1(a) If $X$ and $Y$ are topological spaces, let $\mathcal{C}(X,Y)$ be the set of continuous maps from $X$ to $Y$. Give the definition of the compact-open topology on $\mathcal{C}(X,Y)$ by giving a subbasis.

(b) Now suppose that $X$ is locally compact and Hausdorff, and give $\mathcal{C}(X,Y)$ the compact-open topology. Define the evaluation map $e: X \times \mathcal{C}(X,Y) \to Y$ by the formula $e(x,f) = f(x)$. Prove that this map is continuous.

[Hint: recall that $X$ locally compact implies that for every open neighborhood $U$ of $x$, there is an open neighborhood $U$ with $\overline{U} \subset W$ and $\overline{U}$ compact.]

**Solution.**

(A) For each compact set $C \subset X$ and open set $U \subset Y$ let

$$S(C,U) = \{ f \in \mathcal{C}(X,Y) \mid f(C) \subset U \}.$$ 

The set of all such sets $S(C,U)$ is a subbasis for the compact-open topology on $\mathcal{C}(X,Y)$.

(b) Let $(x,f)$ be a point in $X \times \mathcal{C}(X,Y)$ and $V \subset Y$ a neighborhood of $e(x,f)$. We need to find a neighborhood of $(x,f)$ that is mapped by $e$ into $V$.

Since $f$ is continuous and $f(x) \in V$, the set $W = f^{-1}(V)$ is an open neighborhood of $x$ in $X$. By local compactness and the Hausdorff property, there is a neighborhood $U$ of $x$ such that $\overline{U} \subset W$ and $\overline{U}$ is compact. Now $U \times S(\overline{U},V)$ is an open neighborhood of $(x,f)$ since $f(\overline{U}) \subset V$. Also, any $(x',f') \in U \times S(\overline{U},V)$ is mapped to $V$ by $e$, since $x' \in U$ and $f'$ takes $U$ into $V$.

2(a) Define what it means for $r: X \to A$ to be a retraction, where $A$ is a subspace of $X$.

(b) Let $i: A \to X$ be inclusion and let $r: X \to A$ be a retraction, and pick a basepoint $a_0 \in A$. Show that the induced homomorphism $i_*: \pi_1(A,a_0) \to \pi_1(X,a_0)$ is injective.

(c) Show that there is no retraction of the “solid torus” $S^1 \times D^2$ to the boundary torus $S^1 \times S^1$.

**Solution.**

(A) $r$ is a retraction if $r$ is continuous and $r(a) = a$ for all $a \in A$.

(b) The retraction property means that $r \circ i$ is the identity map on $A$. Hence, $(r \circ i)_* = (i \circ r)_* = (id)_* = id$, the identity homomorphism on $\pi_1(A,a_0)$. In particular, this composition is a bijection, and it follows that $i_*$ is injective and $r_*$ is surjective.

(c) We know that $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(D^2) \cong 1$, and therefore $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(S^1 \times D^2) \cong \mathbb{Z} \times 1 \cong \mathbb{Z}$. By part (b), if there is a retraction then there will be a corresponding injective homomorphism $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. However, this is impossible. For instance, if $(1,0)$ and $(0,1)$ map to $m$ and $n$ respectively, then $(n,0)$ and $(0,m)$ both map to $mn$.

3. Let $p: E \to B$ be a covering map. Choose $e_0 \in E$ and $b_0 \in B$ such that $p(e_0) = b_0$.

(a) Define the lifting correspondence $\Phi: \pi_1(B,b_0) \to p^{-1}(b_0)$.
(b) Show that \( \Phi \) is surjective, and that \( \Phi \) is injective if \( E \) is simply connected. State carefully any results that you use.

**Solution.** Note, we should also assume that \( E \) is path connected.

(A) For each \([f] \in \pi_1(B, b_0)\) let \( \tilde{f} \) be the unique lift of \( f \) starting at \( e_0 \). Then \( \Phi([f]) = \tilde{f}(1) \). This is well defined because if \([f] = [g] \) then any path homotopy from \( f \) to \( g \) lifts to a path homotopy from \( \tilde{f} \) to \( \tilde{g} \), showing that \( \tilde{f} \) and \( \tilde{g} \) have the same endpoints in \( p^{-1}(b_0) \).

(b) For surjectivity, suppose \( e_1 \in p^{-1}(b_0) \). Since \( E \) is path connected, there is a path \( \tilde{f} \) from \( e_0 \) to \( e_1 \). Then \( f \) is a lift of the loop \( f = p \circ \tilde{f} \), and it starts at \( e_0 \), and therefore \( e_1 = \tilde{f}(1) = \Phi([f]) \).

For injectivity, suppose \( \Phi([f]) = \Phi([g]) = e_1 \in p^{-1}(b_0) \). The lifts \( \tilde{f} \) and \( \tilde{g} \) (of \( f \) and \( g \) respectively, starting at \( e_0 \)) are paths in \( E \) from \( e_0 \) to \( e_1 \). Since \( E \) is simply connected, there is a path homotopy \( F \) from \( \tilde{f} \) to \( \tilde{g} \). Then \( p \circ F \) is a path homotopy from \( f \) to \( g \), and therefore \([f] = [g] \).

4. Let \( h: I \to X \) be a path from \( x_0 \) to \( x_1 \).

(a) Give the definition of the change-of-basepoint homomorphism \( \beta_h: \pi_1(X, x_1) \to \pi_1(X, x_0) \). [Or, in Munkres notation, the homomorphism \( \tilde{h}: \pi_1(X, x_0) \to \pi_1(X, x_1) \).]

(b) Prove that \( \beta_h \) (or \( \tilde{h} \)) is a homomorphism, and an isomorphism.

**Solution.** (the For Munkres version see Theorem 52.1.)

(A) We define \( \beta_h([f]) = [h \cdot f \cdot \tilde{h}] \). This is well defined because \([h \cdot f \cdot \tilde{h}] = [h][f][\tilde{h}] \) and multiplication of path homotopy classes is well defined.

(b) First, \( \beta_h([f]) \beta_h([g]) = [h \cdot f \cdot \tilde{h}][h \cdot g \cdot \tilde{h}] = [h][f][\tilde{h}][h][g][\tilde{h}] = [h][f][g][\tilde{h}] = [h \cdot f \cdot g \cdot \tilde{h}] = \beta_h([f \cdot g]) = \beta_h([f])[g] \), and so \( \beta_h \) is a homomorphism. Next we claim that \( \beta_h^{-1} \) and \( \beta_h \) are inverse homomorphisms (and therefore are isomorphisms). We verify: \( \beta_h(\beta_h^{-1}([f])) = [h \cdot \tilde{h} \cdot f \cdot h \cdot \tilde{h}] = [f] \) and \( \beta_h^{-1}(\beta_h([f])) = [\tilde{h} \cdot h \cdot f \cdot \tilde{h} \cdot h] = [f] \).

5. Let \( p: E \to B \) be a covering map with \( B \) connected. Show that if \( p^{-1}(b_0) \) has \( k \) elements for some \( b_0 \in B \) then \( p^{-1}(b) \) has \( k \) elements for every \( b \in B \).

**Solution.**

Define sets \( A \subset B \) and \( C \subset B \) as follows: \( A = \{ b \in B \mid |p^{-1}(b)| = k \} \) and \( C = \{ b \in B \mid |p^{-1}(b)| \neq k \} \). Clearly \( A \cap C = \emptyset \) and \( A \cup C = B \). We claim that \( A \) and \( B \) are both open sets.

If \( b \in A \) then there is an evenly covered neighborhood \( U \) of \( b \). Then \( p^{-1}(U) \cong V_1 \cup \cdots \cup V_k \), with each slice \( V_i \) mapping by \( p \) homeomorphically onto \( U \). There are \( k \) slices, because \( p^{-1}(b) \) has \( k \) elements, and it contains one element of each slice. The same is true of any \( b' \in U \), and therefore \( U \subset A \).

If \( b \in C \) then an evenly covered neighborhood \( U \) of \( b \) has preimage equal to the union of a collection of slices, having the same cardinality as \( p^{-1}(b) \). Every fiber \( p^{-1}(b') \) has this same cardinality, which is not \( k \). Hence, \( U \subset C \).

Thus, \( A \) and \( C \) are open sets. Since \( A \) contains \( b_0 \) and \( B \) is connected, it must be the case that \( C = \emptyset \), and \( B = A \).