Linear Algebra
Homework – 3

1.5

#22 Show that if A is any $n \times n$ matrix, then

(a) $(A+A^T)$ is symmetric.

$(A+A^T)$ is symmetric if
$$(A+A^T)^T = (A+A^T).$$

So, $(A+A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$!

Thus, $A+A^T$ is symmetric.

(b) $(A-A^T)$ is skew-symmetric.

$(A-A^T)$ is skew-symmetric if,
$$(A-A^T)^T = A^T - A$$

$$(A-A^T)^T = A^T - (A^T)^T = A^T - A.$$

Thus, $(A-A^T)$ is skew symmetric.

#34 If A is a non-singular matrix, then,
$A \cdot A^{-1} = I_n$.

Here $A^{-1} = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$

So, $A \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

So, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So, we get

$2a + 4b = 1$ $\qquad$ $2c + 4d = 0$

$a + b = 0$ $\qquad$ $c + d = 1$

On solving, we get

$a = -1/2$ $\qquad$ $c = 2$

$b = 1/2$ $\qquad$ $d = -1$

So, $A = \begin{bmatrix} -1/2 & 1/2 \\ 2 & -1 \end{bmatrix}$.

#45. Theorem 1.7 If A is a non singular matrix, then A^{-1} is non-singular and $(A^{-1})^{-1} = A$.

Proof: If A is a non-singular matrix, then there exists a matrix B such that

$$A \cdot B = B \cdot A = I_n$$

here $B = A^{-1}$.

So, from the definition, we can say that for the matrix B, there is always the matrix A, which is its inverse.

So, $$A \cdot A^{-1} = A^{-1} A = I_n . \quad - (1)$$

$\Rightarrow A^{-1}$ is also a non-singular matrix.

Since A^{-1} is a non-singular matrix, we can write

$$(A^{-1}) \cdot (A^{-1})^{-1} = (A^{-1})^{-1} \cdot A^{-1} = I_n \quad - (2)$$

from (1) & (2)

$$A^{-1} \cdot A = A^{-1} (A^{-1})^{-1}$$

Multiplying both sides by A, we get

$$(A \cdot A^{-1}) \cdot A = (A \cdot A^{-1}) \cdot (A^{-1})^{-1}$$

or, $$I_n \cdot A = I_n (A^{-1})^{-1}$$

$$\Rightarrow A = (A^{-1})^{-1}.$$

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#10 $w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

Given $f(x) = Ax$, where $f : \mathbb{R}^2 \to \mathbb{R}^3$

Need to check if w is in the range of f.

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$

so, $f(x) = w = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 0x_1 + x_2 \\ x_1 + x_2 \end{bmatrix}$

$\Rightarrow w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$ $x_2 = 1$ does not satisfy.

From here, we get $x_2 = 1$. but we get different values for x_1 from the other two equations. So w is not in the range of f.

#16. Let $u = \begin{bmatrix} x \\ y \end{bmatrix}$. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$f(u) = Au$

a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

This is reflection about the line $y = x$.

b) $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$ Reflection about the line $y = -x$.