1(a) Use an integrating factor to solve the first-order linear equation $\frac{dy}{dx} = \frac{-y}{x} + 2$.

$$\underbrace{y' + \frac{1}{x}}_{P(x)} y = \underbrace{2}_{Q(x)} \qquad \rho(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = \underline{x}.$$

$$xy' + y = 2x$$

$$\frac{d}{dx}(xy) = 2x$$

$$xy = \int 2x\,dx = x^2 + c$$

$$\boxed{y = x + \frac{c}{x}}$$

1(b) Find the following items:

(i) $\mathcal{L}\{t\sin(3t)\}$.

$$\mathcal{L}\{\sin(3t)\} = \frac{3}{s^2+9} = 3(s^2+9)^{-1}$$

$$\mathcal{L}\{t\sin(3t)\} = -\frac{d}{ds}\left(3(s^2+9)^{-1}\right) = -\left(-3(s^2+9)^{-2}(2s)\right)$$

$$= \boxed{\frac{6s}{(s^2+9)^2}}$$

(ii) $\mathcal{L}^{-1}\left\{\frac{s}{(s-2)^2+9}\right\}$.

$$= \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+9} + \frac{2}{(s-2)^2+9}\right\}$$

$$= \frac{2}{3}\,\frac{3}{(s-2)^2+9}$$

$$= \boxed{e^{2t}\cos(3t) + \frac{2}{3}e^{2t}\sin(3t)}$$

2. Use Laplace transforms to solve the initial value problem

$$x'' + 4x = \delta(t) + \delta(t-\pi), \quad x(0) = 0, \quad x'(0) = 0.$$

(a) Express your answer using a step function.

(b) Express your answer without using a step function, but as a function with cases.

(a) $\mathcal{L}\{x''\} + 4\mathcal{L}\{x\} = \mathcal{L}\{\delta(t)\} + \mathcal{L}\{\delta(t-\pi)\}$

$s^2 X(s) + 4X(s) = 1 + e^{-\pi s}$

$X(s) = \frac{1}{s^2+4} + e^{-\pi s}\frac{1}{s^2+4}$

$x(t) = \mathcal{L}^{-1}\{X(s)\} = \boxed{\frac{1}{2}\sin(2t) + u(t-\pi)\frac{1}{2}\sin(2(t-\pi))}$

$= \frac{1}{2}\sin(2t) + \frac{1}{2}u(t-\pi)\sin(2t)$

(b)

$$\boxed{x(t) = \begin{cases} \frac{1}{2}\sin(2t) & \text{if } t < \pi \\ \sin(2t) & \text{if } t \geq \pi \end{cases}}$$

3(a) Write down the general solution of the homogeneous linear equation $y'' + y' - 6y = 0$.

$$r^2 + r - 6 = 0 \qquad (r-2)(r+3) = 0$$

$$r = 2, -3$$

$$\boxed{y(x) = Ae^{2x} + Be^{-3x}}$$

(b) Write down the form in which you will look for a particular solution of the non-homogeneous linear equation

$$y'' + y' - 6y = (4 + 3x^2)e^{2x}.$$

Do *not* compute the coefficients!

$$\boxed{y_p(x) = x(A + Bx + Cx^2)e^{2x}}$$

because of duplication

(c) Write down the form in which you will look for a particular solution of the non-homogeneous linear equation

$$y'' + y' - 6y = x\cos x.$$

Do *not* compute the coefficients!

$$\boxed{y_p(x) = (A + Bx)\cos x + (C + Dx)\sin x}$$

(d) Suppose the functions $y^{(b)}(x)$ and $y^{(c)}(x)$ are particular solutions to the equations in parts (b) and (c) above. Assuming that you know these two functions, write down the *general* solution to the non-homogeneous linear equation

$$y'' + y' - 6y = (4 + 3x^2)e^{2x} + x\cos x.$$

$$\boxed{y(x) = \underbrace{Ae^{2x} + Be^{-3x}}_{y_c} + \underbrace{y^{(b)}(x) + y^{(c)}(x)}_{y_p}}$$

4(a) A mass-spring-dashpot system with equation $mx'' + cx' + kx = 0$ has general solution $x(t) = Ae^{-3t} + Bte^{-3t}$.

(i) Is the system underdamped, critically damped, or overdamped? Explain.

(ii) If $m = 1$ and $k = 9$, what can you say about the damping constant c?

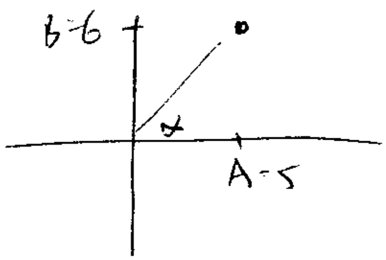
(i) The solution indicates a repeated root, so the system is $\boxed{\text{critically damped}}$.

(ii) The char. equation is $(r+3)(r+3) = 0$

$$\underset{m=1}{r^2} + \underset{c}{6}r + \underset{k=9}{9} = 0$$

$$\boxed{c = 6}$$

4(b) Write the function $x(t) = \underset{A}{5}\cos(3t) + \underset{B}{6}\sin(3t)$ as a single oscillation.



quadrant I $\quad \tan\alpha = B/A$

$$x(t) = C\cos(3t - \alpha) \qquad C = \sqrt{A^2 + B^2}$$

$$\boxed{x(t) = \sqrt{61}\cos\left(3t - \tan^{-1}(6/5)\right)}$$

5. This problem concerns the eigenvalue method and the homogeneous linear system

$$x_1' = 4x_1 + x_2$$
$$x_2' = 6x_1 - x_2.$$

(i) Write the system in matrix notation and find the eigenvalues of the matrix.

(ii) Find the corresponding eigenvectors.

(iii) Write down the general solution $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

(i)

$$\underline{x}'(t) = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} \underline{x}(t)$$

$$\begin{vmatrix} 4-\lambda & 1 \\ 6 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda) - 6$$
$$= -4 + \lambda - 4\lambda + \lambda^2 - 6$$
$$= \lambda^2 - 3\lambda - 10$$
$$= (\lambda+2)(\lambda-5) = 0$$

$$\boxed{\lambda = -2, \ \lambda = 5} \text{ eigenvalues.}$$

(ii)

$\underline{\lambda = -2}$

$$\begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2a \\ -2b \end{bmatrix} \qquad \begin{cases} 4a + b = -2a \\ 6a - b = -2b \end{cases}$$

$$\begin{cases} 6a + b = 0 \\ 6a + b = 0 \end{cases} \qquad \text{take } \boxed{\underset{\sim}{v} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}}$$

(or any multiple of this)

5. (cont)

$\underline{\lambda = 5}$

$$\begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5a \\ 5b \end{bmatrix} \qquad \begin{cases} 4a+b = 5a \\ 6a-b = 5b \end{cases}$$

$$\begin{cases} -a+b = 0 \\ 6a-6b = 0 \end{cases} \qquad \underline{a = b} \quad \text{take} \quad \boxed{\underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

(iii)

$$\underline{x}(t) = c_1 \begin{bmatrix} 1 \\ -6 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}$$

$$= \begin{bmatrix} c_1 e^{-2t} + c_2 e^{5t} \\ -6c_1 e^{-2t} + c_2 e^{5t} \end{bmatrix}$$

or

$$\boxed{\begin{aligned} x_1(t) &= c_1 e^{-2t} + c_2 e^{5t} \\ x_2(t) &= -6c_1 e^{-2t} + c_2 e^{5t} \end{aligned}}$$

6(a) Convert the equation

$$t^3x''' - 2t^2x'' + 3tx' + 5x = \ln t$$

into a first-order system of equations.

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ t^3x_3' - 2t^2x_3 + 3t\,x_2 + 5x_1 &= \ln t \end{aligned}}$$

3 equations in 3 variables, first derivatives only

6(b) The functions $\mathbf{x}_1(t) = \begin{bmatrix} 3e^t \\ 2e^t \end{bmatrix}$ and $\mathbf{x}_2(t) = \begin{bmatrix} 6e^{-t} \\ 4e^{-t} \end{bmatrix}$ are solutions to the system $\mathbf{x}'(t) = \begin{bmatrix} 3 & -3 \\ -2 & 4 \end{bmatrix} \mathbf{x}(t)$.

(i) Are $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ linearly independent? Why or why not?

(ii) Is $\mathbf{x}(t) = A\mathbf{x}_1(t) + B\mathbf{x}_2(t)$ the general solution to the system? Why or why not?

(i) $W(\mathbf{x}_1, \mathbf{x}_2) = \begin{vmatrix} 3e^t & 6e^{-t} \\ 2e^t & 4e^{-t} \end{vmatrix} = 12e^{t-t} - 12e^{t-t} = 0$

They are $\boxed{\text{not linearly independent}}$, because the Wronskian is zero.

(ii) $\boxed{\text{It is not the general solution}}$ because $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly dependent.

7. Use the subsitution $v = \frac{y}{x}$ to convert the homogeneous equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

into a separable equation, and then find the general solution.

$$\frac{dy}{dx} = \frac{4x^2}{2xy} + \frac{3y^2}{2xy} = 2\left(\frac{y}{x}\right)^{-1} + \frac{3}{2}\left(\frac{y}{x}\right)$$

$$v = \frac{y}{x} \Rightarrow vx = y \Rightarrow x\frac{dv}{dx} + v = \frac{dy}{dx}$$

substitute:

$$x\frac{dv}{dx} + v = 2v^{-1} + \frac{3}{2}v$$

$$x\frac{dv}{dx} = \frac{2}{v} + \frac{v}{2} = \frac{4+v^2}{2v}$$

separate:

$$\int \frac{2v}{4+v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\ln|4+v^2| = \ln|x| + C$$

$$4+v^2 = e^{\ln|x|}\cdot e^{c}$$

$$= e^{c}|x| = Kx$$

any constant K

$$\boxed{4+\left(\frac{y}{x}\right)^2 = kx}$$

or

$$\boxed{y^2 = kx^3 - 4x^2}$$

FUNCTION	LAPLACE TRANSFORM
$f(t)$	$F(s)$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f'(t)$	$sF(s)-f(0)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$
$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$
$e^{at}f(t)$	$F(s-a)$
$u(t-a)f(t-a)$	$e^{-as}F(s)$
$(f*g)(t)$	$F(s)G(s)$ where $(f*g)(t)=\int_0^t f(\tau)g(t-\tau)d\tau$
$tf(t)$	$-F'(s)$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)d\sigma$
$f(t)$, period p	$\frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)dt$
t^n	$\frac{n!}{s^{n+1}}$
t^a	$\frac{\Gamma(a+1)}{s^{n+1}}$ where $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$
e^{at}	$\frac{1}{s-a}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$ where $u(t-a)$ is 0 when $t<a$, and 1 when $t\geq a$
$\delta(t-a)$	e^{-as} where $\delta(t-a)$ is a unit impulse at time $t=a$

$f(x)$	y_p
$P_m(x)=b_0+b_1x+b_2x^2+\cdots+b_mx^m$	$x^s(A_0+A_1x+A_2x^2+\cdots+A_mx^m)$
$a\cos kx+b\sin kx$	$x^s(A\cos kx+B\sin kx)$
$e^{rx}(a\cos kx+b\sin kx)$	$x^se^{rx}(A\cos kx+B\sin kx)$
$P_m(x)e^{rx}$	$x^s(A_0+A_1x+A_2x^2+\cdots+A_mx^m)e^{rx}$
$P_m(x)(a\cos kx+b\sin kx)$	$x^s[(A_0+A_1x+\cdots+A_mx^m)\cos kx$ $+(B_0+B_1x+\cdots+B_mx^m)\sin kx]$