1(a) Explain why, if the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent, then so is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$.

The second set cannot be linearly dependent, since that would mean

$a_1 \underline{v}_1 + a_2 \underline{v}_2 + a_4 \underline{v}_4 = \underline{0}$ for some

a_1, a_2, a_4 not all zero. Then

$a_1 \underline{v}_1 + a_2 \underline{v}_2 + 0 \underline{v}_3 + a_4 \underline{v}_4 + 0 \underline{v}_5 = \underline{0}$ with

not all coefficients zero, but this does not happen since $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4, \underline{v}_5\}$ is linearly independent.

1(b) True or false (please circle one for each item):

T **F** The set of all linear combinations of $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}$ is a subspace of R^3.

T **F** The set $S = \left\{ \begin{bmatrix}2\\0\\4\\8\end{bmatrix}, \begin{bmatrix}3\\0\\9\\27\end{bmatrix} \right\}$ is linearly dependent.

neither is a multiple of the other

T **F** The set $S = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix} \right\}$ is linearly dependent.

it includes the zero vector

2. Let V be the vector space P_5 and let W be the set of all polynomials of the form $p(t) = at^3 + bt^2 + c$. Show that W is a subspace of V.

W is closed under addition:

If $p(t) = at^3 + bt^2 + c$ and $q(t) = a't^3 + b't^2 + c'$ are in W then
$p(t) + q(t) = (a + a')t^3 + (b + b')t^2 + (c + c')$
and this is in W.

Closed under scalar multiplication:

If $p(t) = at^3 + bt^2 + c$ is in W and k is a real number then $k \cdot p(t) = (ka)t^3 + (kb)t^2 + (kc)$.
this is in W.

3(a) Compute the product of the following elementary matrices:

$$\underset{E_4}{\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}}\underset{E_3}{\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}}\underset{E_2}{\begin{bmatrix}1&-1&0\\0&1&0\\0&0&1\end{bmatrix}}\underset{E_1}{\begin{bmatrix}1&0&0\\0&1&4\\0&0&1\end{bmatrix}}$$

(b) Let A be the matrix you just found. Give the sequence of row operations needed to convert I_3 into A.

(c) Give the sequence of column operations needed to convert I_3 into A.

(a)

1+'s

$$\begin{bmatrix}1&0&0\\0&0&1\\2&1&0\end{bmatrix}\begin{bmatrix}1&-1&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&4\\0&0&1\end{bmatrix}$$

$$= \begin{bmatrix}1&-1&0\\0&0&1\\2&-1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&4\\0&0&1\end{bmatrix} = \boxed{\begin{bmatrix}1&-1&-4\\0&0&1\\2&-1&-4\end{bmatrix} = A}$$

(b) $A = E_4E_3E_2E_1I_4$ (E_i's = matrices above)

so
$$\boxed{\begin{aligned}A = (I_4)\ & 4r_3+r_2 \to r_2\\ & -r_2+r_1 \to r_1\\ & r_2 \leftrightarrow r_3\\ & 2r_1+r_3 \to r_3\end{aligned}}$$

(c) $A = I_4E_4E_3E_2E_1$ so

$$\boxed{\begin{aligned}A = (I_4)\ & 2c_3+c_1 \to c_1\\ & c_2 \leftrightarrow c_3\\ & -c_1+c_2 \to c_2\\ & 4c_2+c_3 \to c_3\end{aligned}}$$

4. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

or determine that it has no inverse.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 3 & -1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ -1 & 3 & -1 \end{bmatrix}}.$$

<u>check</u>: $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ -1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

5. Let $S = \left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$.

(a) Show that S is linearly dependent.

(b) Write one of the vectors in S as a linear combination of the other two.

(c) Name two vectors in S which span the same subspace of R^2 as S, and say briefly why this is true.

(a) $$a_1\begin{bmatrix}1\\2\end{bmatrix} + a_2\begin{bmatrix}1\\-1\end{bmatrix} + a_3\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$a_1 + a_2 = 0$$
$$2a_1 - a_2 + a_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1/3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ 0 & 1 & -1/3 & 0 \end{array}\right]$$

$$a_1 = -\tfrac{1}{3}a_3$$
$$a_2 = \tfrac{1}{3}a_3$$

take $a_3 = 1$ get $a_1 = -\frac{1}{3}$ $a_2 = \frac{1}{3}$ $a_3 = 1$

$$-\frac{1}{3}\begin{bmatrix}1\\2\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1\\-1\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

coeffs not all 0 $\Rightarrow$ linearly dependent.

(b) $$\boxed{\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\2\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\-1\end{bmatrix}}.$$

(c) $\boxed{\text{the set } \left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix} \right\}}$ spans the same subspace since $\begin{bmatrix}0\\1\end{bmatrix}$ can be written in terms of them.