FUNCTION	LAPLACE TRANSFORM
$f(t)$	$F(s)$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f'(t)$	$sF(s)-f(0)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$
$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$
$e^{at}f(t)$	$F(s-a)$
$u(t-a)f(t-a)$	$e^{-as}F(s)$
$(f*g)(t)$	$F(s)G(s)$ where $(f*g)(t)=\int_0^t f(\tau)g(t-\tau)d\tau$
$tf(t)$	$-F'(s)$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)d\sigma$
$f(t)$, period p	$\frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)dt$
t^n	$\frac{n!}{s^{n+1}}$
t^a	$\frac{\Gamma(a+1)}{s^{n+1}}$ where $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$
e^{at}	$\frac{1}{s-a}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$ where $u(t-a)$ is 0 when $t<a$, and 1 when $t\geq a$
$\delta(t-a)$	e^{-as} where $\delta(t-a)$ is a unit impulse at time $t=a$

1. Compute the following:

(a) $\mathcal{L}\left\{e^{7t}\sin(7t)\right\}$

$$\mathcal{L}\{\sin(7t)\} = \frac{7}{s^2+49}, \text{ so}$$

$$\mathcal{L}\{e^{7t}\sin(7t)\} = \boxed{\frac{7}{(s-7)^2+49}}$$

(b) $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} \cos t & t < 2\pi \\ 0 & t \geq 2\pi. \end{cases}$ (Hint: write $f(t)$ using a step function.)

$$\begin{aligned} f(t) &= \cos t\,(1 - u(t-2\pi)) \\ &= \cos t - u(t-2\pi)\cos t \\ &= \cos t - u(t-2\pi)\cos(t-2\pi), \text{ so} \end{aligned}$$

$$\mathcal{L}\{f(t)\} = \boxed{\frac{s}{s^2+1} - e^{-2\pi s}\frac{s}{s^2+1}}$$

(c) $\mathcal{L}^{-1}\left\{\frac{s}{(s-3)^2+9}\right\}$

$$\frac{s}{(s-3)^2+9} = \frac{s-3}{(s-3)^2+9} + \frac{3}{(s-3)^2+9}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-3)^2+9}\right\} = \boxed{e^{3t}\cos(3t) + e^{3t}\sin(3t)}$$

2(a) Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$ by using the convolution formula and calculating the convolution.

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = 1 * \sin t$$

$$= \int_0^t \sin(t-\tau)\,d\tau \qquad u = t-\tau \qquad \tau = t-u$$

$$du = -d\tau$$

$$= \int_t^0 -\sin u\,du = \int_0^t \sin u\,du = -\cos u\Big|_0^t$$

$$= -\cos t + \cos(0)$$

$$= \boxed{1 - \cos t}$$

2(b) Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$ by using a different formula involving an integral.

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t, \quad \text{so}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin\tau\,d\tau$$

$$= -\cos\tau\Big|_0^t$$

$$= \boxed{1 - \cos t}$$

3. Use the Laplace transform to solve the following initial value problem:

$$x'' - 3x' - 4x = 0, \quad x(0) = 0, \; x'(0) = 5.$$

$$\mathcal{L}\{x''\} - 3\mathcal{L}\{x'\} - 4\mathcal{L}\{x\} = 0$$

$$\mathcal{L}\{x''\} = s^2 X(s) - s\,x(0) - x'(0)$$
$$= s^2 X(s) - 5$$
$$\mathcal{L}\{x'\} = s\,X(s) - x(0) = s\,X(s), \text{ so}$$

$$s^2 X(s) - 5 - 3sX(s) - 4X(s) = 0$$
$$X(s) = \frac{5}{s^2 - 3s - 4} = \frac{5}{(s-4)(s+1)}$$

$$\frac{5}{(s-4)(s+1)} = \frac{A}{(s-4)} + \frac{B}{(s+1)} = \frac{As + A + Bs - 4B}{(s-4)(s+1)}$$

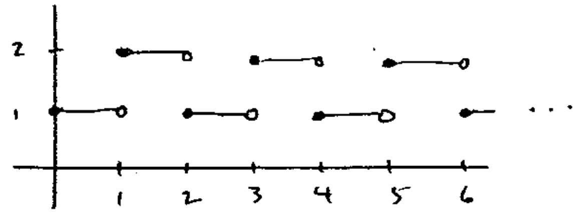
$A + B = 0$, $A - 4B = 5$
$-B - 4B = 5$ $\quad -5B = 5$ $\quad B = -1$

$\Rightarrow A = 1$

$$X(s) = \frac{1}{s-4} - \frac{1}{s+1}$$

$$\boxed{x(t) = e^{4t} - e^{-t}}$$

4(a) Compute the Laplace transform of the function shown here:



periodic, period $= 2$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}\,dt + \int_1^2 2e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{-1}{s} e^{-st}\Big|_0^1 + \frac{-2}{s} e^{-st}\Big|_1^2\right]$$

$$= \boxed{\frac{1}{1-e^{-2s}}\left[\frac{-e^{-s}}{s} + \frac{1}{s} - \frac{2e^{-2s}}{s} + \frac{2e^{-s}}{s}\right]}$$

(b) Find $\mathcal{L}\{te^{4t}\sin t\}$. (Hint: find $\mathcal{L}\{t\sin t\}$ first.)

$$\mathcal{L}\{t\sin t\} = -\frac{d}{ds}\mathcal{L}\{\sin t\} = \frac{-d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{2s}{(s^2+1)^2}, \quad \text{so}$$

$$\boxed{\mathcal{L}\{e^{4t}t\sin t\} = \frac{2(s-4)}{((s-4)^2+1)^2}}$$

5(a) A mass-spring system with an external oscillation force satisfies the differential equation

$$x'' + 9x = 12\sin(3t).$$

Does resonance occur? Explain why or why not.

$r^2 + 9 = 0, \quad r = \pm 3i$

$x_c = A\cos(3t) + B\sin(3t)$

natural frequency of system is 3

Since the frequency of the external force matches the natural frequency, resonance will occur.

5(b) Find the steady periodic solution to the system

$$x'' - x' + 4x = 4\sin(2t).$$

The steady periodic solution is the particular solution

$x_{sp}(t) = A\cos(2t) + B\sin(2t)$

$x_{sp}' = -2A\sin(2t) + 2B\cos(2t)$

$x_{sp}'' = -4A\cos(2t) - 4B\sin(2t)$

$x_{sp}'' - x_{sp}' + 4x_{sp} = (-4A - 2B + 4A)\cos(2t) + (-4B + 2A + 4B)\sin(2t)$

Get $-2B\cos(2t) + 2A\sin(2t) = 4\sin(2t)$

$B = 0, \quad A = 2$

$$\boxed{x_{sp}(t) = 2\cos(2t)}$$