1(a) Solve the initial value problem $y'' - 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 10$.

$$r^2 - 5r + 6 = 0$$

$$(r-2)(r-3) = 0 \qquad r = 2, 3$$

$$y(x) = Ae^{2x} + Be^{3x}$$

$$2 = Ae^0 + Be^0, \quad 2 = A+B.$$

$$y'(x) = 2Ae^{2x} + 3Be^{3x}$$

$$10 = 2A + 3B$$

$$-\quad 4 = 2A + 2B$$

$$6 = B, \quad A = -4$$

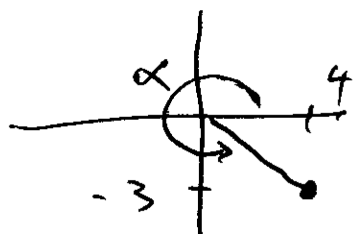
$$\boxed{y(x) = -4e^{2x} + 6e^{3x}}$$

1(b) Convert the function $x(t) = 4\cos(11t) - 3\sin(11t)$ into the form $x(t) = C\cos(\omega t - \alpha)$. Use an exact expression (possibly involving $\tan^{-1}$) for α, rather than a decimal value.

$$A = 4, \quad B = -3$$

$$C = \sqrt{4^2 + (-3)^2} = 5$$

$$\tan\alpha = -3/4, \text{ quad } 4$$



quad. 4

$$\alpha = \tan^{-1}(-3/4) + 2\pi$$

$$\boxed{x(t) = 5\cos\left(11t - \left(\tan^{-1}(-3/4) + 2\pi\right)\right)}$$

2. Recall the equation $mx'' + cx' + kx = 0$ for the motion of a mass on a spring with damping.

(a) Find the position function $x(t)$ when $m = 1$, $c = 2$, and $k = 2$. Is the system overdamped, critically damped, or underdamped?

$$x'' + 2x' + 2x = 0$$

$$r^2 + 2r + 2 = 0$$

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$\boxed{\begin{array}{c} x(t) = e^{-t}(A\cos t + B\sin t) \\ \text{underdamped} \end{array}}$$

(b) You pull the mass to the right 3 units, pause, and let it go. Find the specific position function in this situation.

$$x(0) = 3, \quad x'(0) = 0$$

$$x'(t) = -e^{-t}(A\cos t + B\sin t) + e^{-t}(-A\sin t + B\cos t)$$

$$3 = x(0) = e^{0}(A\cos(0) + B\sin(0)) = A$$

$$0 = x'(0) = -e^{0}(A) + e^{0}(B) = -A + B$$

$$\Rightarrow A = B$$

$$\Rightarrow A = 3, \ B = 3$$

$$\boxed{x(t) = e^{-t}(3\cos t + 3\sin t)}$$

3(a) Explain why the functions $f_1(x) = \sin^2 x$, $f_2(x) = \cos^2 x$, $f_3(x) = 15$ are not linearly independent.

$\sin^2 x + \cos^2 x = 1$, so

$$\boxed{15 \underbrace{\sin^2 x}_{f_1} + 15 \underbrace{\cos^2 x}_{f_2} + (-1) \underbrace{15}_{f_3} = 0}$$

hence, not linearly independent.

(b) Use the Wronskian to determine whether the functions $g_1(x) = x$, $g_2(x) = x^2$, $g_2(x) = x^4$ are linearly independent.

$$W(x, x^2, x^4) = \begin{vmatrix} x & x^2 & x^4 \\ 1 & 2x & 4x^3 \\ 0 & 2 & 12x^2 \end{vmatrix}$$

$$= x \cdot 2x \cdot 12x^2 + 2x^4 - x^2 \cdot 12x^2 - 2 \cdot x \cdot 4x^3$$

$$= (24 + 2 - 12 - 8)x^4 = 6x^4.$$

not zero, hence linearly independent.

4. Consider the differential equation

$$y^{(6)} - 6y^{(5)} + 17y^{(4)} - 48y^{(3)} + 88y'' - 96y' + 144y = x^2e^{3x} + \sin(2x).$$

(a) Using the fact that $r^6 - 6r^5 + 17r^4 - 48r^3 + 88r^2 - 96r + 144 = (r-3)^2(r^2+4)^2$, find a general solution to the associated homogeneous equation. (That is, find the complementary solution y_c.)

roots: $3, 3, \pm 2i, \pm 2i$

$$\boxed{y_c(x) = Ae^{3x} + Bxe^{3x} + C\cos(2x) + D\sin(2x) + Ex\cos(2x) + Fx\sin(2x)}$$

(b) Using the table below, set up the appropriate form of a particular solution (but do not determine the values of the coefficients).

$f(x)$	y_p
$P_m(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$	$x^s(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)$
$a\cos kx + b\sin kx$	$x^s(A\cos kx + B\sin kx)$
$e^{rx}(a\cos kx + b\sin kx)$	$x^se^{rx}(A\cos kx + B\sin kx)$
$P_m(x)e^{rx}$	$x^s(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)e^{rx}$
$P_m(x)(a\cos kx + b\sin kx)$	$x^s[(A_0 + A_1x + \cdots + A_mx^m)\cos kx + (B_0 + B_1x + \cdots + B_mx^m)\sin kx]$

$$\boxed{y_p(x) = x^2\left[(A + Bx + Cx^2)e^{3x}\right] + x^2\left[D\cos(2x) + E\sin(2x)\right]}$$

5. This problem concerns the differential equation $y'' + 25y = 4e^{4x}$.

(a) Using the method of undetermined coefficients, find a particular solution to the equation.

$r^2 + 25 = 0, \quad r = \pm 5i \qquad y_c(x) = A\cos(5x) + B\sin(5x)$

try $y_p = Ae^{4x}$ (no duplication ✓)

$y_p'' = 16Ae^{4x}$

$y_p'' + 25y_p = 16Ae^{4x} + 25Ae^{4x} = 4e^{4x}$

$41A = 4$

$A = \frac{4}{41}$ $\qquad \boxed{y_p(x) = \frac{4}{41}e^{4x}}$

(b) Find the general solution to the equation.

$y(x) = y_c(x) + y_p(x)$

$$\boxed{y = A\cos(5x) + B\sin(5x) + \frac{4}{41}e^{4x}}$$

Extra Credit Use Euler's formula $e^{ix} = \cos x + i\sin x$ to derive a formula for $\cos(\alpha + \beta)$.

$e^{i(\alpha+\beta)} = \underline{\cos(\alpha+\beta)} + i\sin(\alpha+\beta)$

$\parallel$

$e^{i\alpha}e^{i\beta}$

$\parallel$

$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$= \underline{(\cos\alpha\cos\beta - \sin\alpha\sin\beta)} + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$

take real parts, get

$$\boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$$