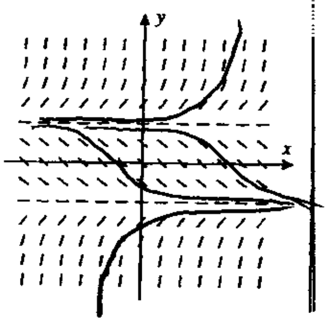
1. The slope field for the equation $y' = y^2 - 1$ is shown below.



Suppose $y(x)$ is a solution to this equation.
(a) What can you say about the value $y(250)$? Be as precise as you can. Are there different cases to consider?

(b) Similarly, what can be said about the value $y(-250)$, possibly in different cases?

(a) $y(250)$ can be any number. However, based on initial conditions, say $y(0)$, we can say

- if $-1 < y(0) < 1$ then $y(250) \approx -1$
- if $y(0) = 1$ then $y(250) = 1$
- if $y(0) = -1$ then $y(250) = -1$
- if $y(0) > 1$ then $y(250)$ is large
- if $y(0) < -1$ then $y(250)$ is ≈ -1

(b) $y(-250)$ is $\begin{cases} \approx 1 & \leftarrow \text{if } y(0) > 1 \text{ or } -1 < y(0) < 1 \\ = 1 & \leftarrow \text{if } y(0) = 1 \\ = -1 & \leftarrow \text{if } y(0) = -1 \\ \text{very negative} & \leftarrow \text{if } y(0) < -1 \end{cases}$

2. A cake is removed from an oven at 210°F and left to cool at room temperature, which is 70°F. After 30 minutes the temperature of the cake is 140°F. When will it be 100°F?

Recall that Newton's law of heating and cooling says the following: the rate of change of the temperature is proportional to the difference between the temperature and the ambient temperature. (The previous sentence is the differential equation; start by writing it down.)

$$\frac{dT}{dt} = k(70 - T) \qquad T(t) = \text{temp. time } t$$

Separate: $\int \frac{1}{70-T}\,dT = \int k\,dt$

$$-\ln|70-T| = kt + C$$

$$|70-T| = e^{-kt} e^{-C}$$

$$70 - T = \pm e^{-C} e^{-kt} = D e^{-kt}$$

$$\underline{T = 70 - De^{-kt}}$$

$\underline{T(0)=210}$ $\Rightarrow$ $210 = 70 - De^{-k(0)}$, $140 = -D$

So $\underline{T = 70 + 140e^{-kt}}$

$\underline{T(30)=140}$ $\Rightarrow$ $140 = 70 + 140e^{-k(30)}$

$$\frac{1}{2} = e^{-30k}, \qquad k = \frac{-\ln(\frac{1}{2})}{30}$$

$$\underline{T = 70 + 140e^{\frac{\ln(1/2)t}{30}}}$$

$$100 = 70 + 140e^{\frac{\ln(1/2)t}{30}}$$

$$3/14 = e^{\frac{\ln(1/2)t}{30}} \qquad \ln\left(\frac{3}{14}\right) = \frac{\ln(\frac{1}{2})t}{30}$$

$$\boxed{t = 30\,\frac{\ln(3/14)}{\ln(1/2)} \text{ minutes}}$$

3(a) Use the substitution $p = y'$ to transform the equation $yy'' = 3(y')^2$ into a first-order equation. Do not solve this equation.

(b) Use a substitution to transform the equation $2xy' + y^3e^{-2x} = 2xy$ into a linear equation. Do not solve this equation.

(a) $p = \frac{dy}{dx}$, $y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy}p$

$$\boxed{y\frac{dp}{dy}p = 3p^2}$$

(b) $y' - y = \frac{-e^{-2x}}{2x}y^3$ Bernoulli, $n = 3$

use $v = y^{1-3} = y^{-2}$, $y = v^{-1/2}$

$\frac{dy}{dx} = \frac{-1}{2}v^{-3/2}\frac{dv}{dx}$

substitute:

$$\frac{-1}{2}v^{-3/2}\frac{dv}{dx} - v^{-1/2} = -\frac{e^{-2x}}{2x}v^{-3/2}$$

multiply by $-2v^{3/2}$:

$$\boxed{\frac{dv}{dx} + 2v = \frac{e^{-2x}}{x}}$$ linear in v.

4. Use an integrating factor to solve the linear equation $\underbrace{y' - 2y}_{P(x)} = \underbrace{3e^{2x}}_{Q(x)}$.

Integrating Factor is $\rho(x) = e^{\int P(x)\,dx}$

$= e^{\int -2\,dx} = e^{-2x}$

$\underbrace{e^{-2x} y' - 2y e^{-2x}} = 3$

$\frac{d}{dx}\left(e^{-2x} y\right) = 3$

$e^{-2x} y = \int 3\,dx = 3x + C$

$$\boxed{y = e^{2x}(3x + C)}$$

5. Consider the differential equation $y' = 3y^{2/3}$.

(a) Find the general solution to the equation.

(b) Find a solution $y(x)$ satisfying the initial condition $y(2) = 0$.

(c) Find a *second* solution satisfying the same initial condition.

(d) Can you find or describe a third one?

(a) $\frac{dy}{dx} = 3y^{2/3}$ separate:

$$\int y^{-2/3}\,dy = \int 3\,dx$$

$$3y^{1/3} = 3x + C$$

$$y^{1/3} = x + \frac{C}{3}$$

$$\boxed{y = \left(x + \frac{C}{3}\right)^3}$$

(b) $0 = \left(2 + \frac{C}{3}\right)^3$ $\qquad \frac{C}{3} = -2$

$C = -6$

$$\boxed{y = (x-2)^3}$$

(c) singular solution, $\boxed{y = 0}$ works

(d) also

$$\boxed{y(x) = \begin{cases} 0 & x \le 2 \\ (x-2)^3 & x \ge 2 \end{cases}}$$