1. True or false (please circle one for each item):

T F For any matrix A, the row rank of A equals the row rank of A^T. = col. rank of A

T F If A is upper triangular and B is lower triangular, then A and B are not similar. diag. matrices are similar to themselves

T F Any eigenvector for A is also an eigenvector for A^2.

T F Any eigenvalue for A is also an eigenvalue for A^2.

T F If $n > m$ then any linear transformation $L\colon R^n \to R^m$ has a non-trivial kernel. rank $\leq m$, so nullity $\geq n-m$

T F If $A^2 = A$ then $\det(A) = 1$. could also be 0

T F If A and B are similar then they have the same eigenvalues.

T F If $n < m$, then every linear transformation $L\colon R^n \to R^m$ has rank less than m. always, rank $\leq n, m$ and $n < m$

T F For any matrix A, the matrix AA^T is symmetric. $(AA^T)^T = A^{T^T} A^T = AA^T$

T F The permutation 3 6 2 5 1 4 is even.

2(a) Find all solutions to the system

$$
\begin{aligned}
x_1 + 2x_2 - x_3 + 3x_4 &= 0\\
2x_1 + 2x_2 - x_3 + 2x_4 &= 0\\
x_1 + 3x_3 + 3x_4 &= 0
\end{aligned}
$$

(b) Give a basis for the space of solutions.

(a)

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0\\ 2 & 2 & -1 & 2 & 0\\ 1 & 0 & 3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0\\ 0 & -2 & 1 & -4 & 0\\ 0 & -2 & 4 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0\\ 0 & 1 & -1/2 & 2 & 0\\ 0 & -1 & 2 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0\\ 0 & 1 & -1/2 & 2 & 0\\ 0 & 0 & 3/2 & 2 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0\\ 0 & 1 & -1/2 & 2 & 0\\ 0 & 0 & 1 & 4/3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 8/3 & 0\\ 0 & 0 & 1 & 4/3 & 0 \end{array}\right]$$

$x_4 = a$, $\quad x_1 = x_4 = a$

$x_2 = -8/3\, x_4 = -\frac{8}{3}a$

$x_3 = -4/3\, x_4 = -\frac{4}{3}a$

solutions are $\boxed{\bar{x} = \begin{bmatrix} a \\ -\frac{8}{3}a \\ -\frac{4}{3}a \\ a \end{bmatrix} \text{ for all numbers } a.}$

(b) taking $a = 1$:

$$\boxed{\text{a basis is } \left\{ \begin{bmatrix} 1 \\ -8/3 \\ -4/3 \\ 1 \end{bmatrix} \right\}}$$

3. Find the determinant of the matrix A in two different ways:

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(a) using reduction to triangular form

(b) using cofactor expansion along a suitable column or row

(a) $\det A = 2\begin{vmatrix} 1 & 1/2 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix}$

$= \frac{3}{2}\cdot 2\begin{vmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = 3\begin{vmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 4/3 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix}$

$= \frac{4}{3}\cdot 3\begin{vmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 1 & 3/4 \\ 0 & 0 & 1 & 2 \end{vmatrix} = 4\begin{vmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 1 & 3/4 \\ 0 & 0 & 0 & 5/4 \end{vmatrix} = 4\cdot\frac{5}{4} = \boxed{5}$

(b)

first row:

$\det A = 2\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$

$= 2(8-2-2) - (4-1)$

$= \boxed{5}$

4. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

(a) If $P_{T\leftarrow S} = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 6 & 1 \\ 5 & 1 & 3 \end{bmatrix}$ and $\mathbf{v} = 2\mathbf{v}_1 + 2\mathbf{v}_2 - 2\mathbf{v}_3$, what is $[\mathbf{v}]_T$? What are $[\mathbf{v}_1]_T$, $[\mathbf{v}_2]_T$, and $[\mathbf{v}_3]_T$?

(b) If $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, find $P_{S\leftarrow T}$.

(a)

$$[v]_T = P_{T\leftarrow S}[v]_S = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 6 & 1 \\ 5 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 18 \\ 6 \end{bmatrix}}$$

$$[v_1]_T = P_{T\leftarrow S}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$$[v_2]_T = P_{T\leftarrow S}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}$$

$$[v_3]_T = P_{T\leftarrow S}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

(b)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\boxed{P_{S\leftarrow T} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}}$$

5. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

(a) Find all eigenvalues and eigenvectors of A.

(b) Is A diagonalizable? If so, find a diagonal matrix equivalent to A. If not, explain why not.

(a) $p(\lambda) = \begin{vmatrix} \lambda-2 & -1 \\ 0 & \lambda-2 \end{vmatrix} = (\lambda-2)^2$

eigenvalues: $\boxed{\lambda = 2}$

$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \Rightarrow \begin{cases} 2x + y = 2x \\ 2y = 2y \end{cases}$

$\Rightarrow \begin{cases} y = 0 \\ 0 = 0 \end{cases}$

$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ Soln: $x = a$, $y = 0$

eigenvectors: $\boxed{\begin{bmatrix} a \\ 0 \end{bmatrix} \text{ for all numbers } a \neq 0}$

(b) any two eigenvectors are multiples of each other, hence cannot be linearly independent.
So there does not exist a basis of eigenvectors.

$\boxed{\text{So } A \text{ is not diagonalizable.}}$

6. Find a basis for the subspace of R^3 spanned by the vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\\3\end{bmatrix}$, $\begin{bmatrix}-1\\1\\1\end{bmatrix}$. Be sure to say why your basis is a basis.

First, are they linearly independent already?

$$a\begin{bmatrix}1\\0\\1\end{bmatrix} + b\begin{bmatrix}1\\1\\3\end{bmatrix} + c\begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1 & 1 & -1 & 0\\0 & 1 & 1 & 0\\1 & 3 & 1 & 0\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 1 & -1 & 0\\0 & 1 & 1 & 0\\0 & 2 & 2 & 0\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 0 & -2 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

then, c = anything, $b = -c$, $a = 2c$

e.g. can have $c = 1$, $b = -1$, $a = 2$, i.e.

$$2\begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}1\\1\\3\end{bmatrix} + \begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Now, $\begin{bmatrix}1\\1\\3\end{bmatrix} = 2\begin{bmatrix}1\\0\\1\end{bmatrix} + \begin{bmatrix}-1\\1\\1\end{bmatrix}$

So $\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}-1\\1\\1\end{bmatrix}$ span the same subspace.

Neither is a multiple of the other, so they are linearly independent.

So $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}$ is a basis of the subspace.

7. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. Find all eigenvectors associated to the eigenvalue $\lambda = 2$, and find a basis for this space of eigenvectors.

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} \qquad \begin{cases} 2x + y + z = 2x \\ 2y = 2y \\ -y + z = 2z \end{cases}$$

$$\begin{cases} 0x + y + z = 0 \\ 0y = 0 \\ -y - z = 0 \end{cases} \qquad \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\uparrow x = a$ $\quad \uparrow z = b$

$$y = -z = -b$$

eigenvectors: $\boxed{\text{all } \begin{bmatrix} a \\ -b \\ b \end{bmatrix} \text{ with } a, b \text{ not both zero}}$

basis: $\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}}$

8. Let L: $R^2 \to R^3$ be the linear transformation given by

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ x-y \\ x+y \end{bmatrix}.$$

Let $S = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ and $T = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$ be ordered bases of R^2 and R^3. Find the matrix of L with respect to these bases.

$L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. Now express these in the basis T:

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

$\left[L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\right]_T$ $\left[L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right]_T$

$$\boxed{\text{The matrix for } L \text{ is } \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}}$$