Study guide

Linear algebra (Math 3333)

MATRICES AND LINEAR SYSTEMS

1. Define or describe the following concepts.

(a) solutions of a linear system of equations
(b) consistent and inconsistent systems
(c) homogeneous systems
(d) trivial and non-trivial solutions
(e) equivalent systems

2. For each of the following systems, (i) write the corresponding augmented matrix and (ii) find all solutions to the system (if any).

(a)

$$\begin{aligned} x-2y-z&=-3\\ -2x+5y+4z&=7\\ 2x+y+8z&=-1 \end{aligned}$$

(b)

$$\begin{aligned} 3x-4y+z&=-4\\ x-y+z&=0\\ 2x-2y+3z&=1 \end{aligned}$$

(c)

$$\begin{aligned} 4x-2y+6z&=5\\ y+z&=1\\ 3x+2y+5z&=5 \end{aligned}$$

3. Define or describe the following concepts.

(a) row echelon form of a matrix
(b) reduced row echelon form of a matrix
(c) the transpose of a matrix
(d) symmetric and skew-symmetric matrices
(e) non-singular matrices
(f) elementary matrices of three types; explain what each type looks like, and how it relates to row and column operations

(g) equivalent matrices
(h) similar matrices
(i) lower-triangular matrices
(j) diagonal matrices

4. Know the properties of transpose, inverses, determinants, etc.
(a) Is $A + A^T$ symmetric? skew-symmetric?
(b) Is $A - A^T$ symmetric? skew-symmetric?
(c) If A is skew-symmetric, is A^T skew-symmetric?
(d) if $\det(AB) = 5$, what is $\det(BA)$?
(e) if $\det(A) = 5$, what is $\det(A^{-1})$?
(f) if $\det(A) = 5$, what is $\det(A^T)$?
(g) If A is skew-symmetric, what is $\det(A)$?
(h) if $A^2 = A$, what can $\det(A)$ be?
(i) verify that the inverse of A^T is $(A^{-1})^T$
(i) if B is similar to A, is B^T similar to A^T? explain.

5. There are theorems saying that various statements are equivalent to the matrix A being non-singular. List as many of these statements as you can. (eg. one of them is "$A\mathbf{x} = \mathbf{0}$ has no non-trivial solutions")

6. (a) Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$.
(b) Use part (a) to solve the system

$$\begin{aligned} x + 2y + 3z &= 1 \\ 2y + 3z &= -3 \\ x + 2y + 4z &= -10. \end{aligned}$$

7. For each matrix below, find a matrix of the form $\left[\begin{array}{c|c} I_r & O_{r\ n-r} \\ \hline O_{m-r\ r} & O_{m-r\ n-r} \end{array}\right]$ that is equivalent to it.

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 0 & 2 & 3 \\ 3 & 4 & 8 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & 3 \\ 3 & 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Vector spaces and subspaces

8. Give some examples of vector spaces other than R^n. List as many as you can.

9. Recall that a subset W of a vector space is a *subspace* if it is (i) closed under addition, and (ii) closed under scalar multiplication. For each of the following subsets of V, show that it is a subspace. Also, say what vector space W is a subspace of.

(a) the space of solutions to the system $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix

(b) the space of eigenvectors for the eigenvalue $\lambda = 2$ of an $m \times n$ matrix A

(c) the space of all linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in V

(d) the kernel of the linear transformation $L\colon R^n \to R^m$

(e) the span of a set S of vectors in V

(f) the range of a linear transformation $L\colon R^n \to R^m$

(g) the space of all vectors of the form $\begin{bmatrix} a-2b \\ a \\ b \end{bmatrix}$

(h) the space of all symmetric 2×2 matrices

10. Define or describe the following concepts.

(a) a linear combination of vectors

(b) the span of a set of vectors

(c) a set S spanning a subspace W (or the vector space V)

(d) a set S being linearly independent

(e) a basis of a vector space

(f) a basis of a subspace

(g) the dimension of a vector space or subspace

(h) the nullity of a matrix

(i) the row rank and column rank of a matrix

11. Determine whether $\begin{bmatrix} -3 \\ 10 \\ 2 \\ 7 \end{bmatrix}$ is in the span of $\begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \\ 3 \end{bmatrix}$.

12. For each collection of vectors, determine whether it is linearly independent.

(a) $\begin{bmatrix} 1 & 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 3 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

13. For each collection of vectors, determine whether it is a basis of R^3.

(a) $\begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

(c) $\begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\4\end{bmatrix}, \begin{bmatrix}2\\8\\5\end{bmatrix}, \begin{bmatrix}3\\1\\7\end{bmatrix}$

(d) $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}4\\3\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}$

14. In the previous problem you should have found that only collection (b) was a basis. Call it S. If $\mathbf{v} = \begin{bmatrix}2\\8\\5\end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix}1\\1\\0\end{bmatrix}$, find $[\mathbf{v}]_S$ and $[\mathbf{u}]_S$.

15. Give a basis for each of the subspaces below.

(a) the space of vectors of the form $\begin{bmatrix}a+b\\b+c\\c+d\\d\end{bmatrix}$

(b) the space of vectors of the form $\begin{bmatrix}a\\-1\\3+b\\0\\b\end{bmatrix}$

(c) the space of vectors of the form $\begin{bmatrix}a-5b+2c\\a+5b\\c\\1\end{bmatrix}$

16. Find a basis for the span of $\begin{bmatrix}1 & 0 & 1 & 3\end{bmatrix}, \begin{bmatrix}4 & 3 & 1 & 0\end{bmatrix}, \begin{bmatrix}2 & 1 & 1 & 1\end{bmatrix}, \begin{bmatrix}2 & 1 & 3 & 7\end{bmatrix}$.

17. Suppose $A = \begin{bmatrix}1 & 3 & 1 & 2 & 4\\-2 & -6 & -1 & -4 & -1\\2 & 6 & 2 & 5 & 6\\1 & 3 & -2 & 0 & -13\end{bmatrix}$.

(a) Find a basis for the null space of A.

(b) What is the nullity of A?

(c) Find a basis for the row space of A. What is the rank of A? Could you find the rank of A using only part (b)?

(d) Find a basis for the column space of A. Before computing anything, how many basis vectors will you get?

18. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

(a) If $\mathbf{v} = 3\mathbf{v}_1 - 6\mathbf{v}_2 + 43\mathbf{v}_3$, what is $[\mathbf{v}]_S$?

(b) What is $[\mathbf{w}_2]_T$?

(c) If $P_{S\leftarrow T} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix}$, what is $[\mathbf{w}_2]_S$? What is $[4\mathbf{w}_1 + 3\mathbf{w}_2 - 2\mathbf{w}_3]_S$?

(d) If $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{w}_1 = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$, $\mathbf{w}_3 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$, find $P_{S\leftarrow T}$.

LINEAR TRANSFORMATIONS

19. Define or describe the following concepts.

(a) a linear transformation

(b) the kernel of a linear transformation

(c) the range of a linear transformation

(d) L being one-to-one or onto

(e) the linear transformation $L\colon\ R^n \to R^m$ given by an $m \times n$ matrix A

20. Verify that the following functions are linear transformations.

(a) $L\colon\ M_{22} \to M_{22}$ defined by $L(A) = A^T$

(b) $L\colon\ P_4 \to P_3$ defined by $L(p(t)) = p'(t)$ (the derivative)

(c) $L\colon\ R_3 \to R_3$ given by $L(\begin{bmatrix} x & y & z \end{bmatrix}) = \begin{bmatrix} x-y & x+z & 0 \end{bmatrix}$

21. Let $L\colon\ R^3 \to R^4$ be given by the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 3 \\ 1 & -2 & -1 \\ 0 & 1 & 3 \end{bmatrix}$. Determine whether $\begin{bmatrix} -3 \\ 10 \\ 2 \\ 7 \end{bmatrix}$ is in the range of L. [Recall that for any matrix A, if $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are the columns of A, then

$$A\mathbf{x} \;=\; x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n.$$

Therefore, the range is the span of the columns of A.]

22. Let $L\colon\ P_2 \to P_2$ be the linear transformation defined by

$$L(at^2 + bt + c) = (a + 2c)t^2 + (b - c)t + (a - c).$$

Let $S = \{1, t, t^2\}$ and $T = \{t^2 - 1, t, t - 1\}$ be ordered bases for P_2.

(a) Find the matrix of L with respect to S and T.

(b) If $p(t) = 2t^2 - 3t + 1$, compute $L(p(t))$ using the matrix obtained in part (a).

23. (a) Find the following determinants using reduction to triangular form (ie. using row and column operations):

$$\begin{vmatrix} 3 & -2 \\ 5 & 4 \end{vmatrix}, \qquad \begin{vmatrix} 2 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 2 & 4 \end{vmatrix}, \qquad \begin{vmatrix} 2 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{vmatrix}$$

(b) Compute the determinants above using cofactor expansion along the second column.

(c) Compute the determinants above using cofactor expansion along the bottom row.

24. The formula for the determinant of an $n \times n$ matrix has $n!$ terms, each with a plus or minus sign. For each term below, determine whether it has a plus or minus sign.

(a) $\pm a_{11}a_{22}a_{34}a_{43}$

(b) $\pm a_{15}a_{22}a_{31}a_{43}a_{54}$

(c) $\pm a_{14}a_{23}a_{36}a_{42}a_{51}a_{65}$

Which of the following terms appear in the determinant formula for a 5×5 matrix?

(d) $\pm a_{14}a_{23}a_{33}a_{45}a_{52}$

(e) $\pm a_{13}a_{24}a_{35}a_{42}a_{51}$

(f) $\pm a_{11}a_{23}a_{35}a_{41}a_{55}$

(g) $\pm a_{34}a_{23}a_{41}a_{52}a_{15}$

(h) $\pm a_{23}a_{51}a_{42}a_{14}a_{45}$

25. (a) Find the characteristic polynomial, all eigenvalues, and their associated eigenvectors for the following matrices:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} -2 & -4 & -8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}.$$

(b) Which of the above matrices are similar to a diagonal matrix? When this occurs, find a matrix P such that $P^{-1}AP$ is diagonal.

26. For each of these matrices, say why they are diagonalizable, and give the diagonal they matrix are equivalent to.

$$\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}, \qquad \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}.$$