,\mathbb{Z}^2)^*$ consists of $1/2$-integral symmetric matrices

$$T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$$

  with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ acting by $T * \gamma = \gamma^t T \gamma$.

Binary Quadratic Forms and the Siegel Parabolic P_s

- The **symplectic group**

$$\mathrm{Sp}_4 = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_4 : AB^t = BA^t, CD^t = DC^t, AD^t - BC^t = 1 \right\}$$

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$$N_s = \left\{ \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} : B \in M_2 \quad \text{and} \quad B^t = B \right\} \subseteq \mathrm{Sp}_4.$$

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$$\mathrm{Hom}(N_s(\mathbb{Z})\backslash N_s(\mathbb{R}), \mathbb{C}^\times) \simeq$$

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$$\mathrm{Hom}(N_s(\mathbb{Z})\backslash N_s(\mathbb{R}), \mathbb{C}^\times) \simeq (\mathrm{Sym}^2 \mathbb{Z}^2)^*$$

  via the map sending $T \in (\mathrm{Sym}^2 \mathbb{Z}^2)^*$ to the **character**

$$\chi_T : N_s(\mathbb{Z})\backslash N_s(\mathbb{R}) \to \mathbb{C}^\times, \qquad \chi_T\left(\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}\right) = \exp(2\pi i\, \mathrm{tr}(BT)).$$

Siegel Modular Forms

The symplectic group $\mathrm{Sp}_4(\mathbb{R})$ acts on the **complex three-fold**

$$\mathcal{H}_2 = \left\{ \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) \colon\ \mathrm{im}(\tau), \mathrm{im}(\tau'), \mathrm{im}(\tau)\,\mathrm{im}(\tau') - \mathrm{im}(z)^2 > 0 \right\}$$

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Definition (Siegel Modular Forms)

Let $\ell \in \mathbb{Z}_{\geqslant 0}$ and write $M_\ell(\mathrm{Sp}_4(\mathbb{Z}))$ for the space of holomorphic functions

$$F \colon \mathcal{H}_2 \to \mathbb{C}$$

such that $F(\gamma \cdot Z) = \det(CZ + D)^\ell F(Z) \ \forall\, \gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_4(\mathbb{Z})$ and $Z \in \mathcal{H}_2$.

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- If $\gamma = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$, the **transformation law** of $F \in M_\ell(\mathrm{Sp}_4(\mathbb{Z}))$ implies

$$F(Z+B) = F(Z)$$

for all $Z \in \mathcal{H}_2$ and $B \in M_2(\mathbb{Z})$ such that $B^t = B$.

The Fourier Expansion of Siegel Modular Forms

Fourier expanding $F \in M_\ell(\mathrm{Sp}_4(\mathbb{Z}))$ in characters of $N_s(\mathbb{Z})\backslash N_s(\mathbb{R})$ gives

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- Since $M_s^{\mathrm{der}}(\mathbb{Z})$ normalizes $N_s(\mathbb{Z})$, changing variables in (1) gives

$$A_F[T * \gamma] = A_F[T]$$

for all $T \in (\mathrm{Sym}^2 \mathbb{Z}^2)^*$ and $\gamma \in M_s^{\mathrm{der}}(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$.

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Corollary

Suppose $F_1, F_2 \in M_\ell(\mathrm{Sp}_4(\mathbb{Z}))$. Then $F_1 = F_2$ if and only

$$A_{F_1}[T] = A_{F_2}[T]$$

for all primitive integral binary quadratic forms $T \in (\mathrm{Sym}^2 \mathbb{Z}^2)^*$.

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- **Yamana (09')** *(Holomorphic modular forms on classical groups are determined by their primitive Fourier coefficients)*.

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- For $m > 0$, ϕ_m inherits a **transformation law** with respect to

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Theorem (Zagier 80')

Suppose $\ell > 0$ *&* $A_F[T] = 0 \; \forall$ ***primitive*** $T \in (\mathrm{Sym}^2 \mathbb{Z}^2)^*$. *Then* $F = 0$.

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- Let $f_{k_0}(\tau) = \sum_{n=0}^{\infty} a_n q^n$ and $n \geqslant 1$ satisfy $\gcd(n, m_0) = 1$. Then a_n is a **finite sum** of Fourier coefficients $A_F[T]$ for primitive Ts.

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- Therefore, under the hypothesis of the theorem, $a_n = 0$ for all $n \geqslant 1$ satisfying $\gcd(n, m_0) = 1$. Hence $f_{k_0} = 0$, which is a contradiction.

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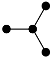
- Φ is $Z(\mathfrak{spin}_8(\mathbb{C}))$-finite.

The Heisenberg Parabolic P_h

Let $P_h = M_h N_h$ be a parabolic associated a **central** node in  .

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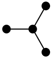
Let $P_h = M_h N_h$ be a parabolic associated a **central** node in  .

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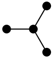
Let $P_h = M_h N_h$ be a parabolic associated a **central** node in  .

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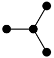
Let $P_h = M_h N_h$ be a parabolic associated a **central** node in  .

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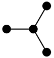
Let $P_h = M_h N_h$ be a parabolic associated a **central** node in  .

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Let $\Phi \in M_\ell(\mathrm{Spin}_8(\mathbb{Z}))$ and $U \leqslant \mathrm{Spin}_8$ be a unipotent subgroup.

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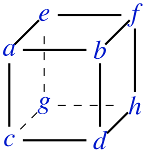

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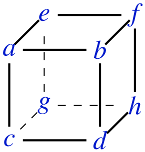

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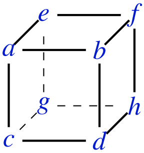

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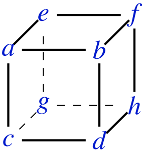

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- Then $4\det(T(B)) = \det(B)$ where $T(B) \in (\mathrm{Sym}^2 \mathbb{Z}^2)^*$ is defined as

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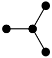
- $P_o = M_o N_o$ the parabolic associated to an outer node in  .

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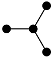
- $P_o = M_o N_o$ the parabolic associated to an outer node in  .
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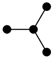
- $P_o = M_o N_o$ the parabolic associated to an outer node in  .
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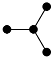
- $P_o = M_o N_o$ the parabolic associated to an outer node in  .
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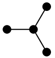
- $P_o = M_o N_o$ the parabolic associated to an outer node in  .
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- For $\Phi \in M_\ell(\mathrm{Spin}_8(\mathbb{Z}))$, define $\mathcal{F}_y(\Phi) \colon \mathrm{Spin}_8(\mathbb{R}) \to \mathrm{Sym}^\ell(\mathbb{V})$ as

$$\mathcal{F}_y(\Phi)(g) = \int_{N_o(\mathbb{Z})\backslash N_o(\mathbb{R})} \Phi(ng)\overline{\chi_y(n)}\, dn$$

Proposition (M. 25')

An element $\Phi \in M_\ell(\mathrm{Spin}_8(\mathbb{Z}))$ Fourier expands along $N_o(\mathbb{Z})\backslash N_o(\mathbb{R})$ as

$$\Phi(g) = \Phi_{N_o}(g) + \sum_{y \in V_6(\mathbb{Z})\backslash\{0\} \colon \langle y, y \rangle \geqslant 0} \mathcal{F}_y(\Phi)(g).$$

Orthogonal Fourier-Jacobi Expansion: Non Degenerate Terms

- Recall $\Phi \in M_\ell(\mathrm{Spin}_8(\mathbb{Z}))$ has Fourier-Jacobi expansion

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Let $y \in V_6(\mathbb{Z})$ **satisfy** $\langle y, y\rangle > 0$, and $H_y \simeq \mathrm{Sp}_4$ equal $\mathrm{Stab}_{\mathrm{Spin}_6}(y)$.

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Proposition (Johnson-Leung, M., Negrini, Pollack, Roy 24', M. 25')

(i) There exists a non-zero linear functional $L\colon\ \mathrm{Sym}^\ell(\mathbb{V}) \to \mathbb{C}$ *and an element* $g_y \in \mathrm{Spin}_8(\mathbb{R})$ *such that if* $\Phi \in M_\ell(\mathrm{Spin}_8(\mathbb{Z}))$ *then the function*

$$\xi_y\colon H_y(\mathbb{R}) \to \mathbb{C}, \qquad \xi_y(h) := L(\mathcal{F}_y(\Phi)(hg_y))$$

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(ii) We have the implication

$$\xi_y \equiv 0 \Longrightarrow \mathcal{F}_y(\Phi) \equiv 0.$$

Reduction to the Case of Holomorphic Modular Forms I

Let N_y be the unipotent radical of a **Siegel parabolic** in $H_y \simeq \mathrm{Sp}_4$.

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$$A_\xi[\chi] = \int_{N_y(\mathbb{Z})\backslash N_y(\mathbb{R})} \xi(n)\overline{\chi(n)}\, dn, \quad \chi \in \mathrm{Hom}(N_y(\mathbb{Z})\backslash N_y(\mathbb{R}), \mathbb{C}^\times).$$

*If $A_\xi[\chi] = 0$ for all **primitive** $\chi \in \mathrm{Hom}(N_y(\mathbb{Z})\backslash N_y(\mathbb{R}), \mathbb{C}^\times)$ then $\xi = 0$.*

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- Therefore, under the hypothesis that $\Lambda_\Phi[B] = 0$ for all primitive B, the primitive Fourier coefficients of ξ_y vanish. Hence, $\xi_y = 0$.

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- Therefore, under the hypothesis that $\Lambda_\Phi[B] = 0$ for all primitive B, the primitive Fourier coefficients of ξ_y vanish. Hence, $\xi_y = 0$.
- Therefore, part (ii) of the previous proposition implies $\mathcal{F}_y(\Phi) = 0$.

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- Recall $\Phi \in M_\ell(\mathrm{Spin}_8(\mathbb{Z}))$ Fourier Jacobi expands as

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Let $y \in V_6(\mathbb{Z}) - \{0\}$ be isotropic. There exists $\gamma \in M_o^{\mathrm{der}}(\mathbb{Z}) = \mathrm{Spin}_6(\mathbb{Z})$ such that if $y' = \gamma \cdot y$ then the Fourier Jacobi coefficient

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This concludes the proof of part (a) of our main theorem for Φ cuspidal .

Analysis of the Constant Term Φ_{N_h}

Let $\Phi \in M_\ell(\mathrm{Spin}_8(\mathbb{Z}))$, write $U = \{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) : x \in \mathbb{R}\}$, and recall $M_h^{\mathrm{der}} = \mathrm{SL}_2^3$.

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Theorem (Pollack 20')

There exists a non-zero linear functional $L\colon \mathrm{Sym}^\ell(\mathbb{V}) \to \mathbb{C}$ such that if $\Phi \in M_\ell(\mathrm{Spin}_8(\mathbb{Z}))$ is such that $\Phi_{N_h} \not\equiv 0$ then the function

$$\phi\colon M_h^{\mathrm{der}}(\mathbb{R}) \to \mathbb{C}, \quad \phi(g) = L(\Phi_{N_h}(g))$$

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In classical notation the modular form ϕ admits a q-expansion

$$\phi(z_1, z_2, z_3) = \sum_{n_1, n_2, n_3 \geqslant 0} a_\phi(n_1, n_2, n_3) e^{2\pi i(n_1 z_1 + n_2 z_2 + n_3 z_3)}$$

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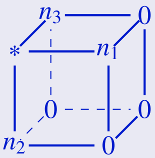
*If $n := (n_1,n_2,n_3) \neq 0$ then $a_\phi(n)$ is a **finite sum** of Fourier coefficients $\Lambda_\Phi[B]$ where B is of the form*



Conclusion to the Proof of Statement (a)

Lemma (M. 25')

Suppose $\ell > 0$ and let $\phi \in M_{(\ell,\ell,\ell)}(\mathrm{SL}_2(\mathbb{Z})^3)$ with Fourier Expansion

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If $a_\phi(n_1, n_2, n_3) = 0$ for all ***jointly coprime*** *$(n_1, n_2, n_3) \in \mathbb{Z}^3$ then $\phi = 0$.*

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- Hence, $\mathcal{F}_y(\Phi) \equiv 0$ for all $y \in V_6(\mathbb{Z}) \backslash \{0\}$ such that $\langle y, y \rangle \geqslant 0$.

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- Therefore, by the above lemma $\phi = 0$, and thus $\Phi_{N_h} \equiv 0$.
- Hence, $\mathcal{F}_y(\Phi) \equiv 0$ for all $y \in V_6(\mathbb{Z}) \setminus \{0\}$ such that $\langle y, y \rangle \geqslant 0$.
- Therefore, $\Phi = \Phi_{N_o}$ is left $N_o(\mathbb{R})$-invariant.

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- Hence, $\mathcal{F}_y(\Phi) \equiv 0$ for all $y \in V_6(\mathbb{Z}) \backslash \{0\}$ such that $\langle y, y \rangle \geqslant 0$.
- Therefore, $\Phi = \Phi_{N_o}$ is left $N_o(\mathbb{R})$-invariant. However, since $\mathrm{Spin}_8(\mathbb{R}) = \langle N_o(\mathbb{R}), \mathrm{Spin}_8(\mathbb{Z}) \rangle$, it follows that Φ is zero.

Conclusion to the Proof of Statement (b)

Lemma (Johnson-Leung, M., Negrini, Pollack, Roy 24', M. 25')

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Suppose Φ is cuspidal. If $B \in W_{\geqslant 0}$ satisfies $\det(B) = 0$ then $\Lambda_\Phi[B] = 0$.

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- For $y \in V_6(\mathbb{Z})\backslash\{0\}$ such that $\langle y, y\rangle > 0$, recall the Fourier coefficients

$$A_{\xi_y}[\chi] = \int_{N_y(\mathbb{Z})\backslash N_y(\mathbb{R})} \xi_y(n)\overline{\chi(n)}\, dn, \quad \chi \in \mathrm{Hom}(N_y(\mathbb{Z})\backslash N_y(\mathbb{R}), \mathbb{C}^\times).$$

Conclusion to the Proof of Statement (b)

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Lemma (M. 25')

(i) *If $B \in W_{>0}$ is* ***primitive****, then there exists a* ***primitive*** *$y \in V_6(\mathbb{Z})$ satisfying $\langle y, y\rangle > 0$, and a character χ, such that $A_{\xi_y}[\chi] = \Lambda_\Phi[B]$.*

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(i) *If $B \in W_{>0}$ is **primitive**, then there exists a **primitive** $y \in V_6(\mathbb{Z})$ satisfying $\langle y, y\rangle > 0$, and a character χ, such that $A_{\xi_y}[\chi] = \Lambda_\Phi[B]$.*

(ii) *Let $y \in V_6(\mathbb{Z})$ be **primitive** with $\langle y, y\rangle > 0$. If χ is **primitive**, then there exists an **s-primitive** $B \in W_{>0}$ such that $A_{\xi_y}[\chi] = \Lambda_\Phi[B]$.*

- Assuming the **s-primitive** Fourier coefficients of Φ vanish, (ii) implies that $\xi_y \equiv 0$ for all **primitive** $y \in V_6(\mathbb{Z})$ satisfying $\langle y, y\rangle > 0$.
- Applying (i), we conclude that $\Lambda_\Phi[B] = 0$ for all **primitive** $B \in W_{>0}$.

Thank You for Listening