

THE DEHN FUNCTIONS OF STALLINGS–BIERI GROUPS

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ABSTRACT. We show that the Stallings–Bieri groups, along with certain other Bestvina–Brady groups, have quadratic Dehn function.

1. INTRODUCTION

Every simplicial graph Γ defines a *right-angled Artin group* A_Γ , whose generators correspond to vertices of Γ and where two such generators commute if and only if their vertices bound an edge in Γ ; see Section 2 for details.

Given A_Γ , there is a surjective homomorphism $\varphi: A_\Gamma \rightarrow \mathbb{Z}$ sending each standard generator to 1. The *Bestvina–Brady group* associated to A_Γ , denoted BB_Γ , is defined to be the kernel of φ .

In this paper, we are concerned with the isoperimetric behavior of certain Bestvina–Brady groups. Dison [12] has shown that every Bestvina–Brady group satisfies a quartic isoperimetric inequality. The Dehn function of such a group (that is, the *optimal* isoperimetric function) may be quartic [1], or it may be smaller. The examples of primary interest to us are the *Stallings–Bieri groups*. Some time ago, Gersten [15] established a quintic isoperimetric bound for these groups and inquired whether their Dehn functions might be smaller. Bridson argued in [8] that these Dehn functions should be quadratic. However, Bridson and Groves later found an error and observed that the method only provided a cubic bound [7, 17]. In this paper, we give a proof of Bridson’s claim.

For $n \geq 1$, the *Stallings–Bieri group* SB_n is defined to be the Bestvina–Brady group associated with the right-angled Artin group $F_2 \times \cdots \times F_2$, where there are n factors F_2 . That is, SB_n is equal to BB_Γ , where Γ is the join of n copies of S^0 (the graph with two vertices and no edges). The groups SB_n are notable for their homological finiteness properties. Recall that a group G is said to be *of type* \mathcal{F}_n if there is a $K(G, 1)$ with finite n -skeleton. Bieri [5] has shown that SB_n is of type \mathcal{F}_{n-1} but not of type \mathcal{F}_n .

Note that SB_1 and SB_2 are not finitely presented. The first of the groups SB_n for which the Dehn function is defined is SB_3 , also known as Stallings’ group [18]. Dison, Elder, Riley, and Young [13] proved that SB_3 has quadratic Dehn function. Their method makes use of a particular presentation for SB_3 given in [3, 15], and is essentially algebraic in nature. Their technique does not appear to generalize easily to the other groups SB_n .

In this paper, we approach the Dehn function of SB_n from a geometric point of view, by considering these groups as level sets in products of CAT(0) spaces and using the ambient CAT(0) geometry.

The specific setting of our main result is that of cube complexes with height functions. Our notion of height function is specific to cube complexes and is slightly more restrictive than the combinatorial Morse functions of [4]; see Section 2. If $h: X \rightarrow \mathbb{R}$ is a height function on a cube complex X , we denote by $[X]_0$ the level set $h^{-1}(0)$. Our main theorem is the following:

Theorem 4.2. *Suppose $\alpha \geq 2$ and let $X_1, X_2,$ and X_3 be simply connected cube complexes with height functions such that each X_i is admissible and has finite-valued Dehn function $\preceq n^\alpha$. Then $[X_1 \times X_2 \times X_3]_0$ is simply connected and has Dehn function $\preceq n^\alpha$.*

Since right-angled Artin groups act naturally on CAT(0) cube complexes with height functions, one easily obtains the following corollary.

Corollary 4.3. *Suppose $\Gamma = \Gamma_1 * \Gamma_2 * \Gamma_3$, so that A_Γ is the product $A_{\Gamma_1} \times A_{\Gamma_2} \times A_{\Gamma_3}$. Then the Bestvina-Brady group BB_Γ has quadratic Dehn function.*

In particular, SB_n has quadratic Dehn function for every $n \geq 3$. This yields new information on the class of groups having quadratic Dehn function:

Corollary 1.1. *For each $n \geq 3$ there exist groups with quadratic Dehn function that are of type \mathcal{F}_{n-1} but not of type \mathcal{F}_n .*

Methods. As mentioned above, our basic viewpoint is the study of level sets in products of CAT(0) spaces, or of more general spaces. The case of horospheres in products has been well studied, for instance by Gromov [16] and Druţu [14], and our approach here uses similar ideas.

If one is looking at a product of CAT(0) spaces with a height function, it is important to note that the zero-level set is not generally CAT(0). However, thanks to the product structure, we will be able to find many overlapping CAT(0) *subspaces* within the zero-level set. An example of this phenomenon, discussed in [16, 2.B(f)], is the solvable Lie group Sol_5 considered as a horosphere in $X = \mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^2$. The height function in this case is a Busemann function on X . While Sol_5 is certainly not non-positively curved, it contains many isometrically embedded copies of $\mathbb{H}^2 \times \mathbb{H}^2$. Indeed, there are three transverse copies passing through every point.

In our setting of cube complexes with height functions, the appropriate subspaces are found using the Embedding Lemma (3.6). We insist on using a particular cell structure (the *sliced cell structure*, see Section 3). Then the lemma produces subcomplexes of

the level set that are *combinatorially isomorphic* to the factors of the ambient product space. These subcomplexes can be found in abundance, if the factor cube complexes are “admissible” (see Section 4).

The next basic technique that we use comes from Gromov [16, 5.A₃’], in which a disk is filled using triangular regions whose areas are controlled by a geometric series. Doing so depends on using a particular triangulation of the disk, shown in Figure 3. We note that Young has formalized this idea in his notion of a *template* [19], although here we do not require the full generality of Young’s notion.

The main body of our argument entails showing how to fill the triangular regions of the template, using the subcomplexes of the level set provided by the Embedding Lemma. This is achieved using the scheme shown in Figure 1.

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2. PRELIMINARIES

Dehn functions. Let X be a simply connected cell complex. Given a closed edge path $p: S^1 \rightarrow X$, the *filling area* of p , informally, is the minimal number of 2-cells of X that must be crossed in a nullhomotopy of p in the 2-skeleton of X . One way to formalize this notion is to use *admissible maps* as in [6]. A map $f: D^2 \rightarrow X$ is *admissible* if its image lies in $X^{(2)}$ and the preimage of each open 2-cell is a disjoint union of open disks in D^2 , each mapping homeomorphically to its image. The *area* of f is the total number of preimage disks. We define the filling area of p to be

$$\text{FA}(p) = \min\{\text{Area}(f) \mid f \text{ is an admissible map extending } p\}.$$

The *Dehn function* of X is the function $\delta_X: \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ given by

$$\delta_X(n) = \sup\{\text{FA}(p) \mid p \text{ is a closed edge path of length } \leq n\},$$

where *length* is the number of edges traversed by p .

There is an equivalence relation on monotone functions $f: \mathbb{N} \rightarrow \mathbb{N}$, where we say that $f \simeq g$ if $f \preceq g$ and $g \preceq f$. Here, $f \preceq g$ means that there is a constant C such that $f(n) \leq Cg(Cn + C) + Cn + C$ for all n .

If G is a finitely presented group, then the Dehn function of any Cayley 2-complex for G takes values in \mathbb{N} . Moreover, any two Cayley 2-complexes will be quasi-isometric and their Dehn functions will be equivalent. The equivalence class of this function is, by definition, the Dehn function of G . See [9] for more background on Dehn functions, including various alternative definitions.

Right-angled Artin groups. Given a simplicial graph Γ with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, the *right-angled Artin group* A_Γ is the group with generating set $\{a_v\}_{v \in V(\Gamma)}$ and relations $\mathcal{R} = \{[a_v, a_u] \mid \{v, u\} \in E(\Gamma)\}$.

Following [11], there is a natural model for $K(A_\Gamma, 1)$ which is a subcomplex of a torus. Let $T^{V(\Gamma)}$ denote the product of copies of the circle, one for each vertex of Γ . For any subset $U \subset V(\Gamma)$ let T^U be the sub-torus spanned by the circle factors corresponding to U . Let $K(\Gamma)$ denote the set of subsets of $V(\Gamma)$ which span complete subgraphs of Γ . Then we define

$$T_\Gamma = \bigcup_{U \in K(\Gamma)} T^U.$$

This subcomplex of $T^{V(\Gamma)}$ is aspherical and has fundamental group A_Γ . It has a piecewise Euclidean cubical structure satisfying Gromov's link condition. Thus it has non-positive curvature, and the universal cover is a CAT(0) cube complex. We will denote this universal cover by X_Γ .

For each $U \in K(\Gamma)$ the preimage of T^U in X_Γ is a disjoint union of isometrically embedded copies of \mathbb{R}^U , which we will call *U-flats*, or *coordinate flats*.

Height functions. Let X be a cube complex. A *height function* on X is a continuous map $h: X \rightarrow \mathbb{R}$ which is affine on each cube and takes each edge to an interval of the form $[n, n+1]$ with $n \in \mathbb{Z}$.

If X_1, \dots, X_n are cube complexes with height functions h_i on X_i , then

$$h(x_1, \dots, x_n) = h_1(x_1) + \dots + h_n(x_n) \tag{2.1}$$

defines a height function on $X_1 \times \dots \times X_n$. Unless stated otherwise, a product of cube complexes with height functions will be given this height function by default.

If $h: X \rightarrow \mathbb{R}$ is a height function, we denote by $[X]_0$ the level set $h^{-1}(0)$.

In the case of X_Γ , a height function can be defined as follows. Choose a base vertex in X_Γ . Consider the linear map $\mathbb{R}^{V(\Gamma)} \rightarrow \mathbb{R}$ which takes each standard basis vector to 1 in \mathbb{R} . This map descends to a map $T^{V(\Gamma)} \rightarrow S^1$, which restricts to a map $T_\Gamma \rightarrow S^1$. This latter map induces the homomorphism $\varphi: A_\Gamma \rightarrow \mathbb{Z}$ sending each generator a_v to 1. The desired height function

$$h_\Gamma: X_\Gamma \rightarrow \mathbb{R}$$

is the unique lift of the map $T_\Gamma \rightarrow S^1$ which takes the base vertex of X_Γ to 0. Moreover, this height function is φ -equivariant. For more details on h_Γ and φ , see [4, 5.12].

Remark 2.2. If Γ is an n -fold join $\Gamma_1 * \dots * \Gamma_n$, then $A_\Gamma = A_{\Gamma_1} \times \dots \times A_{\Gamma_n}$ and X_Γ is the product cube complex $X_{\Gamma_1} \times \dots \times X_{\Gamma_n}$. Choosing basepoints in each X_{Γ_i} defines height

functions h_{Γ_i} . Using the product basepoint, h_{Γ} then agrees with the height function (2.1) built from the functions h_{Γ_i} .

3. THE EMBEDDING LEMMA

The sliced cell structure. Let X be a cube complex with a height function h . The *sliced cell structure* on X is obtained by subdividing each cube of X along the hyperplanes $h^{-1}(n)$ for each $n \in \mathbb{Z}$. Each d -dimensional cube is split into d convex polytopes of dimension d , which are affinely equivalent to *hypersimplices* (see Remark 3.4 below).

There are two types of cells in the sliced cell structure. *Horizontal cells* are those whose image under h is a point. The rest are *transverse cells*; each of these is a piece of a cube of the same dimension, and maps to an interval $[n, n + 1]$ under h .

Whenever we have a cube complex X with a height function, we will assume that X has been given the sliced cell structure, unless stated otherwise. We may refer to it as a *sliced cube complex* to emphasize this assumption.

Note that the level set $[X]_0$ is a subcomplex of X with this structure.

Lemma 3.1. *Let X be a sliced cube complex with height function h . Give \mathbb{R} the structure of a cube complex with vertices at the integers. Then $X \times \mathbb{R}$ is a cube complex with height function $H(x, t) = h(x) + t$. Define the function $f: X \rightarrow X \times \mathbb{R}$ by $f(x) = (x, -h(x))$. Then f is a combinatorial isomorphism of X onto the subcomplex $[X \times \mathbb{R}]_0$.*

Proof. The claim that H is a height function follows from (2.1), since the identity is a height function on \mathbb{R} .

For the main conclusion, it is clear that f is a homeomorphism from X to $[X \times \mathbb{R}]_0$, with inverse given by projection onto the first factor. It remains to show that each d -cell in X maps bijectively to a d -cell in $[X \times \mathbb{R}]_0$. We will show that this holds for each *transverse d -cell*, and moreover that the polyhedral structure of the cell is preserved. Then, since horizontal cells are faces of transverse cells, it follows that horizontal cells also map as desired.

Let σ be a transverse d -cell contained in a d -dimensional cube $C \subset X$. There is a parametrization of C as $[0, 1]^d$ such that $h|_C$ is given by $h(x_1, \dots, x_d) = x_1 + \dots + x_d + N$ for some $N \in \mathbb{Z}$. Then σ is defined by the inequalities

$$0 \leq x_i \leq 1 \quad (i = 1, \dots, d), \quad k \leq x_1 + \dots + x_d \leq k + 1 \quad (3.2)$$

for some $k \in \{0, \dots, d - 1\}$.

The image $f(\sigma)$ lies in the set $C \times \mathbb{R}$ with coordinates x_1, \dots, x_d, t . On the cube C , f is given by

$$(x_1, \dots, x_d) \mapsto (x_1, \dots, x_d, -x_1 - \dots - x_d - N). \quad (3.3)$$

Under this map, the region (3.2) maps onto the region

$$\begin{aligned} 0 \leq x_i \leq 1 \quad (i = 1, \dots, d), \quad -k - N - 1 \leq t \leq -k - N, \\ x_1 + \dots + x_d + N + t = 0. \end{aligned}$$

But this is simply the 0-level set of the cube $[0, 1]^d \times [\ell, \ell + 1]$, for $\ell = -k - N - 1$, with respect to the height function on $X \times \mathbb{R}$. That is, $f(\sigma)$ is a horizontal d -cell of $X \times \mathbb{R}$ at height 0.

Finally, note that the description (3.3) of f shows that $f|_C: C \rightarrow C \times \mathbb{R}$ is the restriction of an injective affine linear map $\mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$, and such a map will preserve the combinatorial structure of any convex polyhedron. \square

Remark 3.4. In the case $X = \mathbb{R}^n$ with its standard cubical structure and height function $h(x_1, \dots, x_n) = \sum_i x_i$, the image $[\mathbb{R}^n \times \mathbb{R}]_0$ is \mathbb{R}^n tessellated by hypersimplices. See [2, Section 3.3] for a description of this tessellation and its cells. The map $f: \mathbb{R}^n \rightarrow [\mathbb{R}^n \times \mathbb{R}]_0$ is affine linear.

Definition 3.5. Let X be a cube complex with height function h . A *monotone line* in X is a 1-dimensional subcomplex $L \subset X$ such that $h|_L: L \rightarrow \mathbb{R}$ is a homeomorphism.

Lemma 3.6 (Embedding Lemma). *Let X and Y be sliced cube complexes with height functions h_X, h_Y , and let $L \subset Y$ be a monotone line. The function*

$$f_L(x) = (x, (h_Y|_L)^{-1}(-h_X(x)))$$

is a combinatorial embedding of X into $[X \times Y]_0$, with image $[X \times L]_0$.

In particular, $[X \times L]_0$ is combinatorially isomorphic to X .

Proof. The map f_L is the composition of the combinatorial embedding $f: X \rightarrow X \times \mathbb{R}$ with image $[X \times \mathbb{R}]_0$ given by Lemma 3.1, and the height-preserving combinatorial embedding $X \times \mathbb{R} \rightarrow X \times Y$ given by $\text{id} \times (h_Y|_L)^{-1}$. \square

4. FILLING DISKS IN LEVEL SETS

Definition 4.1. A cube complex X with a height function is *admissible* if every vertex of X is contained in a monotone line.

For any right-angled Artin group A_Γ , the cube complex X_Γ is admissible. Pick any vertex $v \in V(\Gamma)$, and note that every vertex of X_Γ has a $\{v\}$ -flat passing through it, and such a coordinate flat will be a monotone line for the height function h_Γ .

The following result is our main theorem.

Theorem 4.2. *Suppose $\alpha \geq 2$ and let X_1 , X_2 , and X_3 be simply connected cube complexes with height functions such that each X_i is admissible and has finite-valued Dehn function $\preceq n^\alpha$. Then $[X_1 \times X_2 \times X_3]_0$ is simply connected and has Dehn function $\preceq n^\alpha$.*

Corollary 4.3. *Suppose $\Gamma = \Gamma_1 * \Gamma_2 * \Gamma_3$, so that A_Γ is the product $A_{\Gamma_1} \times A_{\Gamma_2} \times A_{\Gamma_3}$. Then the Bestvina-Brady group BB_Γ has quadratic Dehn function.*

In particular, SB_n has quadratic Dehn function for every $n \geq 3$, since it equals BB_Γ where Γ is the join of n copies of S^0 .

Proof. We have $X_\Gamma = X_{\Gamma_1} \times X_{\Gamma_2} \times X_{\Gamma_3}$ where each X_{Γ_i} is CAT(0), and therefore has Dehn function which is at most quadratic, by [10, Proposition III.Γ.1.6]. Theorem 4.2 then says that the Dehn function of $[X_\Gamma]_0$ is at most quadratic. It is at least quadratic because it contains 2–dimensional quasi-flats, namely the zero-level sets of any product of three monotone lines in the factors. Finally, note that $[X_\Gamma]_0$ is a geometric model for BB_Γ , as follows. It is simply connected, by Theorem 4.2, and is acted on freely by BB_Γ , with quotient a finite cell complex. Hence its Dehn function is the Dehn function of BB_Γ . \square

For the rest of this section, let X_1 , X_2 , and X_3 be as in the statement of the theorem. These cube complexes will be left *unsliced*, for the purpose of estimating distances, though products will always be given the sliced cell structure.

Let $X = X_1 \times X_2 \times X_3$ and let $d_0(\cdot, \cdot)$ be the combinatorial metric on the 1–skeleton of $[X]_0$. This is the path metric obtained by declaring each edge to be isometric to an interval of length 1. Let $d_{X_i}(\cdot, \cdot)$ be the combinatorial metric on the 1–skeleton of the (unsliced) cube complex X_i .

Consider for a moment the 1–skeleton of the unsliced cube complex $X_1 \times X_2 \times X_3$. Its combinatorial metric is given by $d(a, b) = d_{X_1}(a_1, b_1) + d_{X_2}(a_2, b_2) + d_{X_3}(a_3, b_3)$ where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$. Given an edge path in $[X]_0$, every edge in the path can be replaced by a path of length 2 in $X_1 \times X_2 \times X_3$, which yields the following inequality:

$$d_{X_1}(a_1, b_1) + d_{X_2}(a_2, b_2) + d_{X_3}(a_3, b_3) \leq 2d_0(a, b). \quad (4.4)$$

Spanning triangles. Here we give a construction of a triangular loop in $[X]_0$ and a filling of that loop by a topological disk in $[X]_0$. The starting data are: three vertices $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$, and $c = (c_1, c_2, c_3)$ in $[X]_0$ which will be the corners of the triangle, and three monotone lines $L_i \subset X_i$ ($i = 1, 2, 3$) such that $a_1 \in L_1$, $b_2 \in L_2$, and $c_3 \in L_3$.

Figure 1 shows the triangular loop and some vertices and paths which will form part of the 1–skeleton of the filling disk. In addition to the original three corner vertices, there are six “side vertices” and three interior vertices. Their coordinates in X are as indicated in the figure. The side vertices have the property that two of their coordinates are points

lying in the monotone lines. The interior vertices have all three coordinates lying in the monotone lines.

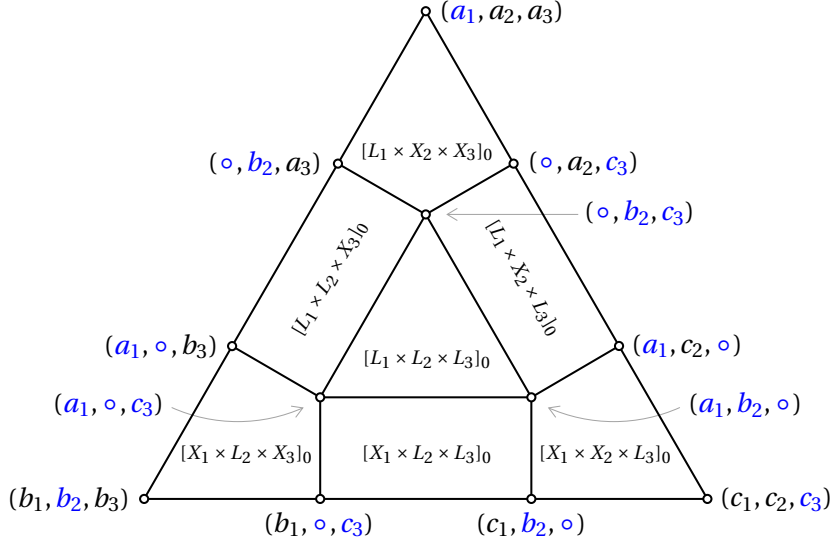


FIGURE 1. A spanning triangle for the vertices $a, b, c \in [X]_0$. A blue entry indicates a point which is known to lie in the monotone line in that factor. A blue circle represents the unique point in the monotone line for which the triple has height 0. The labels in the regions indicate that each path in the boundary of that region lies within the indicated subspace.

There are four types of paths in this figure: six paths joining corner vertices to side vertices, three paths between adjacent side vertices, six paths from side vertices to interior vertices, and three interior paths. Consider first the path between (a_1, a_2, a_3) and (\circ, b_2, a_3) in the upper left part of the figure. We may write the second vertex as (a'_1, b_2, a_3) where a'_1 is the unique point on L_1 such that the triple has height 0. There is an edge path in X_2 from a_2 to b_2 of length $d_{X_2}(a_2, b_2)$. Combine this with the constant path a_3 in X_3 , and the path in L_1 from a_1 to a'_1 which compensates for the height changes in X_2 , keeping the path in $[X]_0$. Put another way, this path in $[X]_0$ is the image of the path in $X_2 \times X_3$ under the identification with $[L_1 \times X_2 \times X_3]_0$ given by the Embedding Lemma.

Next consider the path from (a'_1, b_2, a_3) to the interior vertex $(\circ, b_2, c_3) = (a''_1, b_2, c_3)$. This is defined by combining the constant path b_2 in L_2 with an edge path in X_3 from a_3 to c_3 of length $d_{X_3}(a_3, c_3)$, and interpreting as a path in $[L_1 \times L_2 \times X_3]_0$ via the isomorphism with $L_2 \times X_3$.

To define the path in $[L_1 \times L_2 \times X_3]_0$ from (a'_1, b_2, a_3) to $(a_1, \circ, b_3) = (a_1, b'_2, b_3)$ we proceed in a somewhat non-canonical manner. First use a path in X_3 from a_3 to b_3 , of length

$d_{X_3}(a_3, b_3)$, combined with the constant path b_2 in L_2 , and a compensating motion in L_1 , ending at a point (a_1''', b_2, b_3) . Then (a_1, b_2') and (a_1''', b_2) have the same height in $L_1 \times L_2$, so we can move horizontally from one to the other, in $d_{L_2}(b_2', b_2)$ steps. This path combines with the constant path b_3 in X_3 to give a path in $[X]_0$. Note that $d_{L_2}(b_2', b_2) \leq d_{X_1}(a_1, b_1)$ since the path from (a_1, b_2', b_3) to (b_1, b_2, b_3) (of length $d_{X_1}(a_1, b_1)$) projects to a path in X_2 from b_2' to b_2 .

Lastly consider a path in $[L_1 \times L_2 \times L_3]_0$ between the interior vertices (a_1'', b_2, c_3) and $(a_1, \circ, c_3) = (a_1, b_2'', c_3)$. Use a path in $L_1 \times L_2$ from (a_1'', b_2) to (a_1, b_2'') of length at most $d_{X_1}(a_1, a_1'') + d_{X_2}(b_2, b_2'')$, and embed as a path in $[L_1 \times L_2 \times L_3]_0$. We have seen already that $d_{X_1}(a_1, a_1'') \leq d_{X_2}(a_2, b_2) + d_{X_3}(a_3, c_3)$ and $d_{X_2}(b_2, b_2'') \leq d_{X_1}(a_1, b_1) + d_{X_3}(b_3, c_3)$.

The rest of the paths are defined analogously according to their types. The length information just discussed is collected in Figure 2. We use the shorthand $|a_i - b_i| = d_{X_i}(a_i, b_i)$.

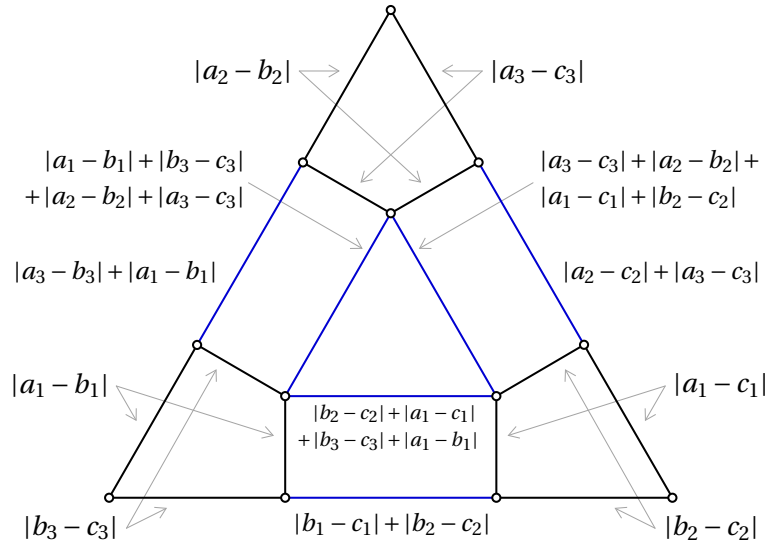


FIGURE 2. Lengths in the spanning triangle. For the blue paths, the lengths given are upper bounds only.

Definition 4.5. The quantity $d_0(a, b) + d_0(b, c) + d_0(c, a)$ will be called the *taut perimeter* of the spanning triangle (not to be confused with the actual perimeter).

Remark 4.6. From Figure 2 and (4.4) we can conclude:

- (1) The side from a to b has length at most $4d_0(a, b)$. Similarly, the other two sides have lengths at most $4d_0(b, c)$ and $4d_0(c, a)$.

- (2) Each of the seven regions has perimeter at most $4P$, where P is the taut perimeter of the spanning triangle.

Remark 4.7. Each of the subcomplexes labelling a region in Figure 1 is combinatorially isomorphic to one of the (sliced) cell complexes $X_i \times X_j$, $L_i \times X_j$, or $L_i \times L_j$, by the Embedding Lemma. All of these have Dehn functions that are $\preceq n^\alpha$ (here we use the assumption that $\alpha \geq 2$). Let C be chosen so that $f(n) = Cn^\alpha$ is an upper bound for these Dehn functions. Then, by Remark 4.6(2), the triangle has filling area at most $7C(4P)^\alpha$ in $[X]_0$, where P is the taut perimeter.

Remark 4.8. The definition of the path along any side of the triangle depends only on its endpoints, and a choice of direction along the side (for the non-canonical path in the middle segment). Given two triples of vertices a, b, c and a, b, c' , spanning triangles for both can be made to agree along their sides from a to b , by choosing the same direction on those sides.

A path along a side of a spanning triangle will be called a *spanning path*.

Short spanning paths. Consider the spanning path from a to b in Figure 1, and suppose that a and b have distance at most 1 in $[X]_0$. If $a = b$ then the spanning path is a constant path, of length 0. If $a \neq b$, the geodesic from a to b is a single edge in $[X]_0$, and we need to examine how this path may differ from the spanning path.

Lemma 4.9. *If a and b have distance 1 in $[X]_0$ then the spanning path from a to b and the geodesic edge from a to b together form a loop with filling area at most 4 in $[X]_0$.*

Proof. Since their distance is 1, the points a and b differ in either one or two coordinates. If they differ in only one coordinate, then one finds that two of the three segments making up the spanning path are constant paths. (There are three cases, according to the coordinate where $a_i \neq b_i$.) The remaining segment has length 1, as shown by Figure 2. Thus, up to reparametrization, the spanning path agrees with the geodesic path and the filling area is 0.

Now suppose that a and b differ in two coordinates. Recall that the spanning path joins the following points, in order: (a_1, a_2, a_3) , (a'_1, b_2, a_3) , (a'''_1, b_2, b_3) , (a_1, b'_2, b_3) , and (b_1, b_2, b_3) . There are now three cases. If $a_1 = b_1$, then one also finds that $a'''_1 = a_1 = b_1$ and $b'_2 = b_2$. The image of the spanning path consists of two edges in the boundary of the cube $[a'_1, a_1] \times [a_2, b_2] \times [b_3, a_3] \subset X$, from (a_1, a_2, a_3) to (a'_1, b_2, a_3) to (a_1, b_2, b_3) . Together with the geodesic edge, these edges are the boundary of a horizontal 2-cell in $[X]_0$, and the filling area is 1.

If $a_2 = b_2$ then one also has $a'_1 = a_1$. Let $[a'''_1, b_1]$ denote the length two path in X_1 from a'''_1 to b_1 , with midpoint a_1 . Then $[a'''_1, b_1] \times [b'_2, a_2] \times [a_3, b_3]$ is a union of two cubes in X .

The image of the spanning path consists of three edges on the boundary of these cubes. Together with the geodesic edge, they form a quadrilateral that bounds two triangles in $[X]_0$.

If $a_3 = b_3$ then one also has $a'_1 = a'''_1$. Let $[a'_1, b_1] \subset X_1$ and $[a_2, b'_2] \subset X_2$ denote the paths of length 2, with midpoints a_1 and b_2 respectively. The image of the spanning path consists of three edges in $[a'_1, b_1] \times [a_2, b'_2] \times \{a_3\}$, and the geodesic edge also lies in this subset. Choose a vertex $a'_3 \in X_3$ such that $d_{X_3}(a_3, a'_3) = 1$ and $(a_1, b_2, a'_3) \in [X]_0$. Such a vertex exists because X_3 is admissible. Then $[a'_1, b_1] \times [a_2, b'_2] \times [a_3, a'_3] \subset X$ is a union of four cubes in which the spanning path and geodesic path bound a disk made of four horizontal triangles, with common vertex (a_1, b_2, a'_3) . \square

Proof of Theorem 4.2. Simple connectedness of $[X]_0$ will follow from the remainder of the proof, in which we construct disks in $[X]_0$ filling any given loop.

Let p be a closed edge path in $[X]_0$ of length $n > 3$. There is a number $k \in \mathbb{N}$ such that $3 \cdot 2^{k-1} < n \leq 3 \cdot 2^k$. Let \hat{p} be a path of length $\hat{n} = 3 \cdot 2^k$ obtained by padding p with steps that move distance 0. Note that p and \hat{p} have the same filling area.

Now consider the triangulated disk D shown in Figure 3. It has \hat{n} vertices along its boundary, \hat{n} bigons around the outside, and $3 \cdot 2^k - 2$ triangles. Each triangle has a *depth*, where the central triangle has depth 0, its neighbors have depth 1, and so on. For $i = 1, \dots, k$ there are $3 \cdot 2^{i-1}$ triangles of depth i , and k is the maximum depth that occurs.

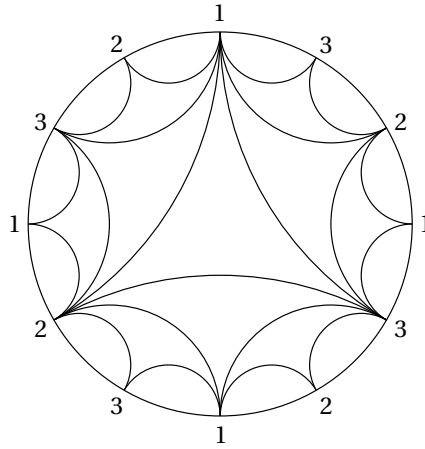


FIGURE 3. Filling the loop with spanning triangles. A 3-coloring of the vertices is shown.

There is a 3-coloring of the vertices of D : each vertex v can be assigned a coordinate $\kappa(v) \in \{1, 2, 3\}$ such that $\kappa(v) \neq \kappa(u)$ whenever u, v bound an edge in D . Now identify the boundary of D with the path \hat{p} . Each vertex of D is identified with a vertex in $[X]_0$ and

has coordinates in X . Writing $v = (v_1, v_2, v_3)$, choose a monotone line L_v in $X_{\kappa(v)}$ which contains the point $v_{\kappa(v)}$.

Each triangle in D can now be filled with a spanning triangle for its vertices, using the three vertices and the three monotone lines chosen for those vertices. The 3-coloring ensures that this data conforms to the requirements of the starting data for spanning triangles. We start by filling the central triangle, and then proceed to fill triangles in order of depth. Each new triangle to be filled meets the previously filled triangles in a single edge, so by Remark 4.8, the spanning triangle can be chosen to use the same spanning path for that edge. Then the spanning triangles fit together to yield a filling of D , minus the bigons.

Declare the *depth* of an interior edge in D to be the minimum of the depths of its neighboring triangles. Note that an edge of depth i joins points on the boundary that bound a boundary arc of length 2^{k-i} , and so these points have distance at most 2^{k-i} in $[X]_0$.

The central triangle has taut perimeter at most \hat{n} , and the taut perimeter of a depth i spanning triangle ($i = 1, \dots, k$) is at most $2^{k-(i-1)} + 2^{k-i} + 2^{k-i} = 2^{k-i+2}$. By Remark 4.7 the central spanning triangle has area at most $7C(4\hat{n})^\alpha = 7 \cdot 12^\alpha C \cdot 2^{k\alpha}$ and a depth i spanning triangle has area at most $7 \cdot 4^\alpha C (2^{k-i+2})^\alpha$. Now the total area of the spanning triangles is at most

$$\begin{aligned} 7 \cdot 12^\alpha C \cdot 2^{k\alpha} + \sum_{i=1}^k 3 \cdot 2^{i-1} \cdot 7 \cdot 4^\alpha C (2^{k-i+2})^\alpha &= 7 \cdot 12^\alpha C \cdot 2^{k\alpha} + 21C \cdot 2^{k\alpha+4\alpha-1} \sum_{i=1}^k 2^{(1-\alpha)i} \\ &< 7 \cdot 12^\alpha C \cdot 2^{k\alpha} + 21C \cdot 2^{k\alpha+4\alpha-1} \\ &< 28C \cdot 12^{4\alpha} \cdot 2^{k\alpha}. \end{aligned}$$

Each bigon has filling area at most 4, by Lemma 4.9. Then the filling area of p is at most

$$\begin{aligned} 28C \cdot 12^{4\alpha} \cdot 2^{k\alpha} + 4 \cdot 3 \cdot 2^k &< (28C + 1) \cdot 12^{4\alpha} \cdot 2^{k\alpha} \\ &= (28C + 1) \cdot 12^{4\alpha} \cdot 2^\alpha \cdot 3^{-\alpha} \cdot (3 \cdot 2^{k-1})^\alpha \\ &< ((28C + 1) \cdot 12^{4\alpha} \cdot 2^\alpha \cdot 3^{-\alpha}) n^\alpha \end{aligned}$$

since $3 \cdot 2^{k-1} < n$. Therefore $\delta_{[X]_0}(n) \leq Kn^\alpha$ where $K = (28C + 1) \cdot 12^{4\alpha} \cdot 2^\alpha \cdot 3^{-\alpha}$. \square

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