1(a) Consider the ordered bases $S = \{t+1, t-2\}$ and $T = \{t-5, t-2\}$ for the vector space P_1.

(i) If $[\mathbf{v}]_T = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, what polynomial is $\mathbf{v}$?

(ii) Determine $[\mathbf{v}]_S$.

(i) $\underline{v}$ is $-1(t-5)+3(t-2) = \boxed{2t-1}$

(ii) $2t-1 = a_1(t+1) + a_2(t-2)$

$$a_1 + a_2 = 2 \qquad a_1 - 2a_2 = -1$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -2 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -3 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] \quad a_1 = 1,\ a_2 = 1$$

so $\boxed{[\underline{v}]_S = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$.

(check: $1\cdot(t+1) + 1\cdot(t-2) = 2t-1$ ✓)

1(b) True or false (please circle one for each item):

T **F** If the $n \times n$ matrix A is singular, then rank $A \leq n-1$.

T **F** The only matrix similar to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is itself. $P^{-1} I_4 P = P^{-1}P = I_4$

T **F** For *every* basis S of R^3, $[\mathbf{0}]_S = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

2. Consider the homogeneous system $A\mathbf{x} = \mathbf{0}$ where

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 5 \\ 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 2 & 6 & 2 \end{bmatrix}.$$

(a) Find all solutions to the system.

(b) Write down a basis for the space of solutions.

(c) What is the nullity of A?

(d) What is the rank of A?

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 5 & 0 \\ 1 & 2 & 2 & 4 & 6 & 0 \\ 0 & 0 & 2 & 6 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 5 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 5 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -2 & 4 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = a \qquad x_4 = b \qquad x_5 = c$

then solution is

$$\boxed{\underline{x} = \begin{pmatrix} -4c + 2b - 2a \\ a \\ -c - 3b \\ b \\ c \end{pmatrix} \quad \text{for all } a, b, c,}$$

$$\boxed{\text{basis : } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

$\boxed{\text{nullity} = 3}$ (dim. of solution space)

$\boxed{\text{rank} = 2}$ (5 − 3, or # non-zero rows)

3. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be ordered bases for R^3, where

$$\mathbf{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \mathbf{w}_1 = \begin{bmatrix}2\\1\\2\end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix}0\\3\\0\end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix}1\\2\\3\end{bmatrix}.$$

(a) Compute the change of basis matrix $P_{S \leftarrow T}$.

(b) Let $\mathbf{u} = 3\mathbf{w}_1 - 5\mathbf{w}_2 + 2\mathbf{w}_3$. Find $[\mathbf{u}]_S$.

(c) Express $\mathbf{w}_1$ as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

(a)
$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 & 3 & 2 \\ 0 & 1 & 0 & 2 & 0 & 3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & 3 \\ 1 & 0 & 0 & 1 & 3 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 & 0 & 3 \\ 0 & -1 & -1 & -1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 1 & -3 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 & -3 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\boxed{\begin{matrix} 1 & 3 & 2 \\ 2 & 0 & 3 \\ -1 & -3 & -4 \end{matrix}}}_{P_{S \leftarrow T}} \\ 0 & 0 & 1 & \end{array}\right]$$

(b) $[u]_T = \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}$ so

$$[u]_S = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 3 \\ -1 & -3 & -4 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} -8 \\ 12 \\ 4 \end{bmatrix}}$$

(c) $[w_1]_T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ so $[w_1]_S = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 3 \\ -1 & -3 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$

Hence $\boxed{w_1 = v_1 + 2v_2 - v_3}$

4. Let $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2\}$ be ordered bases for R^3 and R^2, where

$$\mathbf{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}, \quad \mathbf{w}_1 = \begin{bmatrix}1\\2\end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix}1\\3\end{bmatrix}.$$

Let $L\colon R^3 \to R^2$ be the linear transformation which takes $\begin{bmatrix}x\\y\\z\end{bmatrix}$ to $\begin{bmatrix}x-y\\y-z\end{bmatrix}$. Find the matrix representing L with respect to the ordered bases S and T.

$$L(e_1) = \begin{bmatrix}1\\0\end{bmatrix}, \quad L(e_2) = \begin{bmatrix}-1\\1\end{bmatrix}, \quad L(e_3) = \begin{bmatrix}0\\-1\end{bmatrix}.$$

$$\left[\begin{array}{cc|c|c|c} 1 & 1 & 1 & -1 & 0 \\ 2 & 3 & 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{cc|c|c|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 3 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c|c|c} 1 & 0 & 3 & -4 & 1 \\ 0 & 1 & -2 & 3 & -1 \end{array}\right]$$

$$[L(e_1)]_T \quad [L(e_2)]_T \quad [L(e_3)]_T$$

So the matrix is

$$\begin{bmatrix} | & | & | \\ [L(e_1)]_T & [L(e_2)]_T & [L(e_3)]_T \\ | & | & | \end{bmatrix} = \boxed{\begin{bmatrix} 3 & -4 & 1 \\ -2 & 3 & -1 \end{bmatrix}}$$

5. NOTE: this problem should not require calculations.
Let $L\colon P_3 \to P_3$ be the linear transformation given by $L(at^3+bt^2+ct+d) = (a-b)t^3 + (c-d)t$.

(a) Name some vectors in P_3 that are in the the range of L.

(b) Name some vectors in P_3 that are in kernel of L.

(c) Give bases for both the kernel and the range of L. How can you be sure these are bases?

(d) What are the rank and nullity of L?

(a) $t,\ t^3,\ t^3+t,\ 3t^3-2t,\ \ldots$

(b) $t^3+t^2,\ t+1,\ 5t^3+5t^2+t+1,\ 3t^3+3t^2-11t-11,\ \ldots$

(c) $\boxed{\{t^3, t\}}$ is a basis for the range.

- it's lin. indep, so far

$\boxed{\{t^3+t^2,\ t+1\}}$ is a basis for the kernel.

- it's lin. indep (not multiples of each other)

Since rank + nullity $= 4$ and rank, nullity are both ≥ 2 they must both equal 2. Hence the sets above span the range and kernel.

(d) $\boxed{\begin{array}{l}\text{rank } L = 2 \\ \text{nullity } L = 2\end{array}}$ $\begin{array}{l}(= \dim \text{range } L) \\ (= \dim \text{kernel } L)\end{array}$