The adjunction problem over torsion-free groups

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Abstract We prove injectivity and relative asphericity for "layered" systems of equations over torsion-free groups, when the exponent matrix is invertible over $\mathbb{Z}$.

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A long-standing problem in group theory is the *adjunction problem* of deciding when a given group injects into the group obtained by adjoining new generators and relators [8]. This note solves the adjunction problem over torsion-free groups in the special case in which new generators and relators are added in pairs and such that the exponent matrix is invertible. We prove that in this case the group does inject. The case of one such pair was proved by Klyachko [7]. The extension uses our previous paper [3] and an additional result that follows from a theorem of Bogley and Pride [1], which in turn is based on an old theorem of Serre [6]. We shall give a direct elementary proof of the result that we need.

Let (L, K) be a relative 2–complex (a CW–pair such that $L - K$ is at most 2–dimensional). We say that (L, K) is *relatively aspherical* if the map

$$\pi_2(K \cup L^{(1)}, K) \to \pi_2(L, K)$$

is surjective. As shown in [3, 3.1–3.3], this occurs if and only if conclusions (a) and (b) of Theorem 1 below hold. This is the natural topological notion of asphericity but it should be noted that it differs from the combinatorial notion introduced in [1]. The difference concerns the definition of irreducibility of diagrams representing elements of $\pi_2(L, K)$; see [3].

The fundamental group of L is obtained from $G = \pi_1(K)$ by adding generators $\{t_i\}$ and relators $\{r_j\}$ corresponding to the 1–cells and 2–cells respectively of $L - K$. The relators $r_j \in G * \langle t_1, \ldots, t_n\rangle$ can then be viewed as a system of equations in the variables $\{t_i\}$ with coefficients in G. It is well known (see Howie [5] for example) that the map $\pi_1(K) \to \pi_1(L)$ is injective if and only if the system has a solution in an overgroup of G.

The *exponent matrix* of the system (or of the pair (L,K)) has entries m_{ij} equal to the exponent sum of t_i in the relator r_j. In topological terms it is the 2–dimensional boundary map in the relative cellular chain complex of (L,K).

A long-standing conjecture [5] states that for any relative 2–complex (L,K), if the exponent matrix is nonsingular, then $\pi_1(K) \to \pi_1(L)$ is injective. If we assume further that $\pi_1(K)$ is torsion-free and the exponent matrix is invertible over $\mathbb{Z}$ then we conjecture that (L,K) is also relatively aspherical. (As shown in [3] this conclusion can fail if either of the additional hypotheses is omitted.) Our main result proves this in a special case:

Theorem 1 *Let (L,K) be a layered relative 2–complex with $\pi_1(K)$ torsion-free. If the exponent matrix is invertible over $\mathbb{Z}$ then*

(a) *$\pi_1(K) \to \pi_1(L)$ is injective, and*

(b) *the inclusion-induced map $\mathbb{Z}\pi_1(L) \otimes_{\mathbb{Z}\pi_1(K)} \pi_2(K) \to \pi_2(L)$ is an isomorphism.*

Here, (L,K) is *layered* if $L - K$ has equal numbers of 1– and 2–cells and L is formed from K by alternately adding 1–cells and 2–cells. In terms of the associated relative presentation it means that the generators and relators can be added alternately.

A special case of Theorem 1 was proved in [3]: the theorem was proved when $L - K$ consists of one 1–cell and one 2–cell. In this note we observe that the special case can be applied inductively. It is worth stressing that part (b) of Theorem 1, for the case of one new generator and one new relator, is a non-trivial extension of Klyachko's theorem and it is the key to allowing the inductive argument of this note to proceed.

Proof The layered hypothesis implies that there is a nested sequence of subcomplexes $K = K_0 \subset K_1 \subset \cdots \subset K_n = L$ where $K_{i+1} - K_i$ has one 1–cell and one 2–cell for each i. Note that the exponent matrix for (L,K) is triangular with diagonal entries equal to ± 1, and these diagonal entries represent the 1×1 exponent matrices for the pairs (K_{i+1}, K_i). In particular each pair (K_{i+1}, K_i) is *amenable* in the sense of Fenn and Rourke [2]. Then by the main theorem of [3], the pair (K_{i+1}, K_i) is relatively aspherical provided $\pi_1(K_i)$ is torsion-free.

We are given that $\pi_1(K_0)$ is torsion-free, and so (K_1, K_0) is relatively aspherical. By Proposition 1 below $\pi_1(K_1)$ is then torsion-free. Proceeding inductively, using [3] and Proposition 1, we find that every pair (K_{i+1}, K_i) is relatively aspherical.

It remains to verify that relative asphericity is transitive. Given $K \subset L \subset M$ with $M - K$ at most 2–dimensional, relative asphericity of (M, L) and (L, K) implies

$$\begin{aligned}\pi_2(M) &= \mathbb{Z}\pi_1(M) \otimes_{\mathbb{Z}\pi_1(L)} (\mathbb{Z}\pi_1(L) \otimes_{\mathbb{Z}\pi_1(K)} \pi_2(K)) \\ &= \mathbb{Z}\pi_1(M) \otimes_{\mathbb{Z}\pi_1(K)} \pi_2(K)\end{aligned}$$

so condition (b) holds for (M, K). Condition (a) for (M, K) is clear. □

Remark The proof shows that the exponent matrix hypothesis can be relaxed to allow layered relative 2–complexes for which each pair (K_{i+1}, K_i) is amenable, ie, that the relator given by the new 2–cell has an "amenable t–shape" in terms of the new generator; see [2] or [3]. The result also solves the adjunction problem for systems of generators and relators which can be transformed, by a change of variables, into a layered amenable system.

Proposition 1 *If (L, K) is relatively aspherical and $\pi_1(K)$ is torsion-free then $\pi_1(L)$ is also torsion-free.*

Proof By adding cells of dimension $\geqslant 3$ we can arrange that all the homotopy groups of K vanish in dimensions 2 and above. This does not change the fact that (L, K) is relatively aspherical. The easiest way to see this is to use the diagram interpretation used in [3]: relative asphericity means that there are no irreducible diagrams over $\pi_1(K)$ using the cells of $L - K$. This only depends on $\pi_1(K)$ and the form of the added relators and hence is unchanged by a change in the higher homotopy groups of K. After adding the new cells $\pi_2(L)$ is trivial.

Let $\widetilde{L}$ be the universal cover of L and $\widetilde{K}$ the preimage of K in $\widetilde{L}$. Let $\widehat{L}$ be the 2–complex obtained from $\widetilde{L}$ by collapsing each connected component of $\widetilde{K}$ to a vertex. Since each of these components is contractible, the map $\widetilde{L} \to \widehat{L}$ is a homotopy equivalence, and so $\pi_1(\widehat{L})$ and $\pi_2(\widehat{L})$ are trivial. Then since $\widehat{L}$ is 2–dimensional, it is contractible.

Note that the induced action of $\pi_1(L)$ on $\widehat{L}$ is free away from the 0–skeleton, and the vertices have stabilisers equal to the conjugates of $\pi_1(K)$ in $\pi_1(L)$. We claim that any element $g \in \pi_1(L)$ of prime order fixes a vertex of L. It follows that it lies in a congugate of $\pi_1(K)$, which is torsion-free, and hence must be trivial. Thus $\pi_1(L)$ has no elements of prime order and hence is torsion-free.

Suppose that g has prime order and fixes no vertex. Then we have $\mathbb{Z}/p\mathbb{Z}$ acting freely on $Q = \widehat{L}$ which is a contractible 2–complex. This is well known to be impossible, as $Q/(\mathbb{Z}/p\mathbb{Z})$ would then be a finite-dimensional $K(\mathbb{Z}/p\mathbb{Z}, 1)$ space. □

Remark Proposition 1 can be extended to show that if (L, K) is relatively aspherical then every finite subgroup of $\pi_1(L)$ is contained in a unique conjugate of $\pi_1(K)$. This result (in a more general form, based on the weaker notion of relative asphericity) was first proved by Bogley and Pride [1], using a theorem of Serre [6] whose proof depends on Tate cohomology. In [4] we give elementary geometric proofs (along the lines of the proof above) of the Bogley–Pride result and of Serre's theorem.

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