FINITE-INDEX ACCESSIBILITY OF GROUPS

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ABSTRACT. We prove an accessibility theorem for finite-index splittings of groups. Given a finitely presented group G there is a number $n(G)$ such that, for every reduced locally finite G–tree T with finitely generated stabilizers, T/G has at most $n(G)$ vertices and edges. We also show that deformation spaces of locally finite trees (with finitely generated stabilizers) are maximal in the partial ordering of domination of G–trees.

1. INTRODUCTION

Given a group G, one may wish to understand the various ways in which G decomposes as the fundamental group of a graph of groups. For simplicity, we shall call a graph of groups decomposition a *splitting* of G. There are several theorems in the literature, known as *accessibility* theorems, which state that all splittings of G of a particular kind have underlying graphs of bounded complexity. That is, there is a bound on the total number of vertices and edges.

An early result of this kind concerns free products. By Grushko's theorem, rank is additive under free products. It follows easily that for a finitely generated group G, there is a bound on the complexity of any reduced splitting of G having trivial edge groups.

Dunwoody's accessibility theorem [Dun85] provides a complexity bound on all reduced splittings of G having finite edge groups, if G is finitely presented. A fundamental structure theorem then follows from this result: every such group admits a decomposition whose edge groups are finite and whose vertex groups each have at most one end. Dunwoody accessibility does not hold for all finitely generated groups [Dun93], though Linnell had shown earlier that it does if one assumes a bound on the orders of the finite subgroups of G [Lin83].

Bestvina and Feighn [BF91a, BF91b] generalized Dunwoody's theorem by bounding the complexity of all reduced splittings of G having small edge groups. Here, a group is *small* if it does not admit an irreducible action on a simplicial tree. (Note that every group with no non-abelian free subgroup is small.) Bestvina and Feighn also gave examples showing that once one allows the free group of rank 2 as an edge group, then no such bounds can exist.

Sela's acylindrical accessibility theorem [Sel97] generalizes the free product case in a different way. A graph of groups is called *k–acylindrical* if its associated Bass-Serre tree has the property that the stabilizer of every segment of length greater than k is trivial. Sela proved that for every finitely generated group G and $k \geqslant 1$, there is a bound on the complexity of all reduced k–acylindrical splittings of G.

Sela's result has been generalized to (k, C)–acylindrical splittings of groups by Delzant and Weidmann [Del99, Wei12], where a splitting is called (k, C)*–acylindrical* if every segment of length greater than k has stabilizer of cardinality at most C.

Finally there is a notion of *strong accessibility*, which concerns finiteness of hierarchies of splittings of a given type. See Louder and Touikan [LT17] for results on hierarchies of splittings over slender groups of finitely presented groups.

In the current paper we prove an accessibility theorem for *finite-index splitting*s of groups. These are graph of groups decompositions in which every edge group includes as a finite-index subgroup of its neighboring vertex groups. The Bass-Serre trees of such splittings are exactly the G–trees that are locally finite.[1]

Each of the earlier results has a hypothesis limiting the edge groups or segment stabilizers in size or complexity. In our theorem the limitation works in the other direction, with edge groups being as large as possible relative to the vertex groups.

Recall from [Sta68] that a group G is *almost finitely presented* if there is a connected simplicial complex K with $H^1(K; \mathbb{Z}_2) = 0$ on which G acts freely and cocompactly. Every finitely presented group is almost finitely presented, but not conversely (see [BS80] for a counterexample).

Theorem 1.1. *Let G be an almost finitely presented group. Then there is an integer $n(G)$ such that the following holds: if T is a reduced locally finite G–tree with finitely generated stabilizers, then T/G has at most $n(G)$ vertices and edges.*

Note that in any locally finite G–tree, all vertex and edge stabilizers are commensurable. Hence either they are all finitely generated, or none of them are.

Remark 1.2. The assumption of finitely generated stabilizers cannot be dropped. At the end of the paper we give examples of finite-index splittings of the free group F_2 of arbitrarily large complexity. The vertex and edge groups are all free of infinite rank.

Our main technical result leading to Theorem 1.1 is a statement about deformation spaces of locally finite trees. Let X and Y be cocompact G–trees. We say that X *dominates* Y if there is a simplicial G–map $X \to Y$ (after possibly subdividing X). This occurs if and only if every elliptic subgroup for X is elliptic for Y. The trees are in the same *deformation space* if they dominate each other, or equivalently, have the same elliptic subgroups. There is then an induced partial ordering of domination of deformation spaces. The next result says that deformation spaces of non-trivial locally finite trees (with finitely generated stabilizers) are maximal in this partial ordering.

[1] Here we are only considering graphs of groups with finite underlying graphs.

Theorem 1.3. *Let Y be a non-trivial locally finite G–tree with finitely generated stabilizers. If $f\colon X \to Y$ is any surjective simplicial map of G–trees, with X cocompact, then X and Y are in the same deformation space.*

If X is minimal then this implies that it must also be locally finite. Like Theorem 1.1, this result fails if one does not assume Y to have finitely generated stabilizers. Consider any HNN extension $G = A*_C$ whose Bass–Serre tree X is not a line. Let $H \subset G$ be the subgroup generated by all conjugates of A. This subgroup is not finitely generated and is not elliptic for X. However, there is a morphism $X \to Y$ onto a linear G–tree whose vertex and edge stabilizers are all H. These trees are in different deformation spaces.

The assumption of non-triviality is also essential, as every G–tree admits a simplicial G–map to a point.

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2. Preliminaries

For more detail on the background material given here, see [Bas93, For02, BF91a]. Our convention for graphs and trees is that every edge e is oriented, with initial vertex $\partial_0(e)$ and terminal vertex $\partial_1(e)$. Then $\overline{e}$ denotes the same geometric edge with the opposite orientation, so that $\partial_0(\overline{e}) = \partial_1(e)$ and $\partial_1(\overline{e}) = \partial_0(e)$.

By a *G–tree* we mean a simplicial tree together with a G–action by simplicial automorphisms, with no inversions. A G–tree is *trivial* it it has a global fixed point; otherwise, it is *non-trivial*. Let X be a G–tree. An element $g \in G$ is *elliptic* if it fixes a vertex and *hyperbolic* otherwise. If g is hyperbolic, then there is a g–invariant line $X_g \subset X$ called the *axis* of g on which g acts as a translation.

An *end* of X is an equivalence class of rays, where two rays are considered equivalent if their intersection is a ray. The space of ends of X is denoted by ∂X. A subtree *contains* the end $\varepsilon \in \partial X$ if it contains a ray representing ε.

Lemma 2.1 ([Tit70], (3.4))**.** *Let X be a G–tree and $H \subset G$ a subgroup such that every element of H is elliptic. Then either:*

1. *H fixes a vertex of X, or*
2. *H has no fixed vertex, but there is a unique end $\varepsilon_H \in \partial X$ fixed by H.*

In case (1), we say that H is *elliptic*. In case (2), we say that H is *elliptic at infinity*.

If X is a G–tree, a *minimal subtree* is a subtree that is G–invariant and does not properly contain any smaller G–invariant subtree. If G is elliptic, then any vertex fixed by G is a minimal subtree. If G contains a hyperbolic element, then a minimal subtree X_G exists and is unique: it is equal to the union of the axes of hyperbolic elements. The third possibility is that G is elliptic at infinity; then there is no minimal subtree. Instead, there is a descending chain of G–invariant subtrees, with empty intersection, all containing the fixed and ε_G (for instance, take a sequence of horoballs at ε_G).

Lemma 2.2. *Let X' be a G–invariant subtree of the G–tree X.*

(1) *X and X' have the same elliptic subgroups.*
(2) *If G is elliptic at infinity for X then X' contains the fixed end ε_G.*
(3) *If G has a hyperbolic element then X' contains the minimal subtree X_G.*

Proof. First note that X and X' have the same elliptic and hyperbolic elements; each g has either a fixed vertex or an axis in X', and then has the same in X. Item (3) follows immediately since X' must contain every axis. Next, a subgroup cannot fix both a vertex of X and an end of X' without also fixing a vertex of X', so (1) holds. Finally, if G is elliptic at infinity for X, it is also elliptic at infinity for X', by (1). So G fixes a unique end ε of X'. Uniqueness of ε_G for X implies that $\varepsilon_G = \varepsilon$. ☐

Recall that two subgroups H, H' of a group G are *commensurable* if $H \cap H'$ has finite index in both H and H'. In any locally finite G–tree, stabilizers of vertices and edges are all commensurable to each other.

Lemma 2.3. *Let X be a G–tree and let H, H' be commensurable subgroups of G.*

(1) *If H is elliptic then so is H'.*
(2) *If H is elliptic at infinity, then so is H' and $\varepsilon_H = \varepsilon_{H'}$.*
(3) *If H contains a hyperbolic element, then so does H' and $X_H = X_{H'}$.*

Proof. Item (1) is immediate from Example 6.3.4 of [Ser80]. Item (3) is Corollary 7.7 of [Bas93]. For (2), H' being elliptic at infinity follows from (1) and (3). Further, if $\varepsilon_H \neq \varepsilon_{H'}$, then $H \cap H'$ fixes pointwise the geodesic line joining these two ends. But then $H \cap H'$ is elliptic, contradicting (1). ☐

Definition 2.4 (Collapse maps)**.** Let $f \colon X \to Y$ be an equivariant simplicial map of G–trees. It is called a *collapse map* if it is surjective and $f^{-1}(v)$ is connected for every vertex $v \in V(Y)$. In terms of graphs of groups, this means that Y/G is obtained from X/G by collapsing the components of a subgraph of X/G to vertices. The vertex groups associated to these new vertices are the fundamental groups of the collapsed subgraphs of groups.

Definition 2.5 (Morphisms)**.** An equivariant map $f \colon X \to Y$ is a *morphism* if it takes vertices to vertices and edges to edges (and respects the relation of incidence between vertices and edges). Equivalently, it is a simplicial map such that no edge is collapsed to a vertex.

Definition 2.6 (Elementary moves)**.** Let X be a G–tree and e an edge of X with endpoints in distinct G–orbits, such that $G_{\partial_0(e)} = G_e$. We obtain a new G–tree by collapsing each edge in the orbit of e to a vertex. This operation is called an *elementary collapse* move. It is a special case of a collapse map from X to the resulting tree. What makes it different from a generic collapse map is that the elliptic subgroups do not change. The reverse move is called an *elementary expansion*.

Definition 2.7 (Deformations)**.** An *elementary deformation* between G–trees X and Y is a finite sequence of elementary collapse and expansion moves taking X to Y. When this occurs, we say that they are in the same *deformation space*. Elementary deformations preserve the elliptic subgroups of G, and moreover, cocompact G–trees are in the same deformation space if and only if they define the same elliptic subgroups [For02, Theorem 1.1].

Definition 2.8 (Reduced trees)**.** A G–tree is *reduced* if it admits no elementary collapse moves. Any cocompact G–tree can be made reduced by a sequence of such moves. Note that reduced trees are minimal; one easily verifies that any edge outside of a G–invariant subtree admits an elementary collapse.

Warning: our definition of "reduced" is not the same as the one used in [BF91a]. However, every reduced tree is also reduced in the sense of [BF91a]. Theorem 1.1 will hold both for reduced trees and for "BF-reduced" trees; see Remark 3.3.

Definition 2.9 (Folds)**.** Let X be a G–tree and suppose e, e' are edges with $\partial_0(e) = \partial_0(e') = v$. Let $u = \partial_1(e)$ and $u' = \partial_1(e')$. Define an equivalence relation $\sim$ on X to be the smallest equivalence relation satisfying:

- $u \sim u'$, $e \sim e'$, and $\overline{e} \sim \overline{e}'$
- $g(x) \sim g(y)$ whenever $x \sim y$ and $g \in G$.

The resulting quotient graph is a simplicial tree with a G–action. There could be inversions, in which case we subdivide the inverted edge orbit to obtain a G–tree. The map $X \to X/{\sim}$ is called a *fold*.

In [BF91a] folds are organized into seven types, according to how e, e', v, u, and u' are related to each other by the group action. Three of the types are called "type A" (types IA, IIA, IIIA). The fold is of type A if and only if $v \notin Gu \cup Gu'$.

Lemma 2.10. *Let X and Y be G–trees in the same deformation space and suppose that X contains a locally finite subtree that is G–invariant. Then Y also contains a G–invariant locally finite subtree.*

Note that local finiteness itself is not preserved by elementary deformations. An expansion move outside of a minimal subtree can create infinite-valence vertices.

Proof. It suffices to consider a single elementary collapse move $q\colon X \to Y$ along the edge $e \in E(X)$ with $G_{\partial_0(e)} = G_e$. First suppose that X has a G–invariant locally finite subtree X'. Let $Y' = q(X')$, a G–invariant subtree of Y. If $e \notin X'$ then X' maps isomorphically to Y', and so Y' is locally finite. Otherwise, the restriction $q\colon X' \to Y'$ is an elementary collapse. In the proof of [For02, Theorem 7.3] it was observed that q is a $(3,2/3)$–quasi-isometry, and that $\frac{1}{3}(d(x,x')-2) \leqslant d(q(x),q(x'))$ for all $x,x' \in X'$. It follows that the preimage of a ball of radius 1 is contained in a ball of radius 5 in X'. The latter is finite, and so every ball of radius 1 in Y' is finite.

Next consider the same collapse move $q\colon X \to Y$ but suppose that Y contains a locally finite G–invariant subtree Y'. We wish to find the same in X. Let $Z \subset X$ be the G–invariant subgraph whose edges are $\{e' \in E(X) - (Ge \cup G\overline{e}) \mid q(e') \in E(Y')\}$. If Z is connected then let $X' = Z$; it maps isomorphically to Y' by q and hence is locally finite.

Otherwise, $\partial_0(e)$ is in Z. We define X' to be $Z \cup (Ge \cup G\overline{e}) = q^{-1}(Y')$. For any vertex $v \in V(X')$ consider the ball $B_v(1)$ of radius 1 at v. Edges of Z in $B_v(1)$ map injectively to Y', so there are finitely many. It remains to bound the number of edges of $Ge \cup G\overline{e}$ in $B_v(1)$. Note that each component of $Ge \cup G\overline{e}$ is a cone on some subset $S \subset G\partial_0(e)$ (with cone point in $G\partial_1(e)$). Each vertex of S is incident to an edge of Z and these edges are all distinct. Hence S is finite, because collapsing e results in a locally finite tree. Hence $Ge \cup G\overline{e}$ is locally finite, and therefore $B_v(1)$ is finite. □

3. Main results

Here we give two key propositions and then use them to prove Theorems 1.1 and 1.3. In order for the main argument to work, these propositions are stated for a slightly wider class of trees than just locally finite trees.

Proposition 3.1. *Let Y be a G–tree that contains a non-trivial locally finite G–invariant subtree Y' (possibly equal to Y). If $q\colon X \to Y$ is any collapse map then X and Y have the same elliptic subgroups.*

Proof. Note that Y and Y' have the same elliptic subgroups, by Lemma 2.2(1). First we show that q preserves hyperbolicity of elements of G. Suppose g is hyperbolic in X and elliptic in Y, with fixed vertex $y \in Y$. We may assume that $y \in Y'$. Let $H = G_y$ and consider the H–action on X. Since q is a collapse map, $q^{-1}(y)$ is a subtree of X, and it is H–invariant. The hyperbolic element g is in H, and so X_H is non-empty and unique, and is contained in $q^{-1}(y)$.

Now pick any vertex $y' \in Y'$ distinct from y, and let $H' = G_{y'}$. Since Y' is locally finite, the subgroups H and H' are commensurable. By Lemma 2.3(3), H' contains a hyperbolic element (so that $X_{H'}$ is non-empty and unique) and $X_{H'} = X_H$. However, $X_{H'}$ is contained in the H'–invariant subtree $q^{-1}(y')$ which is disjoint from $q^{-1}(y)$. Thus we have a contradiction.

Next we show that X and Y have the same elliptic subgroups. Let $y \in Y'$ be any vertex and let $H = G_y$. If H is not elliptic in X, it must be elliptic at infinity since X and Y have the same elliptic elements (by the previous paragraphs). Its fixed end ε_H is an end of the H–invariant subtree $q^{-1}(y)$. Now let y' be any other vertex of Y' and let $H' = G_{y'}$. By Lemma 2.3(2), H' is elliptic at infinity and $\varepsilon_{H'} = \varepsilon_H$. But $\varepsilon_{H'}$ is an end of the H'–invariant subtree $q^{-1}(y')$ which is disjoint from $q^{-1}(y)$. We have a contradiction, since two disjoint subtrees cannot have an end in common. Thus H is elliptic. □

Proposition 3.2. *Let Y be a G–tree containing a non-trivial locally finite G–invariant subtree Y'. If $f\colon X \to Y$ is any fold then X and Y are in the same deformation space.*

Proof. Recall from [BF91a] that any fold that is not of type A can be achieved by performing subdivision, two type A folds, and the reverse of subdivision. Since subdivision preserves deformation spaces, we may assume the fold is of type A. Also, by Proposition 3.16 of [For02], it suffices to show that f preserves hyperbolicity of elements of G.

Suppose the fold happens at $e, e' \in E(X)$ where $\partial_0(e) = \partial_0(e') = v$ and $\partial_1(e) = u$, $\partial_1(e') = u'$. Being of type A means that $v \notin Gu \cup Gu'$, and therefore the orbit Gv maps injectively to Y under the fold.

Now suppose that g is hyperbolic in X and elliptic in Y, with fixed vertex $y \in Y$. We may assume that $y \in Y'$. The set $\{g^i v\}_{i\in\mathbb{Z}}$ maps by f to an infinite set of vertices that are equidistant from y. In a locally finite tree, every bounded subtree is finite, and so these vertices cannot lie in Y'. Thus, $f(v) \in Y - Y'$. It follows that $f(e) = f(e') \in Y - Y'$ as well.

Note that f is surjective, and since Y' is not a point, it must contain an edge. Hence there is an edge $e'' \in E(X)$ such that $f(e'') \in Y'$. The orbits Ge and Ge' both land outside of Y', so $e'' \notin Ge \cup Ge'$. Hence Ge'' maps injectively to Y. But now the set $\{g^i e''\}_{i\in\mathbb{Z}}$ maps by f to an infinite set of edges of Y' that lie in a bounded neighborhood of y, a contradiction. □

Proof of Theorem 1.3. Let $f\colon X \to Y$ be as in the statement of the theorem. Any simplicial map factors as a collapse map followed by a morphism. Being surjective, the morphism factors as a finite composition of folds, by [BF91a, Section 2]. This step uses the assumptions that X is cocompact and Y has finitely generated edge stabilizers. Now we have f represented as

$$X \to Y_0 \to Y_1 \to \cdots \to Y_n = Y$$

with $X \to Y_0$ a collapse map and each $Y_i \to Y_{i+1}$ a fold. Note that none of the trees has a fixed point, since Y does not.

Applying Proposition 3.2 to the last fold, we find that Y_{n-1} is related to Y_n by a deformation, and so by Lemma 2.10 it contains a G–invariant locally finite subtree (which is not a point). Proceeding from right to left along the sequence of folds, we infer the same properties for every Y_i. Now Proposition 3.1 says that X and Y_0 have the same elliptic subgroups. Since X and Y_0 are both cocompact, they are in the same deformation space (along with Y). □

To prove Theorem 1.1 we need one more definition, from [DF87]. If X is a G–tree, a vertex x is *inessential* if it is the initial endpoint of exactly two edges e_1 and e_2 and $G_{e_1} = G_x = G_{e_2}$. Otherwise x is called *essential*. Note that if x is inessential and y is an essential vertex closest to x, then $G_x \subseteq G_y$. Hence every elliptic subgroup fixes an essential vertex (unless there are none, meaning that X is a line and G acts by translations).

Proof of Theorem 1.1. Let T be a reduced locally finite G–tree with finitely generated stabilizers. It suffices to bound the number of vertices of T/G, since there will then be at most $|V(T/G)| - 1 + \beta_1(G)$ edges. We may assume T is not a point or a line, since $|V(T/G)| \leqslant 2$ in those cases. Let K be a connected simplicial complex with $H^1(K;\mathbb{Z}_2) = 0$ on which G acts freely and cocompactly. Suppose $L = K/G$ has ℓ_0 vertices and ℓ_2 2–simplices. Define $\delta(G) = 2\dim H^1(L;\mathbb{Z}_2) + \ell_0 + \ell_2$.

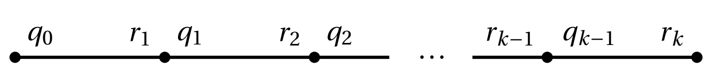
By Theorem 1.6 of [DF87] there is a G–tree T' and a simplicial G–map $\alpha\colon T' \to T$ such that T' has at most $\delta(G)$ orbits of essential vertices. From its construction, it is evident that T' is cocompact. The map is surjective because T is minimal. By Theorem 1.3, T' and T are in the same deformation space.

Now let v be any vertex of T. The elliptic subgroup G_v fixes an essential vertex w of T'. Either $\alpha(w) = v$ or G_v fixes the path from v to $\alpha(w)$. In the latter case, the first edge e along this path satisfies $G_e = G_v$. Since T is reduced, e maps to a loop in T/G. The number of such edge orbits is bounded by $\beta_1(G)$. In particular, the number of vertex orbits that are not images of essential vertex orbits of T' is bounded by $\beta_1(G)$. Hence the total number of vertex orbits of T is bounded by $\delta(G) + \beta_1(G)$. ☐

Remark 3.3. Theorem 1.1 remains true even if T is only "BF-reduced" (i.e. reduced in the sense of [BF91a]). Given a Dunwoody resolution $\alpha\colon T' \to T$ with T and T' in the same deformation space, the proof of the main result of [BF91a, Section 4 "The elliptic case"] applies. This is so because every edge stabilizer of T is elliptic in T'. That argument yields the bound $|V(T/G)| \leqslant 4\delta(G) + 9\beta_1(G) - 5$.

4. Examples

If one drops the assumption that stablilizers are finitely generated, then Theorem 1.1 becomes false. Our examples are based on 2–generator generalized Baumslag–Solitar groups. These are discussed in [Lev15b]. They fall into two families, one of which we describe here. Recall that a *generalized Baumslag–Solitar group* is a group that admits a graph of groups decomposition in which every vertex and edge group is infinite cyclic. The figure below describes such a graph of groups. Each edge-to-vertex morphism $\mathbb{Z} \to \mathbb{Z}$ is multiplication by a non-zero integer q_i or r_i.



We shall assume the graph of groups is reduced, which means that $r_i, q_j \neq \pm 1$ for all i, j. Then the group is 2–generated if and only if r_i and q_j are relatively prime whenever $1 \leqslant i \leqslant j \leqslant k-1$; see the discussion on page 9 of [Lev15b], which depends on [Lev15a, Theorem 1.1]. (One direction is Lemma 2.5 and the converse is an easy exercise.) The two generators are the generators of the vertex groups at the terminal vertices.

Let G be the group shown and assume it is 2–generated. The Bass–Serre tree X is reduced and locally finite, with vertices of valence $|q_0|$, $|r_i| + |q_i|$, and $|r_k|$. The surjection $F_2 \to G$ defines an action of F_2 on X with the same quotient graph. Hence edge stabilizers have index $|q_i|$ or $|r_i|$ in their neighboring vertex stabilizers, and X is reduced as an F_2–tree. These stabilizers are all free groups of infinite rank, by a theorem of Bieri on cohomological dimension in this setting [Bie76, Section 6]. Namely, if these stabilizers were finite rank free groups, then Bieri's theorem says they each have cohomological dimension zero. But then the quotient graph of groups has trivial fundamental group, a contradiction.

To conclude, say by taking $q_i = 2$ and $r_i = 3$ for all i, we have a reduced locally finite F_2–tree with $k+1$ vertex orbits, for arbitrary k. One further adjustment, setting $q_0 = r_k = 5$, results in reduced F_2–actions on the *same* tree, the regular tree of valence 5, of arbitrarily large complexity.

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