

[1] Given  $g(x) = 2x^2 - 3x$ , use the definition of the derivative to find  $g'(x)$ .

$$g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2(x+h)^2 - 3(x+h) - (2x^2 - 3x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2x^2 + 4xh + 2h^2 - 3x - 3h - 2x^2 + 3x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{4xh + 2h^2 - 3h}{h}$$

$$= \lim_{h \rightarrow 0} 4x + 2h - 3 = \boxed{4x - 3}$$

[2] (a) Compute  $\lim_{x \rightarrow \infty} \frac{3x^3 - 2x + 5}{5x^3 + x^2 + x}$ . Give your reasoning.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{3x^3 - 2x + 5}{5x^3 + x^2 + x} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} &= \lim_{x \rightarrow \infty} \frac{3 - \frac{2}{x^2} + \frac{5}{x^3}}{5 + \frac{1}{x} + \frac{1}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{3}{5} \quad \text{since } \lim_{x \rightarrow \infty} \frac{1}{x^n} = 0 \text{ for } n > 0 \\ &= \boxed{\frac{3}{5}} \end{aligned}$$

(b) Compute  $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$ . Give your reasoning.

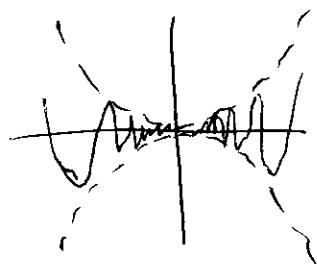
$$\text{Note } -1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$\text{So } -x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

$$\text{Since } \lim_{x \rightarrow 0} -x^2 = 0 \text{ and } \lim_{x \rightarrow 0} x^2 = 0$$

We have  $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = 0$  by

the Squeeze Theorem.



[3] Find the equation of the tangent line to the curve  $y = \sin(\sin(x))$  at the point  $(\pi, 0)$ .

$$y = \sin(\sin(x))$$

$$\frac{dy}{dx} = \cos(\sin(x)) \cos(x)$$

$$\begin{aligned} \text{at } (\pi, 0), \quad \frac{dy}{dx} & \text{ is } \cos(\sin(\pi)) \cos(\pi) \\ &= \cos(0) \cos(\pi) \\ &= 1 \cdot (-1) = -1 \end{aligned}$$

Line of slope  $-1$  through  $(\pi, 0)$ :

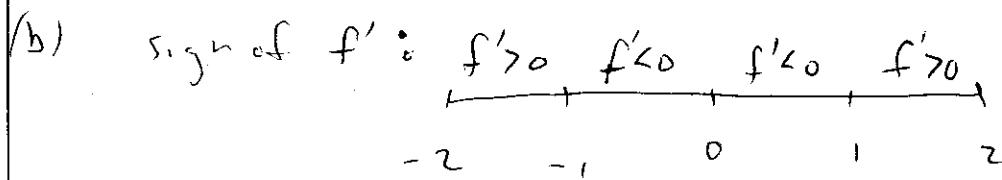
$$y = -(x - \pi)$$

[4] Consider the function  $f(x) = 3x^5 - 5x^3$  on the interval  $[-2, 2]$ .

- Find the critical numbers of  $f(x)$ .
- Use an appropriate test to identify all local maxima and minima.
- Find the absolute maximum and absolute minimum.
- Find all inflection points.

(a)  $f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$   
 $= 15x^2(x+1)(x-1)$

Critical numbers:  $x = 0, 1, -1$



[Local max at  $x = -1$ , Local min at  $x = 1$ ]

(c)  $f(2) = 3 \cdot 32 - 5 \cdot 8 = 56$

$f(-2) = -56$

$f(0) = 0$

$f(-1) = 2$

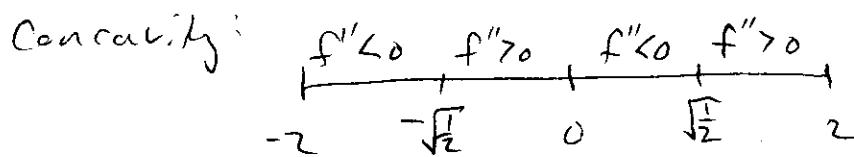
Abs max is 56, at  $x = 2$

Abs min is -56, at  $x = -2$

$f(-2) = -56$

(d)  $f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$

$$f''(x) = 0 \Rightarrow x = 0 \text{ or } 2x^2 - 1 = 0$$
$$x^2 = \frac{1}{2} \quad x = \pm \sqrt{\frac{1}{2}}$$



[Inflection points at  $x = -\sqrt{\frac{1}{2}}, x = 0, x = \sqrt{\frac{1}{2}}$ ]

[5] (a) Find the derivative of  $f(x) = (\ln(2x))^5$ .

$$f'(x) = \boxed{5(\ln(2x))^4 \cdot \frac{1}{2x} \cdot 2}$$

(b) Find the derivatives of  $g(x) = \int_1^x \frac{t^2}{1+t^4} dt$  and  $h(x) = \int_{\sqrt{x}}^1 \frac{t^2}{1+t^4} dt$ .

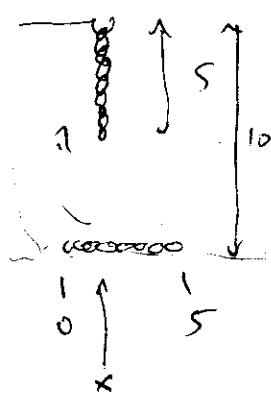
$$g'(x) = \boxed{\frac{x^2}{1+x^4}}$$

$$h(x) = - \int_1^{\sqrt{x}} \frac{t^2}{1+t^4} dt = -g(\sqrt{x})$$

$$\text{so } h'(x) = -g'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

$$h'(x) = \boxed{\frac{-(\sqrt{x})^2}{1+(\sqrt{x})^4} \cdot \frac{1}{2\sqrt{x}}}$$

- [6] A 5 foot chain weighs 20 lbs and is lying on the ground. Find the work done to hang the chain from one end, on a hook which is 10 feet above the ground.



Weight: Chain is 4 lbs/foot.

position  $x$  along chain gets raised  
a height of  $5+x$  feet.

a small piece at  $x$  weighs  $4\Delta x$  lbs

$$\text{Work done} = \int_0^5 4(5+x) dx$$

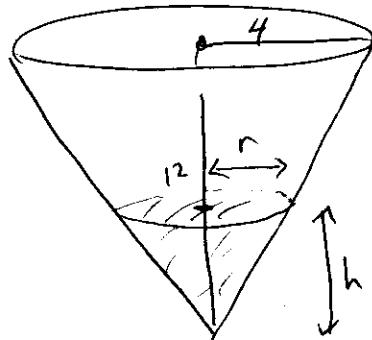
↑  
weight      ↗  
                distance

$$= \int_0^5 (20 + 4x) dx = [20x + 2x^2]_0^5$$

$$= 100 + 50$$

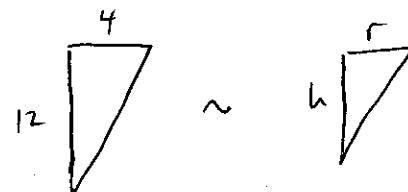
$$= \boxed{150 \text{ ft-lbs}}$$

- [7] A conical paper cup is being filled with water at a constant rate of  $2 \text{ cm}^3/\text{sec}$ . The cup has radius 4 cm and height 12 cm. How fast is the height of the water changing when the water in the cup is at 5 cm? [Note: the volume of a cone is  $\frac{1}{3}$  times area of the base times height.]



$h$  = height of water

$r$  = radius of top of water



$$\text{Similar OS} \Rightarrow r = \frac{h}{4}$$

Volume of water =

$$V = \frac{1}{3} \pi \left(\frac{h}{4}\right)^2 h = \frac{\pi h^3}{27} \quad \text{relation}$$

so

$$\frac{dV}{dt} = \frac{\pi h^2}{9} \frac{dh}{dt} \quad \text{relation of rates}$$

Put in data:

$$\frac{dV}{dt} = 2 \text{ cm}^3/\text{sec}$$

$$h = 5 \text{ cm}$$

$$2 = \frac{25\pi}{9} \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{18}{25\pi}$$

Water height is rising at a rate  
of  $\frac{18}{25\pi} \text{ cm/sec}$ .

[8] (a) Evaluate  $\int_0^\pi (\sin x)^5 \cos x \, dx$

$$u = \sin x, \quad du = \cos x \, dx$$

$$\begin{aligned} \int_0^\pi (\sin x)^5 \cos x \, dx &= \int_0^0 u^5 \, du \\ &= \boxed{0} \end{aligned}$$

(b) Evaluate  $\int \frac{2x^3}{\sqrt{1+x^2}} \, dx$ .

$$u = 1+x^2, \quad du = 2x \, dx$$

$$\Rightarrow x^2 = u - 1$$

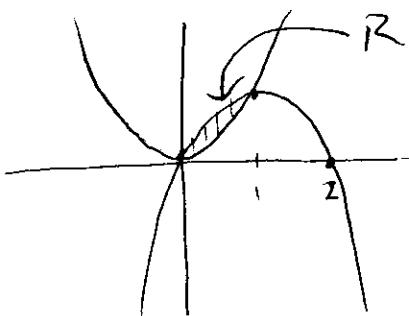
$$\text{get } \int \frac{u-1}{\sqrt{u}} \, du = \int (u^{1/2} - u^{-1/2}) \, du$$

$$= \frac{2}{3} u^{3/2} - 2 u^{1/2} + C$$

$$= \boxed{\frac{2}{3} (1+x^2)^{3/2} - 2 (1+x^2)^{1/2} + C}$$

[9] Let  $R$  be the region between the curves  $y = x^2$  and  $y = 2x - x^2$ .

(a) Find the area of  $R$ .



$$y = 2x - x^2 = x(2-x)$$

roots at 0, 2

$$2x - x^2 = x^2, \quad 2x - 2x^2 = 0$$

$$2x(1-x) = 0$$

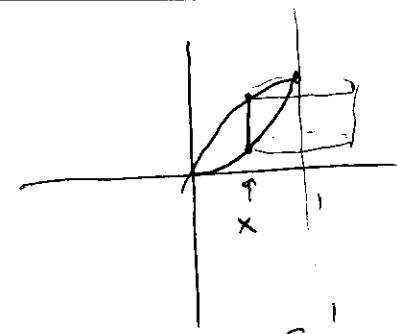
$$x=0, 1$$

intersection.

$$\text{Area} = \int_0^1 (2x - x^2 - x^2) dx = \int_0^1 (2x - 2x^2) dx$$

$$= \left[ x^2 - \frac{2}{3}x^3 \right]_0^1 = \boxed{\frac{1}{3}}$$

(b) Use the method of cylindrical shells to find the volume of the solid that is obtained by rotating  $R$  about the line  $x = 1$ .



$$\text{radius} \equiv 1-x$$

$$A(x) = 2\pi(1-x)(2x - 2x^2)$$

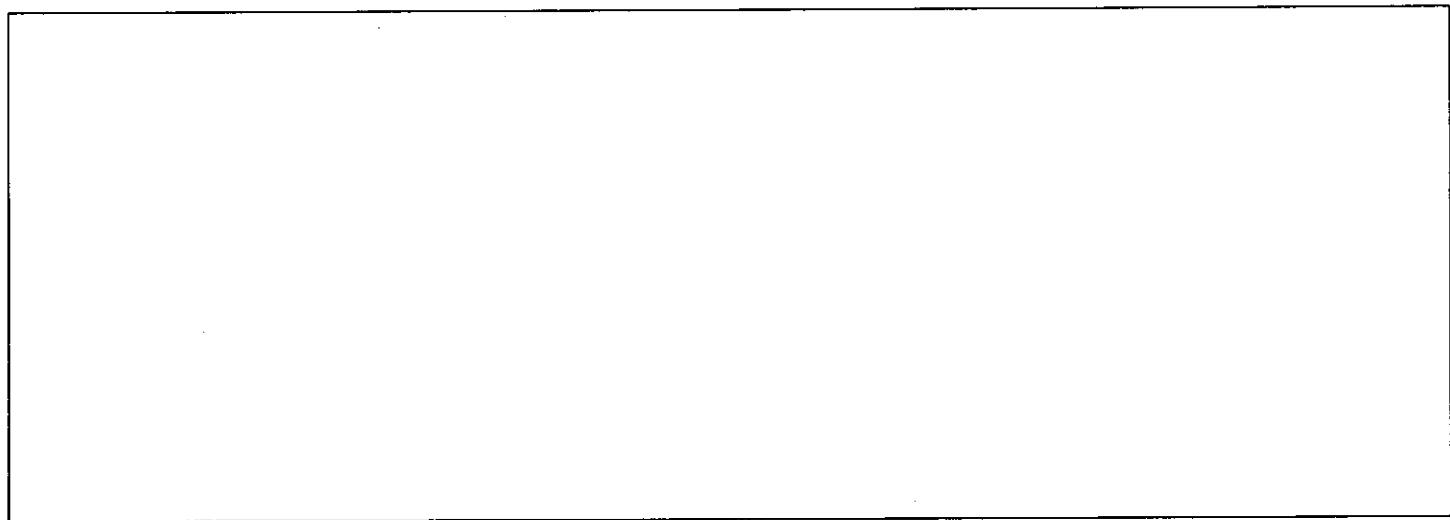
$$\text{Volume} = \int_0^1 2\pi(1-x)(2x - 2x^2) dx$$

$$= 2\pi \int_0^1 (2x - 4x^2 + 2x^3) dx$$

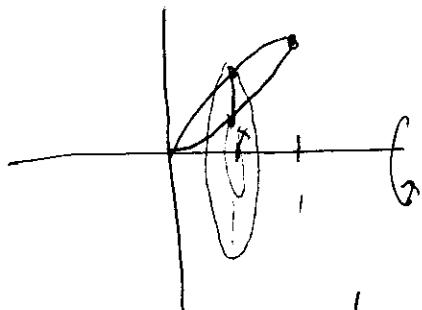
$$= 2\pi \left( x^2 - \frac{4}{3}x^3 + \frac{2}{4}x^4 \right) \Big|_0^1 = 2\pi \left( 1 - \frac{4}{3} + \frac{1}{2} \right)$$

$$= \boxed{\frac{\pi}{3}}$$

(b) (cont)



(c) Use the method of washers to find the volume of the solid that is obtained by rotating  $R$  about the line  $y = 0$ .



$$\text{Outer radius of washer} = 2x - x^2$$

$$\text{Inner radius} = x^2$$

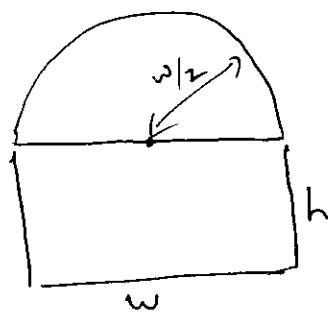
$$A(x) = \pi (2x - x^2)^2 - \pi (x^2)^2$$

$$\text{Volume} = \int_0^1 (\pi (2x - x^2)^2 - \pi (x^2)^2) dx$$

$$= \pi \int_0^1 (4x^2 - 4x^3 + x^4 - x^4) dx$$

$$= \pi \left( \frac{4}{3}x^3 - x^4 \right) \Big|_0^1 = \boxed{\frac{\pi}{3}}$$

- [10] A Norman window has the shape of a semi-circle sitting on top of a rectangle (so the diameter of the semi-circle equals the width of the rectangle). It has perimeter equal to 22 feet (the length of the semi-circle plus three sides of the rectangle). Find the dimensions that maximize the area of the window.



$$\text{Perimeter} = \text{semicircle} + 3 \text{ sides}$$

$$= \frac{\pi w}{2} + w + 2h = 22$$

$$2h = 22 - \frac{\pi w}{2} - w$$

$$h = 11 - \frac{\frac{\pi w}{2} + w}{2}$$

$$\text{Area} = \frac{1}{2} \pi \left(\frac{w}{2}\right)^2 + wh$$

$$A(w) = \frac{1}{2} \pi \left(\frac{w}{2}\right)^2 + w \left(11 - \frac{\frac{\pi w}{2} + w}{2}\right)$$

$$= \frac{\pi w^2}{8} + 11w - \frac{\pi w^2}{4} - \frac{w^2}{2}$$

$$A'(w) = \frac{\pi w}{4} + 11 - \frac{\pi w}{2} - w = 0$$

$$\frac{\pi w}{4} + 11 - \frac{\pi w}{2} - w = 0$$

$$w = \frac{11}{\frac{\pi}{4} + 1} = \frac{44}{\pi + 4}$$

only  
critical  
number

$$h = 11 - \left(\frac{\pi}{4} + \frac{1}{2}\right) \left(\frac{44}{\pi + 4}\right)$$

$$\text{radius of semicircle} = \frac{w}{2} = \frac{22}{\pi + 4}$$