1(a) Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ an $n \times p$ matrix. Suppose $a_{3j} = 0$ for all j.

(i) Describe informally what this says about A.

(ii) What can you say about the matrix AB?

(i) the third row of A is all zeros

(ii) the third row of AB is also all zeros.

The $3j$ entry of $A\underline{B}$ is

$\sum_{k=1}^{n} a_{3k} b_{kj}$ which is 0 because each a_{3k} is 0.

1(b) If A is an $n \times n$ matrix, what are the entries on the main diagonal of $A - A^T$? Explain.

The diagonal entries of A are $a_{11}, a_{22}, \dots, a_{nn}$.

The ij-entry of A^T is a_{ji}, so the ii-entry of A^T is a_{ii}.

The ii entry of $A - A^T$ is $a_{ii} - a_{ii} = 0$.

So the diagonal entries of $A - A^T$ are all 0.

2. Let $A = \begin{bmatrix} 4 & -8 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 9 \end{bmatrix}$. Find a matrix B in reduced row echelon form that is row equivalent to A. Use the standard notation to record the sequence of row operations that are used.

$$B = A_{\frac{1}{4}r_1 \to r_1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$

$$C = B_{2r_2 + r_1 \to r_1} = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$

$$D = C_{\frac{1}{9}r_3 \to r_3} = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E = D_{-6r_3 + r_1 \to r_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F = E_{-3r_3 + r_2 \to r_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3(a) Define what it means for an $n \times n$ matrix A to be *skew-symmetric.*

(b) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is skew-symmetric, what can you say about the entries a, b, c, d? What is the most general form of A?

(c) Show that the product of two 2×2 skew-symmetric matrices is diagonal. [Hint: use (b).]

(a) A is skew-symmetric if $A^T = -A$.
(equivalently, if $a_{ji} = -a_{ij}$ for all i, j)

(b) $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $-A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$ so $a = -a$, $c = -b$, $b = -c$, $d = -d$. That is, $a = 0$, $b = -c$, $d = 0$.

General form: $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$

(c)

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = \begin{bmatrix} -ab & 0 \\ 0 & -ab \end{bmatrix}.$$

two arbitrary skew-symmetric matrices

diagonal

4. Let $f(\mathbf{u}) = A\mathbf{u}$ where

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Determine whether $\mathbf{w} = \begin{bmatrix} -2 \\ 8 \\ -1 \end{bmatrix}$ is in the range of A. Note, $f: R^2 \to R^3$.

This is asking whether there is a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 8 \\ -1 \end{bmatrix}$.

This would mean

$$\begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ -1 \end{bmatrix}, \text{ i.e. } \begin{bmatrix} x \\ -x+2y \\ 2x+y \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ -1 \end{bmatrix}.$$

we have

$$\left.\begin{aligned} x &= -2 \\ -x+2y &= 8 \\ 2x+y &= -1 \end{aligned}\right\} \quad \begin{aligned} &\to \underline{x = -2} \\ &\to 2+2y = 8 \to 2y = 6 \\ &\qquad\qquad\qquad \underline{y = 3} \end{aligned}$$

Check third equation: $2(-2) + (3) = -4+3 = -1$ yes.

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ works:

$$f\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 8 \\ -1 \end{bmatrix}.$$

So $\underline{w}$ is in the range of f.

5. After some row operations, the augmented matrix of the linear system $A\mathbf{x} = \mathbf{b}$ is

$$[C \mid \mathbf{d}] = \left[\begin{array}{cccc|c} 1 & -2 & 4 & 5 & -6 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(a) Is C in reduced row echelon form? Explain.

(b) How many solutions are there to the system $A\mathbf{x} = \mathbf{b}$?

(c) Is A non-singular? Explain.

(d) Determine all possible solutions to the system $A\mathbf{x} = \mathbf{b}$.

(a) It is not, because the second leading 1 does not have zero above it

(b) There are infinitely many - see part (d).

(c) If A were non-singular then $A\underline{x} = \underline{b}$ would have exactly one solution. Then, $C\underline{x} = \underline{d}$ would also have exactly one solution, since these systems are equivalent. Since there are infinitely many solutions to $C\underline{x} = \underline{d}$, A cannot be non-singular.

(d) Columns 2 and 4 have no leading 1s, so variables x_2 and x_4 will be free.

$$x_1 - 2x_2 + 4x_3 + 5x_4 = -6$$

$$x_3 + 3x_4 = 0 \quad \rightarrow \quad \underline{x_3 = -3x_4}$$

$$\rightarrow x_1 = -6 + 2x_2 - 4(-3x_4) - 5x_4$$

$$\underline{x_1 = -6 + 2x_2 + 7x_4}$$

$\underline{x_2 = \text{anything}}$

$\underline{x_4 = \text{anything}}$