

Splittings of generalized Baumslag–Solitar groups

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Abstract. We study the structure of generalized Baumslag–Solitar groups from the point of view of their (usually non-unique) splittings as fundamental groups of graphs of infinite cyclic groups. We find and characterize certain decompositions of smallest complexity (*fully reduced* decompositions) and give a simplified proof of the existence of deformations. We also prove a finiteness theorem and solve the isomorphism problem for generalized Baumslag–Solitar groups with no non-trivial integral moduli.

Introduction

This paper explores the structure of generalized Baumslag–Solitar groups from the point of view of their (usually non-unique) splittings as fundamental groups of graphs of groups. By definition, a generalized Baumslag–Solitar group is the fundamental group of a graph of infinite cyclic groups. Equivalently, it is a group that acts on a simplicial tree with infinite cyclic vertex and edge stabilizers. We call such tree actions generalized Baumslag–Solitar trees. These groups have arisen in the study of splittings of groups, both in the work of Kropholler [12, 11] and as useful examples of JSJ decompositions [8]. They were classified up to quasi-isometry in [16, 6], but their group-theoretic classification is still unknown.

Our approach to understanding generalized Baumslag–Solitar groups is to study the space of all generalized Baumslag–Solitar trees for a given group. In most cases this is a *deformation space*, consisting of G -trees related to a given one by a deformation (a sequence of collapse and expansion moves [10, 1]). Equivalently this is the set of G -trees having the same elliptic subgroups (subgroups fixing a vertex) as the given one [7]. It is important to note that G -trees having the same elliptic subgroups need not have the same vertex stabilizers. This is one of the main issues arising in this paper.

Two notions of complexity for G -trees are the number of edge orbits and the number of vertex orbits. Within a deformation space, the local minima for both notions occur at the reduced trees: those for which no collapse moves are possible. In the first part of this paper we study *fully reduced* G -trees. These are G -trees in which no vertex stabilizer contains the stabilizer of a vertex from a different orbit. Fully reduced trees, when they exist, globally minimize complexity in a deformation space. They are somewhat canonical

(cf. Proposition 4.5) but are not always unique. The slide-inequivalent trees given in [8] are both fully reduced, for example.

One of our main results is Theorem 3.3 which states that every generalized Baumslag–Solitar tree can be made fully reduced by a deformation. After developing properties of fully reduced trees in Section 4 we use these results to give a simplified proof of the fact (originally proved in [7]) that all non-elementary generalized Baumslag–Solitar trees with the same group lie in a single deformation space. These results also lay the groundwork for further study on the classification of generalized Baumslag–Solitar groups.

In the second part of the paper we focus on generalized Baumslag–Solitar groups having no non-trivial integral moduli. It turns out that this class of groups can be understood reasonably well. One key property is given in Theorem 7.4: for such groups, deformations between reduced trees can be converted into sequences of slide moves, which do not change complexity. We then prove a finiteness theorem for such trees (Theorem 8.2), and these two results together yield a solution to the isomorphism problem (Corollary 8.3). This is our second main result.

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1. Preliminaries

We will use Serre’s notation for graphs and trees [15]. Thus a graph A is a pair of sets $(V(A), E(A))$ with maps $\partial_0, \partial_1: E(A) \rightarrow V(A)$ and an involution $e \mapsto \bar{e}$ (for $e \in E(A)$), such that $\partial_i \bar{e} = \partial_{1-i} e$ and $e \neq \bar{e}$ for all e . An element $e \in E(A)$ is to be thought of as an oriented edge with initial vertex $\partial_0 e$ and terminal vertex $\partial_1 e$. We denote by $E_0(v)$ the set of all edges having initial vertex v . An edge e is a *loop* if $\partial_0 e = \partial_1 e$.

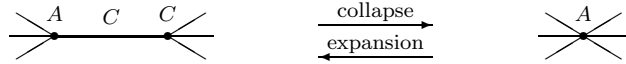
Definition 1.1. Let G be a group. A G -tree is a tree with a G -action by automorphisms, without inversions. A G -tree is *proper* if every edge stabilizer is strictly smaller than its neighboring vertex stabilizers. It is *minimal* if there is no proper G -invariant subtree, and it is *cocompact* if the quotient graph is finite.

Given a G -tree X , an element $g \in G$ is *elliptic* if it fixes a vertex of X and is *hyperbolic* otherwise. If g is hyperbolic then there is a unique G -invariant line in X , called the *axis* of g , on which g acts as a translation. A subgroup H of G is *elliptic* if it fixes a vertex.

Suppose a graph of groups has an edge e which is a loop. Let A be the vertex group and C the edge group, with inclusion maps $i_0, i_1: C \hookrightarrow A$. If one of these maps, say

i_0 , is an isomorphism, then e is an *ascending loop*. The *monodromy* is the composition $i_1 \circ i_0^{-1}: A \hookrightarrow A$.

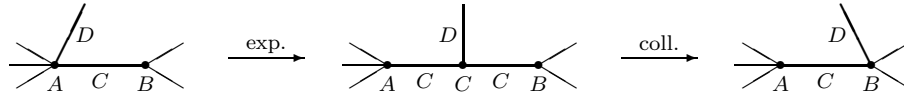
Definition 1.2. In a *collapse move*, an edge in a graph of groups carrying an amalgamation of the form $A *_C C$ is collapsed to a vertex with group A . Every inclusion map having target group C is reinterpreted as a map into A , via the injective map of vertex groups $C \hookrightarrow A$.



An *expansion move* is the reverse of a collapse move. Both of these moves are called *elementary moves*. A *deformation* (also called an *elementary deformation* in [7, 8]) is a finite sequence of such moves.

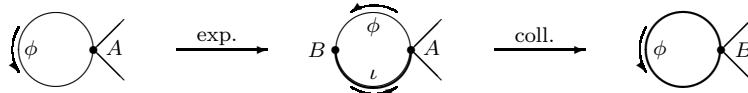
A graph of groups is *reduced* if it admits no collapse moves. This means that if an inclusion map from an edge group to a vertex group is an isomorphism, then the edge is a loop. Correspondingly, a G -tree is reduced if, whenever an edge stabilizer is equal to the stabilizer of one of its endpoints, both endpoints are in the same orbit. Note that reduced G -trees are minimal.

Definition 1.3. The deformation shown below (cf. [10]) is called a *slide move*. In order to perform the move it is required that $D \subseteq C$ (regarded as subgroups of A).



It is permitted for the edge carrying C to be a loop; in this case the only change to the graph of groups is in the inclusion map $D \hookrightarrow A$. See Proposition 2.4 for an example.

Definition 1.4. An *induction move* is an expansion and collapse along an ascending loop. In the diagram below the ascending loop has vertex group A and monodromy $\phi: A \rightarrow A$, and B is a subgroup such that $\phi(A) \subseteq B \subseteq A$. The map $\iota: B \rightarrow A$ is inclusion. The lower edge is expanded and the upper edge is collapsed, resulting in an ascending loop with monodromy the induced map $\phi|_B: B \rightarrow B$.



The reverse of this move is also considered an induction move. Notice that the vertex group changes, in contrast with slide moves.

Definition 1.5. A *fold* is most easily described directly in terms of G -trees. The graph of groups description includes many different cases which are explained in [5]. To perform a fold in a G -tree one chooses edges e and f with $\partial_0 e = \partial_0 f$, and identifies e and f to a single edge. One also identifies ge with gf for every $g \in G$, so that the resulting quotient graph has a G -action. It is not difficult to show that the new graph is a tree.

The following basic result is proved in [7, Proposition 3.16].

Proposition 1.6. *Suppose a fold between G -trees preserves hyperbolicity of elements of G . Then the fold can be represented by a deformation.* \square

2. Generalized Baumslag–Solitar groups

Definition 2.1. A *generalized Baumslag–Solitar tree* is a G -tree whose vertex and edge stabilizers are all infinite cyclic. The groups G that arise are called *generalized Baumslag–Solitar groups*. Basic examples include Baumslag–Solitar groups [4], torus knot and link groups, and finite index subgroups of these groups.

The quotient graphs of groups have all vertex and edge groups isomorphic to \mathbb{Z} , and the inclusion maps are multiplication by various non-zero integers. Thus any example is specified by a graph A and a function $i: E(A) \rightarrow (\mathbb{Z} - \{0\})$. The corresponding graph of groups will be denoted by $(A, i)_{\mathbb{Z}}$. If X is the G -tree above $(A, i)_{\mathbb{Z}}$ then the induced function $i: E(X) \rightarrow (\mathbb{Z} - \{0\})$ satisfies

$$|i(e)| = [G_{\partial_0 e} : G_e] \tag{2.2}$$

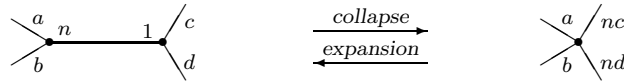
for all $e \in E(X)$.

Remark 2.3. There is generally some choice involved in writing down a quotient graph of groups of a G -tree. This issue is explored fully in [2, Section 4]. Without changing the G -tree it describes, a graph of groups may be modified by twisting an inclusion map by an inner automorphism of the target vertex group. Any two quotient graphs of groups of a G -tree are related by modifications of this type.

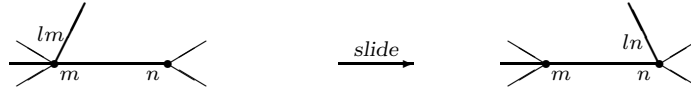
In the case of generalized Baumslag–Solitar trees there are no such inner automorphisms and the quotient graph of groups is uniquely determined by the G -tree. The associated edge-indexed graph is then very nearly uniquely determined; the only ambiguity arises from the choice of generators of edge and vertex groups. One may simultaneously change the signs of all indices at a vertex, or change the signs of $i(e)$ and $i(\bar{e})$ together for any e , with no change in the graph of groups or the G -tree it encodes.

Elementary moves and deformations can be described directly in terms of edge-indexed graphs, as follows. The verifications are left to the reader. In the diagrams below, each index $i(e)$ is shown next to the endpoint $\partial_0 e$. Note in particular that any deformation performed on a generalized Baumslag–Solitar tree results again in a generalized Baumslag–Solitar tree.

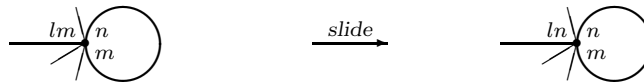
Proposition 2.4. *If an elementary move is performed on a generalized Baumslag–Solitar tree, then the quotient graph of groups changes locally as follows:*



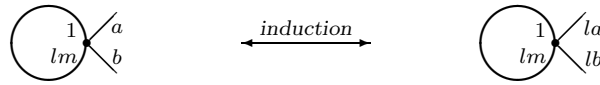
A slide move has the following description:



or



An induction move is as follows (cf. Lemma 3.11):



Definition 2.5. A G -tree is *elementary* if there is a G -invariant point or line, and *non-elementary* otherwise. In [8, Lemma 2.6] it is shown that a generalized Baumslag–Solitar tree is elementary if and only if the group is isomorphic to \mathbb{Z} , $\mathbb{Z} \times \mathbb{Z}$, or the Klein bottle group. Thus we may speak of generalized Baumslag–Solitar groups as being elementary or non-elementary.

A fundamental property of generalized Baumslag–Solitar groups is that the elliptic subgroups are canonical, except in the elementary case. Recall that two subgroups H, K of G are *commensurable* if $H \cap K$ has finite index in both H and K . The following lemma is proved in [7, Corollary 6.10] and [8, Lemma 2.5].

Lemma 2.6. *Let X be a non-elementary generalized Baumslag–Solitar tree with group G . A nontrivial subgroup $H \subseteq G$ is elliptic if and only if it is infinite cyclic and is commensurable with all of its conjugates.* □

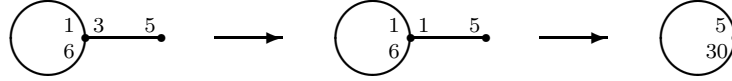
The property of H given in the lemma is rather special. Kropholler showed in [11] that among finitely generated groups of cohomological dimension 2, the existence of such a subgroup exactly characterizes the generalized Baumslag–Solitar groups.

3. Full reducibility

Definition 3.1. A graph of groups is *fully reduced* if no vertex group can be conjugated into another vertex group. Correspondingly, a G -tree is fully reduced if, whenever one vertex stabilizer contains another vertex stabilizer, they are conjugate. Notice that a fully reduced graph of groups is minimal and reduced. Two basic examples of fully reduced trees are proper trees and trees having a single vertex orbit.

We shall see that for generalized Baumslag–Solitar groups, fully reduced decompositions exist and have underlying graphs of smallest complexity (Theorems 3.3 and 4.6 below).

Example 3.2. The G -tree shown on the left is reduced but not fully reduced. The valence three vertex group can be conjugated into the other vertex group (by conjugating around the loop). After performing an induction move and a collapse one finds that G is the Baumslag–Solitar group $BS(5, 30)$, which was perhaps not obvious initially.



The following result generalizes this procedure to arbitrary generalized Baumslag–Solitar trees.

Theorem 3.3. *Every cocompact generalized Baumslag–Solitar tree is related by a deformation to a fully reduced (generalized Baumslag–Solitar) tree.*

The proof relies strongly on the fact that stabilizers are infinite cyclic, and therefore contain a unique subgroup of any given index. Before proving the theorem we establish some preliminary facts concerning generalized Baumslag–Solitar trees. Our first objective (Corollary 3.6) is to characterize the paths that are fixed by vertex stabilizers.

Lemma 3.4. *Let X be a generalized Baumslag–Solitar tree with group G . Suppose $G_x \subseteq nG_{x'}$ for vertices $x \neq x'$. Let (e_1, \dots, e_k) be the path from x to x' , with vertices $x_0 = x$, $x_i = \partial_1 e_i$ for $1 \leq i \leq k$. Define $m_i = i(\bar{e}_i)$, $n_i = i(e_{i+1})$, and $n_k = n$. Then $i(e_1) = \pm 1$ and*

- (i) $G_x = \left(\prod_{i=1}^k m_i / \prod_{i=1}^{k-1} n_i \right) G_{x_k}$
- (ii) $\prod_{i=1}^k n_i$ divides $\prod_{i=1}^k m_i$.

Proof. The statement $i(e_1) = \pm 1$ is clear because G_x fixes e_1 . The other two statements are proved together by induction on k .

If $k = 1$ then (i) says that $G_x = m_1 G_{x'}$, which holds because $i(e_1) = \pm 1$. Then the assumption $G_x \subseteq n_1 G_{x'}$ implies that n_1 divides m_1 , because $G_x = G_{e_1} = m_1 G_{x'}$.

Now let $k > 1$ be arbitrary. Since G_x fixes e_k we have $G_x \subseteq n_{k-1} G_{x_{k-1}}$, and the induction hypothesis gives that $\prod_{i=1}^{k-1} n_i$ divides $\prod_{i=1}^{k-1} m_i$. We also have $G_x = \left(\prod_{i=1}^{k-1} m_i / \prod_{i=1}^{k-2} n_i \right) G_{x_{k-1}}$ and $n_{k-1} G_{x_{k-1}} = G_{e_k} = m_k G_{x_k}$. Therefore

$$G_x = \left(\prod_{i=1}^{k-1} m_i / \prod_{i=1}^{k-1} n_i \right) n_{k-1} G_{x_{k-1}} = \left(\prod_{i=1}^{k-1} m_i / \prod_{i=1}^{k-1} n_i \right) m_k G_{x_k},$$

proving (i).

Next, the assumption $G_x \subseteq n_k G_{x_k}$ becomes $\left(\prod_{i=1}^k m_i / \prod_{i=1}^{k-1} n_i \right) G_{x_k} \subseteq n_k G_{x_k}$ by (i), establishing (ii). \square

Remark 3.5. The lemma is valid in any locally finite G -tree, provided one interprets statements such as $G_x \subseteq n G_{x'}$ correctly. For example this statement would mean that $G_x \subseteq G_{x'}$ and n divides $[G_{x'} : G_x]$. The following corollary, however, is specific to generalized Baumslag–Solitar trees.

Corollary 3.6. *Let (e_1, \dots, e_k) be a path in X and define m_i, n_i as in the previous lemma. Then $G_{\partial_0 e_1}$ fixes the path (e_1, \dots, e_k) if and only if $i(e_1) = \pm 1$ and for every $l \leq (k-1)$*

$$\prod_{i=1}^l n_i \text{ divides } \prod_{i=1}^l m_i. \quad (3.7)$$

Proof. The forward implication is given by Lemma 3.4(ii). The converse is proved by induction on k . Suppose (3.7) holds for each l and that $G_{\partial_0 e_1}$ fixes the path (e_1, \dots, e_{k-1}) . Then $G_{\partial_0 e_1}$ is the subgroup of $G_{\partial_0 e_k}$ of index $\prod_{i=1}^{k-1} m_i / \prod_{i=1}^{k-2} n_i$ by Lemma 3.4(i). Property (3.7) for $l = k-1$ implies that n_{k-1} divides this index, and so $G_{\partial_0 e_1} \subseteq G_{e_k}$. \square

Next we describe the steps needed to construct the deformation of Theorem 3.3. The kind of example one should have in mind is one similar to Example 3.2, but with several loops incident to the left-hand vertex.

Throughout the rest of this section X denotes a generalized Baumslag–Solitar tree with group G and quotient graph of groups $(A, i)_{\mathbb{Z}}$.

Definition 3.8. Let $f \in E(A)$ be an edge with $\partial_0 f \neq \partial_1 f$. Let $\rho = (e_1, \dots, e_k, f)$ be a path in A such that each e_i is a loop at $\partial_0 f$. We say that ρ is an *admissible path for f* if, for some lift $\tilde{\rho} = (\tilde{e}_1, \dots, \tilde{e}_k, \tilde{f}) \subset X$,

$$G_{\partial_0 \tilde{e}_1} \subseteq G_{\tilde{f}}.$$

This condition depends only on the indices along the path ρ by Corollary 3.6, so it is independent of the choice of $\tilde{\rho}$. When dealing with admissible paths we will use the notation $m_i = i(\bar{e}_i)$, $n_i = i(e_{i+1})$, and $n_k = i(f)$; then the path is admissible if and only if $i(e_1) = \pm 1$ and (3.7) holds for each l .

An edge $e \in E(A)$ is *essential* if $i(e) \neq \pm 1$, and *inessential* otherwise. The *length* of ρ is k , and the *essential length* of ρ is the number of essential edges occurring in ρ .

Lemma 3.9. *If $\rho = (e_1, \dots, e_k, f)$ is an admissible path then there is a permutation σ such that the path $\rho_\sigma = (e_{\sigma(1)}, \dots, e_{\sigma(k)}, f)$ is admissible and all of the essential edges of ρ_σ occur after the inessential edges.*

Proof. We show first that if e_j is inessential for some $j > 1$ then

$$\rho' = \rho_{((j-1) j)} = (e_1, \dots, e_{j-2}, e_j, e_{j-1}, e_{j+1}, \dots, e_k, f)$$

is an admissible path for f . Letting m'_i and n'_i be the indices along ρ' , we have $m'_{j-1} = m_j$, $m'_j = m_{j-1}$, and $n'_{j-2} = n_{j-1}$, $n'_{j-1} = n_{j-2}$, with all other indices unchanged. Clearly (3.7) still holds for $l \neq j-2, j-1$. One easily verifies (3.7) for these other two cases as well, using the fact that $n_{j-1} = \pm 1$ (because e_j is inessential). Hence ρ' is admissible for f . Next, by using transpositions of this type, one can move all of the inessential edges in ρ to the front of the path. \square

Lemma 3.10. *Let $\rho = (e_1, \dots, e_k, f)$ be an admissible path such that e_1, \dots, e_j are inessential and e_{j+1}, \dots, e_k are essential. Then there is a sequence of slide moves, after which the path $\rho' = (e_1, \dots, e_1, e_{j+1}, \dots, e_k, f)$ is admissible. The inessential part (e_1, \dots, e_1) of ρ' may have length greater than j , though ρ and ρ' have the same essential length $(k-j)$.*

Proof. First we slide \bar{e}_1 over each edge of $(A, i)_{\mathbb{Z}}$ (other than e_1) that appears in (e_2, \dots, e_j) . Since these edges are all inessential loops, these slides can be performed. The index $i(e_1)$ is unchanged so occurrences of e_1 in the path are still inessential.

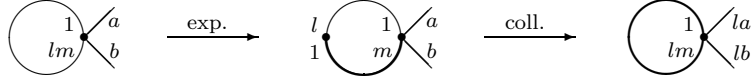
Each slide of \bar{e}_1 over e_i multiplies $i(\bar{e}_1)$ by $\pm i(\bar{e}_i)$. The end result is that m_1 gets multiplied by a product $\pm \prod_{\nu=1}^r n_{i_\nu}$. By itself this change does not violate the conditions (3.7). However if some e_i is equal to \bar{e}_1 (where $i > j$), then n_{i-1} is also multiplied by $\pm \prod_{\nu=1}^r n_{i_\nu}$, and (3.7) may fail. To remedy this we adjoin several copies of e_1 to the front of the path, one for each occurrence of \bar{e}_1 in (e_{j+1}, \dots, e_k) . Then the products $\prod_{i=1}^l m_i$ acquire enough additional factors $\pm \prod_{\nu=1}^r n_{i_\nu}$ to remain divisible by $\prod_{i=1}^l n_i$. This extended path is therefore admissible. As a result of the slide moves, m_i now divides m_1 for each $i \leq j$.

We now replace e_i by e_1 for $i \leq j$. The quantities $\prod_{i=1}^l m_i$ increase and each $\prod_{i=1}^l n_i$ remains unchanged (up to sign), so property (3.7) still holds for every l . \square

Lemma 3.11. *Let $(e_1, \dots, e_1, e_{j+1}, \dots, e_k, f)$ be an admissible path such that e_1 is inessential and e_{j+1}, \dots, e_k are essential. Then there is a sequence of induction moves after which a path of the same form is admissible, has essential length at most $(k - j)$, and satisfies $i(e_{j+1}) = \pm i(\bar{e}_1)^r$ for some r .*

Proof. If $e_{j+1} = \bar{e}_1$ then we can discard it from the path without affecting admissibility. Thus we can assume that $e_{j+1} \neq \bar{e}_1$. Admissibility implies that $i(e_{j+1})$ divides $i(\bar{e}_1)^j$. Let r be minimal so that $i(e_{j+1})$ divides $i(\bar{e}_1)^r$ and let l be any factor of $i(\bar{e}_1)^r/i(e_{j+1})$ that divides $i(\bar{e}_1)$. We show how to make $i(e_{j+1})$ become $l \cdot i(e_{j+1})$. By repeating this procedure the desired result can be achieved.

Writing $i(\bar{e}_1)$ as lm , we perform an induction move along e_1 as follows:



The index of every edge incident to $\partial_0 e_1$ is multiplied by l , except for $i(e_1)$ and $i(\bar{e}_1)$, which remain the same. As a result the indices m_i and n_{i-1} are multiplied by l whenever e_i is not equal to e_1 or \bar{e}_1 . For every such i we adjoin a copy of e_1 to the front of the path, making it admissible as in the proof of the preceding lemma. \square

Remark 3.12. The previous argument is still valid when $j = k$. That is, if the path (e_1, \dots, e_1, f) is admissible, then there is a sequence of induction moves after which $i(f) = i(\bar{e}_1)^r$ for some r .

Proof of Theorem 3.3. We show that if $(A, i)_{\mathbb{Z}}$ is not fully reduced then there is a deformation to a decomposition having fewer edges. Repeating the procedure will eventually produce a fully reduced decomposition.

If $(A, i)_{\mathbb{Z}}$ is not fully reduced then there exist vertices v, w of X such that $G_v \subseteq G_w$ and $v \notin G_w$. The path $(\tilde{e}_1, \dots, \tilde{e}_r)$ from v to w contains an edge that does not map to a loop in A . Let $\tilde{f} = \tilde{e}_{k+1}$ be the first such edge. Since G_v stabilizes $(\tilde{e}_1, \dots, \tilde{e}_k, \tilde{f})$, the image $\rho = (e_1, \dots, e_k, f)$ of this path in A is an admissible path for f .

Next we show how to produce an admissible path for f having essential length smaller than that of ρ , assuming this length is positive. Suppose ρ has essential length s . Applying Lemmas 3.9, 3.10, and 3.11 in succession to the path ρ we can arrange that there is an admissible path $\rho' = (e'_1, \dots, e'_1, e'_{k'-s+1}, \dots, e'_{k'}, f)$ such that e'_1 is inessential, $e'_{k'-s+1}, \dots, e'_{k'}$ are essential, and $i(e'_{k'-s+1}) = i(\bar{e}'_1)^r$ for some r . (The essential length s may have decreased, but then we are done for the moment.) In applying these lemmas the decomposition $(A, i)_{\mathbb{Z}}$

changes by a deformation to $(A', i')_{\mathbb{Z}}$ where A' has the same number of edges as A . Now we can slide $e'_{k'-s+1}$ over \bar{e}'_1 r times to make $i(e'_{k'-s+1}) = \pm 1$.

These slide moves affect the indices of the edges e'_i that are equal to $e'_{k'-s+1}$ or $\bar{e}'_{k'-s+1}$. If $e'_i = e'_{k'-s+1}$ then n_{i-1} is divided by $(m_1)^r = i(\bar{e}'_1)^r$ and this change does not affect the admissibility of ρ' . If $e'_i = \bar{e}'_{k'-s+1}$ then m_i is divided by $(m_1)^r$. In order to keep ρ' admissible we adjoin r copies of e'_1 to the front of the path for each such e'_i . This done, we have produced an admissible path for f of smaller essential length because $e'_{k'-s+1}$ is now inessential.

By repeating this process we can obtain a decomposition $(A', i')_{\mathbb{Z}}$ related by a deformation to $(A, i)_{\mathbb{Z}}$ (with $|E(A')| = |E(A)|$), and an admissible path for f having essential length zero. Applying Lemmas 3.10 and 3.11 once more, this path has the form (e'_1, \dots, e'_1, f) where $i(f) = i(\bar{e}'_1)^r$. Now we slide f over \bar{e}'_1 r times to make $i(f) = \pm 1$, and collapse f . The resulting decomposition has fewer edges than $(A, i)_{\mathbb{Z}}$. \square

4. Vertical subgroups

In this section we link the structure of a fully reduced G -tree to that of the group G , using the notion of a vertical subgroup. We are concerned with the difference between elliptic subgroups (which may be uniquely determined) and vertex stabilizers (which often are not). It turns out that vertical subgroups are a useful intermediate notion. See in particular Example 4.3.

Definition 4.1. Let X be a G -tree. A subgroup $H \subseteq G$ is *vertical* if it is elliptic and every elliptic subgroup containing H is conjugate to a subgroup of H .

Lemma 4.2. *If X is fully reduced then an elliptic subgroup is vertical if and only if it contains a vertex stabilizer.*

Proof. Suppose H contains G_v , and let H' be an elliptic subgroup containing H . Since H' is elliptic, it is contained in G_w for some w , and hence $G_v \subseteq G_w$. Full reducibility implies that $(G_w)^g = G_v$ for some g , and therefore $(H')^g \subseteq H$.

Conversely suppose H is vertical. Then $H \subseteq G_v$ for some v , and so $(G_v)^g \subseteq H$ for some $g \in G$. Hence H contains G_{gv} . \square

Example 4.3. Let G be the Baumslag–Solitar group $BS(1, 6)$ with its standard decomposition $G = \mathbb{Z} *_{\mathbb{Z}}$ and presentation $\langle x, t \mid txt^{-1} = x^6 \rangle$. The vertex stabilizers of the Bass–Serre tree X are the conjugates of the subgroup $\langle x \rangle$. Among the subgroups of the

form $\langle x^n \rangle$, notice that all are elliptic, and only those where n is a power of 6 are vertex stabilizers. According to Lemma 4.2, $\langle x^n \rangle$ is vertical if and only if n divides a power of 6. Now consider the automorphism $\phi: G \rightarrow G$ defined by $\phi(x) = x^3$, $\phi(t) = t$ (with inverse $x \mapsto t^{-1}x^2t$, $t \mapsto t$). If we twist the action of G on X by ϕ then the vertex stabilizers will be the conjugates of $\langle x^2 \rangle$ rather than $\langle x \rangle$. Thus there is no hope of characterizing vertex stabilizers from the structure of G alone. On the other hand, the vertical subgroups are uniquely determined (because the elliptic subgroups are).

In this particular example, the set of vertical subgroups is the smallest $\text{Aut}(G)$ -invariant set of elliptic subgroups containing a vertex stabilizer. Every vertical subgroup can be realized as a vertex stabilizer by twisting by an automorphism.

Definition 4.4. Now we define an equivalence relation on the set of vertical subgroups of G . Set $H \sim K$ if H is conjugate to a subgroup of K . This relation is symmetric: suppose $H \sim K$, so that $H^g \subseteq K$ for some $g \in G$. Then $H \subseteq K^{g^{-1}}$ and so $K^{g^{-1}}$ is conjugate to a subgroup of H , as H is vertical. Therefore $K \sim H$. Reflexivity and transitivity are clear.

Proposition 4.5. *If X is fully reduced then the vertex orbits correspond bijectively with the \sim -equivalence classes of vertical subgroups of G . The bijection is induced by the natural map $v \mapsto G_v$.*

Proof. The induced map is well defined since $G_v \sim (G_v)^g = G_{gv}$ for any g . For injectivity, suppose that $G_v \sim G_w$. Then $G_{gv} \subseteq G_w$ for some $g \in G$. Full reducibility implies that gv and w are in the same orbit, hence v and w are as well.

For surjectivity, suppose H is vertical. It contains a stabilizer G_v by Lemma 4.2, and $G_v \subseteq H$ implies $G_v \sim H$. \square

The following application of Proposition 4.5 explains the choice of the term *fully reduced*.

Theorem 4.6. *A non-elementary cocompact generalized Baumslag–Solitar tree is fully reduced if and only if it has the smallest number of edge orbits among all generalized Baumslag–Solitar trees having the same group.*

Proof. The proof of Theorem 3.3 shows that any generalized Baumslag–Solitar tree with the smallest number of edge orbits is fully reduced. For the converse we show that no tree with more edge orbits can also be fully reduced.

Suppose the given tree is fully reduced. Let $N \subseteq G$ be the normal closure of the set of elliptic elements. This subgroup is uniquely determined since the tree is non-elementary. Note that G/N is the fundamental group of the quotient graph. Hence the homotopy type of this graph is uniquely determined. Proposition 4.5 implies that the number of vertices is also uniquely determined, and so the number of edges is as well. Thus any two fully reduced trees have the same number of edge orbits. \square

5. Existence of deformations

We now know that generalized Baumslag–Solitar trees can be made fully reduced (Theorem 3.3) and that for such trees, the structure of the tree is partially encoded in the set of elliptic subgroups (Proposition 4.5). Using these facts we may now give a quick proof of the existence of deformations between generalized Baumslag–Solitar trees. This result is a special case of Theorem 1.1 of [7].

Theorem 5.1. *Let X and Y be non-elementary cocompact generalized Baumslag–Solitar trees with isomorphic groups. Then X and Y are related by a deformation.*

Definition 5.2. A map between trees is a *morphism* if it sends vertices to vertices and edges to edges (and respects the maps $\partial_0, \partial_1, e \mapsto \bar{e}$). Geometrically it is a simplicial map which does not send any edge into a vertex.

The following result is taken from [5, Section 2].

Proposition 5.3 (Bestvina–Feighn). *Let G be a finitely generated group and suppose that $\phi: X \rightarrow Y$ is an equivariant morphism of G -trees. Assume further that X is cocompact, Y is minimal, and the edge stabilizers of Y are finitely generated. Then ϕ is a finite composition of folds. \square*

Proof of Theorem 5.1. Let G be the common group acting on X and Y . Note that both trees define the same elliptic and vertical subgroups. By Theorem 3.3 we can assume that both trees are fully reduced (and minimal). Applying Propositions 5.3 and 1.6, it now suffices to construct a morphism from X to Y . In fact we shall construct such a map from X' to Y , where X' is obtained from X by subdivision (a special case of a deformation).

Let $x_1, \dots, x_n \in V(X)$ be representatives of the vertex orbits of X . Then there are vertices $y_1, \dots, y_n \in V(Y)$ such that $G_{x_i} \subseteq G_{y_i}$, since each G_{x_i} is elliptic. We define a map $\phi: V(X) \rightarrow V(Y)$ by setting $\phi(x_i) = y_i$ and extending equivariantly. We then extend ϕ to a topological map $X \rightarrow Y$ by sending an edge e to the unique reduced path in Y from $\phi(\partial_0 e)$ to $\phi(\partial_1 e)$. Subdividing where necessary, we obtain an equivariant simplicial map $\phi': X' \rightarrow Y$.

Now we verify that ϕ' is a morphism. It suffices to check that $\phi(x) \neq \phi(x')$ whenever x and x' are vertices of X that bound an edge. There are two cases. If $x' = gx$ for some $g \in G$ then g is hyperbolic, since it has translation length one in X , and equivariance implies that $\phi(x) \neq \phi(x')$. Otherwise, if x and x' are in different orbits, then $G_x \not\sim G_{x'}$ by Proposition 4.5. Here we are using the fact that X is fully reduced. Equivariance yields $G_x \subseteq G_{\phi(x)}$ and $G_{x'} \subseteq G_{\phi(x')}$, and since these are all vertical subgroups (by Lemma 4.2) we now have $G_x \sim G_{\phi(x)}$ and $G_{x'} \sim G_{\phi(x')}$. Hence $G_{\phi(x)} \not\sim G_{\phi(x')}$, and in particular $\phi(x) \neq \phi(x')$. \square

6. The modular homomorphism

Let $\mathbb{Q}_{>0}^\times$ denote the positive rationals considered as a group under multiplication. The following notion was first defined by Bass and Kulkarni [3].

Definition 6.1. The *modular homomorphism* $q: G \rightarrow \mathbb{Q}_{>0}^\times$ of a locally finite G -tree is given by

$$q(g) = [V : V \cap V^g] / [V^g : V \cap V^g]$$

where V is any subgroup of G commensurable with a vertex stabilizer. In this definition we are using the fact that in locally finite G -trees, vertex stabilizers are commensurable with all of their conjugates. One can easily check that q is independent of the choice of V .

In the case of generalized Baumslag–Solitar trees the modular homomorphism may be defined directly in terms of the graph of groups $(A, i)_{\mathbb{Z}}$, as in [3]. First note that q factors through $H_1(A)$ because it is trivial on elliptic subgroups and $\mathbb{Q}_{>0}^\times$ is abelian. Writing q as a composition $G \rightarrow H_1(A) \rightarrow \mathbb{Q}_{>0}^\times$, the latter map is then given by

$$(e_1, \dots, e_k) \mapsto \prod_{j=1}^k |i(e_j)/i(\bar{e}_j)|. \quad (6.2)$$

To verify (6.2) note that given $g \in G$, the corresponding 1-cycle in $H_1(A)$ is obtained by projecting any (oriented) segment of the form $[v, gv]$ to A . One then uses $V = G_v$ to evaluate $q(g)$, by applying (2.2) to the edges of $[v, gv]$.

The next definition is not actually needed in this paper. We mention it for completeness, with the expectation that it will be useful in future work.

Definition 6.3. The *signed modular homomorphism* $\hat{q}: G \rightarrow \mathbb{Q}^\times$ of a generalized Baumslag–Solitar tree with quotient graph of groups $(A, i)_{\mathbb{Z}}$ is defined via the map $H_1(A) \rightarrow \mathbb{Q}^\times$ given by

$$(e_1, \dots, e_k) \mapsto \prod_{j=1}^k i(e_j)/i(\bar{e}_j). \quad (6.4)$$

One should verify that this is well defined in light of Remark 2.3. Clearly, changing the signs of $i(e)$ and $i(\bar{e})$ together, for any e , has no effect. Similarly, since (e_1, \dots, e_k) is a cycle, changing all signs at a vertex will introduce an even number of sign changes in (6.4).

There is also an *orientation homomorphism* $G \rightarrow \{\pm 1\}$ defined by $g \mapsto \hat{q}(g)/q(g)$.

Remark 6.5. The modular homomorphisms are invariant under deformations. For the unsigned case, note that during an elementary move there is a vertex stabilizer that remains unchanged. Taking V to be this stabilizer, one obtains invariance of q . Alternatively, one may verify directly that the homomorphisms defined by (6.2) and (6.4) are invariant, using Proposition 2.4.

7. Deformations and slide moves

In this section we show how to rearrange elementary moves between generalized Baumslag–Solitar trees. Our goal is to replace deformations by sequences of slide moves, which are considerably easier to work with. It should be noted that in general, reduced generalized Baumslag–Solitar trees with the same group G need not be related by slide moves; see [8]. Nevertheless this does occur in a special case, given in Theorem 7.4 below.

Definition 7.1. Suppose $(A, i)_{\mathbb{Z}}$ has a loop e with $(i(e), i(\bar{e})) = (m, n)$. If m divides n then e is a *virtually ascending loop*. It is *strict* if $n \neq \pm m$. Similarly, a *strict ascending loop* is one with indices of the form $(\pm 1, n)$, $n \neq \pm 1$.

The next two propositions are valid for sequences of moves between generalized Baumslag–Solitar trees.

Proposition 7.2. *Suppose an expansion is followed by a slide move. Either*

- (i) *the moves remove a strict virtually ascending loop and create a strict ascending loop, or*
- (ii) *the moves may be replaced by a (possibly empty) sequence of slides, followed by an expansion.*

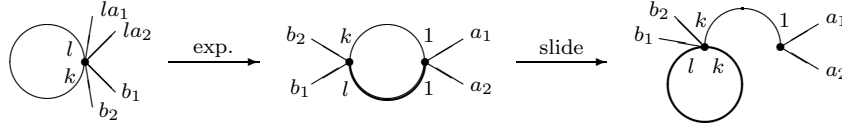
Proof. Suppose the expansion creates e and the second move slides e_0 over e_1 (from $\partial_0 e_1$ to $\partial_1 e_1$). If e is not e_i or \bar{e}_i ($i = 0, 1$) then the moves may be performed in reverse order as they do not interfere with each other.

If $e = e_1$ or \bar{e}_1 then there is no need to perform the slide at all. When performing an expansion at a vertex, the incident edges are partitioned into two sets, which are then separated by a new edge. Sliding e_0 over the newly created edge is equivalent to including e_0 in the other side of the partition before expanding.

If $e = e_0$ or \bar{e}_0 then there are several cases to consider. Orient e so that $i(e) = 1$, and e_1 so that e slides over e_1 from $\partial_0(e_1)$ to $\partial_1(e_1)$. The cases depend on which of the vertices $\partial_0 e$, $\partial_1 e$, $\partial_0 e_1$, $\partial_1 e_1$ coincide (after the expansion and before the slide).

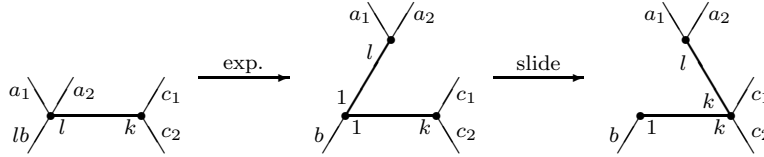
Case 1. After expanding e , e_1 has endpoints $\partial_0(e)$ and $\partial_1(e)$. If $\partial_1(e) = \partial_0(e_1)$ and $\partial_0(e) = \partial_1(e_1)$ then $i(e)$ is still 1 after the slide and e has become an ascending loop. In addition, since the slide takes place we must have $i(e_1) \mid i(\bar{e})$. Writing $i(e_1) = k$ and $i(\bar{e}) = kl$, we must have had $kl \mid i(\bar{e}_1)$ and $i(e_1) = k$ before the expansion, so e_1 was a virtually ascending loop. Writing $i(\bar{e}_1) = klm$ (before the expansion) the modulus of the loop e_1 is lm . If $lm \neq \pm 1$ then alternative (i) holds. If $lm = \pm 1$ then after the expansion, $i(\bar{e}_1) = \pm 1$. The expansion and slide may then be replaced by a single expansion.

Otherwise $\partial_1(e) = \partial_1(e_1)$ and $\partial_0(e) = \partial_0(e_1)$. The two moves have the form:

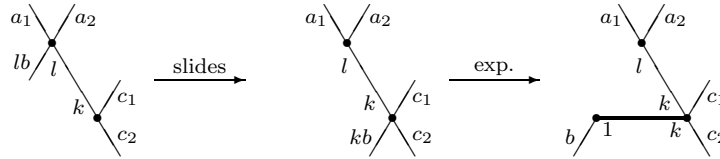


The same G -tree results if we first perform slides and then expand, including $\partial_0(e_1)$ in the same side as $\partial_1(e_1)$, and then exchange the names of e and e_1 .

Case 2. The edge e_1 has distinct endpoints and is incident to only one endpoint of e . Let $\{f_i\}$ be the edges with $\partial_0(f_i) = \partial_0(e)$ just before the slide move (not including e_1). We replace the expansion and slide by slides and an expansion as follows: first slide each f_i over e_1 , then expand at $\partial_1(e_1)$ so that the new expansion edge e separates $\{f_i\}$ from the rest of the edges at $\partial_1(e_1)$. For example:



becomes



As in Case 1, the names of e and e_1 must be exchanged after the new moves. This procedure works whether e_1 is incident to $\partial_0(e)$ or to $\partial_1(e)$.

Case 3. The edge e_1 is a loop incident to $\partial_1(e)$. Then the procedure from Case 2 works. The two moves are replaced by a sequence of slides (around the loop e_1) followed by an expansion.

Case 4. The edge e_1 is a loop incident to $\partial_0(e)$. Let $l = i(\bar{e})$. Since $\partial_0(e)$ is the end of e that slides over e_1 and $i(e) = 1$, the loop e_1 must be an ascending loop (before and after the slide). Note that before the expansion of e , the indices of e_1 were l times their current values; hence e_1 was originally a virtually ascending loop. Let k be the modulus of e_1 . If $k \neq \pm 1$ then alternative (i) holds. If $k = 1$ then the slide move may simply be omitted. If $k = -1$ then first perform the slide moves as described in Case 2, and then expand the edge e as before, but with $i(\bar{e}) = l$ and $i(e) = -1$. \square

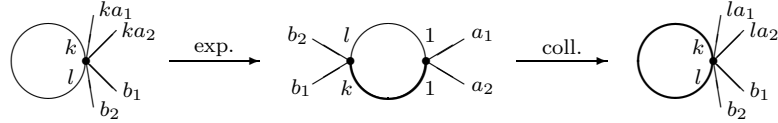
Proposition 7.3. *Suppose an expansion creating the edge e is followed by the collapse of an edge e' . Then either*

- (i) e' is a strict ascending loop before the expansion and e is a strict ascending loop after the collapse,
- (ii) both moves may be deleted,
- (iii) both moves may be replaced by a sequence of slides, or
- (iv) the collapse may be performed before the expansion move.

Proof. If $e = e'$ or $e = \bar{e}'$ then clearly (ii) holds. Otherwise e and e' are distinct, proper edges just after the expansion and before the collapse. If they do not meet then conclusion (iv) holds.

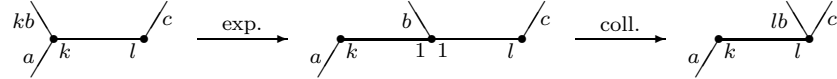
Now assume that e and e' meet in one or two vertices. Orient both edges so that $i(e) = i(e') = 1$.

Case 1. The edges e and e' have two vertices in common. If $\partial_0(e) = \partial_1(e')$ and $\partial_1(e) = \partial_0(e')$ then alternative (i) holds. Otherwise, if $\partial_0(e) = \partial_0(e')$ and $\partial_1(e) = \partial_1(e')$ then set $k = i(\bar{e})$ and $l = (\bar{e}')$. The moves have the form:



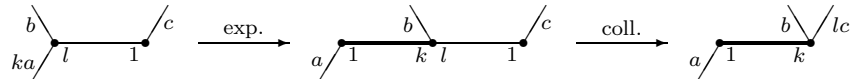
Evidently the moves may be replaced by slides around the loop e' .

Case 2. The edges e and e' meet in one vertex. Again let $k = i(\bar{e})$ and $l = (\bar{e}')$. There are four configurations. If $\partial_0(e) = \partial_0(e')$ then we see:



We may replace the two moves by slides over the edge e' .

In the other three configurations the collapse may be performed before the expansion. To illustrate, the case $\partial_1(e) = \partial_1(e')$ has the following configuration:



and it is easy to see that the collapse may be performed first. The remaining two cases are entirely similar. \square

Theorem 7.4. *Let X and Y be reduced non-elementary cocompact generalized Baumslag–Solitar trees with group G , and suppose that $q(G) \cap \mathbb{Z} = 1$. Then X and Y are related by slide moves.*

Proof. The property $q(G) \cap \mathbb{Z} = 1$ guarantees that no generalized Baumslag–Solitar decomposition of G contains strict virtually ascending loops. Starting with a sequence of moves from X to Y (given by Theorem 5.1) we claim that Propositions 7.2 and 7.3 can be applied to obtain a new sequence consisting of collapses, followed by slides, followed by expansions. To see this, note that case (i) of either proposition cannot occur. Therefore expansions can be pushed forward past slides (by 7.2) and past collapses (by 7.3), and collapses can be pulled back before slides (by 7.2 applied to the reverse of the sequence of moves). That is, we have the replacement rules $ES \rightarrow S^*E$, $EC \rightarrow (S^* \text{ or } CE)$, and $SC \rightarrow CS^*$, where E and C denote expansion and collapse moves respectively and S^* denotes a (possibly empty) sequence of slide moves.

The algorithm for simplifying a sequence of moves is to repeatedly perform either of the following two steps, until neither applies. The first step is to find the first collapse move that is preceded by an expansion or slide, and apply the replacement $EC \rightarrow (S^* \text{ or } CE)$ or $SC \rightarrow CS^*$ accordingly. The second step is to find the last expansion move that is followed by a collapse or slide and apply the replacement $EC \rightarrow (S^* \text{ or } CE)$ or $ES \rightarrow S^*E$. This procedure terminates, in a sequence of the form $C^*S^*E^*$. Then since X and Y are reduced, the new sequence of moves has no collapses or expansions. \square

8. The isomorphism problem

Next we approach the problem of classifying generalized Baumslag–Solitar groups. At the minimum, a classification should include an algorithm for determining when two indexed graphs define the same group. This is the problem considered here.

For certain generalized Baumslag–Solitar groups the isomorphism problem is trivial. This occurs when the deformation space contains only one reduced tree (such a tree is called *rigid*). The basic rigidity theorem for generalized Baumslag–Solitar trees was proved independently in [14, 9, 7] and it states that $(A, i)_{\mathbb{Z}}$ is rigid if there are no divisibility relations at any vertex. Levitt [13] has extended this result by giving a complete characterization of trees that are rigid. Then to solve the isomorphism problem for such groups one simply makes the trees reduced and compares them directly (cf. Remark 2.3).

In this section we solve the isomorphism problem for the case of generalized Baumslag–Solitar groups having no non-trivial integral moduli. The general case is still open.

Lemma 8.1. *Let $Q \subset \mathbb{Q}_{>0}^\times$ be a finitely generated subgroup such that $Q \cap \mathbb{Z} = 1$. Then for any $r \in \mathbb{Q}$ the set $rQ \cap \mathbb{Z}$ is finite.*

Proof. We consider $\mathbb{Q}_{>0}^\times$ as a free \mathbb{Z} -module with basis the prime numbers, via prime decompositions. Note that a positive rational number is an integer if and only if it has nonnegative coordinates in $\mathbb{Q}_{>0}^\times = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$, and so the positive integers comprise the first “orthant” of $\mathbb{Q}_{>0}^\times$.

We are given that Q meets the first orthant only at the origin. By taking tensor products with \mathbb{R} we may think of $\mathbb{Q}_{>0}^\times$ as a vector space and Q a finite dimensional subspace. Since multiplication by r is a translation in $\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$, we have that rQ is an affine subspace parallel to Q . It suffices to show that this affine subspace meets the first orthant in a compact set.

This is clear if Q is a codimension 1 subspace of a coordinate subspace $\mathbb{R} \oplus \cdots \oplus \mathbb{R}$, because the subspace would have a strictly positive normal vector, and then rQ would meet the first orthant of $\mathbb{R} \oplus \cdots \oplus \mathbb{R}$ in a simplex or a point (or not at all). Otherwise we can choose a coordinate subspace $\mathbb{R} \oplus \cdots \oplus \mathbb{R}$ containing Q and then enlarge Q to make it codimension 1, preserving the property that it meets the first orthant only at the origin. The result then follows easily. \square

Theorem 8.2. *Let G be a finitely generated generalized Baumslag–Solitar group. If $q(G) \cap \mathbb{Z} = 1$ then there are only finitely many reduced graphs of groups $(A, i)_{\mathbb{Z}}$ with fundamental group G .*

Proof. If G is elementary then there are only four reduced graphs of groups whose universal covering trees have at most two ends and the result is clear. These are: a single vertex, a loop with indices ± 1 (two cases: equal signs or opposite signs), and an interval with indices ± 2 .

If G is non-elementary then any two reduced trees are related by slide moves, by Theorem 7.4. In particular there are only finitely many possible quotient graphs. Thus we may consider sequences of slide moves in which every edge returns to its original position in the quotient graph. To prove the theorem it then suffices to show that after such a sequence, there are only finitely many possible values for each edge index $i(e)$.

Suppose X is a generalized Baumslag–Solitar tree with group G and e is an edge of X with initial vertex v . Consider a sequence of slide moves after which e has initial vertex gv for some $g \in G$. Then we have $G_e \subset (G_v \cap G_{gv})$. This implies that

$$\begin{aligned} [G_v : G_e] &= [G_v : (G_v \cap G_{gv})] [(G_v \cap G_{gv}) : G_e] \\ &= q(g) [G_{gv} : G_e], \end{aligned}$$

or equivalently $[G_{g^v} : G_e] = [G_v : G_e]q(g^{-1})$. Therefore, since $i(e) = \pm[G_v : G_e]$ before the slide moves, the new index after the moves is an element of the set $\pm i(e)q(G) \cap \mathbb{Z}$. This set is finite by Lemma 8.1. \square

Corollary 8.3. *There is an algorithm which, given finite graphs of groups $(A, i)_{\mathbb{Z}}$ and $(B, j)_{\mathbb{Z}}$ such that $(A, i)_{\mathbb{Z}}$ has no non-trivial integral moduli, determines whether the associated generalized Baumslag–Solitar groups are isomorphic.*

Proof. First make $(A, i)_{\mathbb{Z}}$ and $(B, j)_{\mathbb{Z}}$ reduced. If one or both is elementary then it is a simple matter to check for isomorphism. Among the reduced elementary graphs of groups, a single vertex has group \mathbb{Z} , a loop with both indices equal to 1 has group $\mathbb{Z} \times \mathbb{Z}$, and the remaining two cases yield the Klein bottle group.

Otherwise, by Theorem 7.4, the groups are isomorphic if and only if there is a sequence of slide moves taking $(A, i)_{\mathbb{Z}}$ to $(B, j)_{\mathbb{Z}}$. Now consider the set of graphs of groups related to $(A, i)_{\mathbb{Z}}$ by slide moves. This is the vertex set of a connected graph \mathcal{G} whose edges correspond to slide moves. We claim that every vertex of \mathcal{G} is a reduced graph of groups. This then implies that \mathcal{G} is finite by Theorem 8.2. To prove the claim one observes, using Proposition 2.4, that an edge in a reduced graph of groups cannot be made collapsible during a slide move unless it slides over a strict ascending loop, but there are no such loops because the group has no non-trivial integral moduli.

Now search \mathcal{G} by performing all possible sequences of slide moves of length n , for increasing n until no new graphs of groups are obtained. Then the two generalized Baumslag–Solitar groups are isomorphic if and only if $(B, j)_{\mathbb{Z}}$ has been found by this point. \square

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