ON WELL-ALIGNED ELEMENTS OF GROUPS ACTING ON TREES

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1. INTRODUCTION

Suppose G acts without inversion on a simplicial tree T. In [CFL16] the notion of a *well-aligned* hyperbolic element was defined: g is well-aligned if there does not exist an element $h \in G$ such that $ghgh^{-1}$ fixes an edge of the axis T_g. Equivalently, g is well-aligned if, whenever T_g and $h(T_g)$ overlap with inconsistent orientations, the length of the overlap is at most $|g|$. See Figure 1. It was then shown (Theorem 6.9) that if a hyperbolic element g is well-aligned, then $\mathrm{scl}(g) \geqslant 1/12$.

In [CF10] a lower bound on $\mathrm{scl}(g)$ was established when g is a hyperbolic element of an amalgam $G = A *_C B$ satisfying the "double coset condition". Namely, suppose g is represented by the cyclically reduced word w. If the double coset CgC does not contain any element represented by a cyclic conjugate of w^{-1}, then $\mathrm{scl}(g) \geqslant 1/624$.

In [CFL16] it was claimed that being well-aligned is the G–tree analogue of the double coset condition. The purpose of this note is to explain and justify that statement.

Proposition 1.1. *Let T be the Bass–Serre tree associated with the amalgamated free product $G = A *_C B$. Suppose w is a cyclically reduced word of length > 1 in the generators $A \cup B$, representing $g \in G$. The following are equivalent:*

(1) *g is well-aligned;*
(2) *CgC does not contain any element represented by a cyclic conjugate of w^{-1}.*

2. THE BASS–SERRE TREE OF AN AMALGAM

Every element of $G = A *_C B$ either is in C or can be expressed as an alternating product

$$g = a_1 b_2 a_3 \cdots h_n \quad \text{or} \quad g = b_1 a_2 b_3 \cdots h_n$$

with $a_i \in A - C$ and $b_i \in B - C$ for all i, and $h_n = a_n$ or b_n. Such an expression is called a *reduced word* for g. It is *cyclically reduced* if n is even. The word is of *type AA* if it begins with a_1 and ends with a_n, of *type AB* if it begins with a_1 and ends with b_n, of *type BB* if it begins with b_1 and ends with b_n, and of *type BA* if it begins with b_1 and ends with a_n.

The Bass–Serre tree T for $G = A *_C B$ has a distinguished edge e such that $G_e = C$, $G_{\partial_0 e} = B$, and $G_{\partial_1 e} = A$. Let $g = h_1 h_2 \cdots h_n$ be a non-trivial reduced alternating word. For each i let $g_i = h_1 \cdots h_i$. One verifies easily that if the word $h_1 \cdots h_n$ is of type AA or AB, then the edges

$$e,\ g_1\overline{e},\ g_2 e,\ g_3\overline{e},\ \ldots,\ g_{2j-1}\overline{e},\ g_{2j}e,\ \cdots \qquad \textbf{(2.1)}$$

form an oriented path in T without backtracking, with initial endpoint $\partial_0 e$. If the word is of type BA or BB, then

$$\overline{e},\ g_1 e,\ g_2\overline{e},\ g_3 e,\ \dots,\ g_{2j-1}e,\ g_{2j}\overline{e},\ \cdots \tag{2.2}$$

is an oriented path without backtracking, with initial endpoint $\partial_1 e$. In either case, e and $g_i e$ are coherently oriented if and only if i is even. It follows that $g = h_1 \cdots h_n$ is cyclically reduced if and only if g is hyperbolic and e is on the axis T_g. (Note that g may still be hyperbolic if n is odd; in this case e is not on the axis.)

In the following lemma, e is the distinguished edge discussed above.

Lemma 2.3. *Suppose $g = h_1 h_2 \cdots h_n$ and $g' = h'_1 h'_2 \cdots h'_m$ are elements of $G - C$ expressed as non-trivial alternating words. Suppose $ge = g'e$. Then g and g' are of the same type, $m = n$, and $(h_1 \cdots h_i)^{-1}(h'_1 \cdots h'_i) \in C$ for each i.*

Proof. Let $g_i = h_1 \cdots h_i$ and $g'_i = h'_1 \cdots h'_i$ for each i. The sequence in (2.1) or (2.2) gives the unique edge path in T from e to ge, whose length is $n+1$ (including e and ge). This path has initial endpoint $\partial_0 e$ if and only if h_1 is in A. Thus $ge = g'e$ implies that both words are of the same type and $n = m$. Also evident from (2.1) or (2.2) is that $g_i e$ is the ith edge (or its reverse) along this segment, after e. Thus $g_i e = g'_i e$ for all i. Therefore $g_i^{-1} g'_i \in C$ for all i. □

Remark 2.4. It now makes sense to refer to an element $g \in G - C$ as being cyclically reduced or not, being of one of the four types, or of having length n. (Take $g = g'$ in the Lemma.)

Next, the distinguished edge e separates T into two subtrees T_A and T_B. Namely, T_A is the maximal subtree containing $\partial_1 e$ but not e, and T_B is the maximal subtree containing $\partial_0 e$ but not e. Let T_g denote the *characteristic subtree* for g. This is either the axis of g (if g is hyperbolic) or the subtree of fixed points of g.

Lemma 2.5. *Suppose $g \in G - C$. Then*

1. *if g is of type AB or BA then $e \in T_g$,*
2. *if g is of type AA then $T_g \subset T_A$,*
3. *if g is of type BB then $T_g \subset T_B$.*

Proof. First, consider the following statements:

- if g is of type AB then $g(T_A) \subset T_A$,
- if g is of type BA then $g(T_B) \subset T_B$,
- if g is of type AA then $g(T_B) \subset T_A$,
- if g is of type BB then $g(T_A) \subset T_B$.

The statements follow by induction on the length of g (and direct examination in the cases $g \in A$ and $g \in B$). Next, we have already noted that g is hyperbolic and $e \in T_g$ if and only if g is of type AB or BA, so (1) holds. If g is elliptic then $e \notin T_g$ because $g \notin C$. So in cases (2) and (3), T_g is on one side of e. Since $g(T_g) = T_g$, only the side indicated is consistent with the four statements given above. □

3. PROOF OF PROPOSITION 1.1.

First we prove the easy direction, (1) implies (2). Suppose (2) does not hold, i.e. the element represented by some cyclic conjugate of w^{-1} lies in CgC. Then we can write $dg^{-1}d^{-1} = c_1gc_2$, or equivalently, $gc_2dgd^{-1} = c_1^{-1}$. Then $g(c_2d)g(c_2d)^{-1} = c_1^{-1}c_2^{-1}$. Taking $h = c_2d$ we see that $ghgh^{-1}$ fixes e, which is an edge on the axis of g (because g is cyclically reduced). Thus g is not well-aligned.

For the converse, suppose g is not well-aligned. Thus $ghgh^{-1}$ fixes an edge of T_g. We may assume that this edge is the distinguished edge e. The geometric meaning of this situation is shown in Figure 1. Since $e \in T_g$, the edges e and $g^{-1}e$ are coherently oriented and both in T_g. Since $g^{-1}e = hgh^{-1}e$, these two edges must also be on the axis $T_{hgh^{-1}}$. Therefore $T_g \cap T_{hgh^{-1}} = T_g \cap h(T_g)$ contains the segment $[e, g^{-1}e]$.

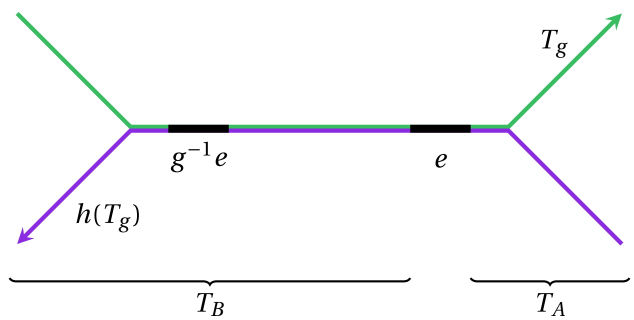

FIGURE 1. When g is not well-aligned. Geometrically, this means that T_g and $h(T_g)$ overlap in a segment of length $> |g|$, inducing opposite orientations on this segment. (The subtrees T_A, T_B are as shown when g is of type AB.)

Note that the starting assumptions do not change if we replace h by hg^k for any k. By doing so, we can arrange that h fixes a vertex between e and $g^{-1}e$.

Now suppose that g is of type AB, so that T_g contains the edge sequence (2.1). Then $T_h \subset T_B$, so by Lemma 2.5, h is of type BB.

Next, let e' be an edge in $h(T_g) \cap T_A$. We claim that for all sufficiently large k, the element g^kh^{-1} has characteristic subtree $T_{g^kh^{-1}}$ inside T_A. If $e' \not\in T_g$ then for all sufficiently large k, the edge $g^kh^{-1}e'$ is coherently oriented with e' and is in $(T_g \cap T_A) - h(T_g)$. Then, the axis of g^kh^{-1} goes through these edges and is inside T_A. If $e' \in T_g$, then for large k the edge $g^kh^{-1}e'$ is incoherently oriented with e', and e' separates it from e. Thus, e' separates e from $T_{g^kh^{-1}}$, and hence the latter is inside T_A. Now we can conclude, by Lemma 2.5, that g^kh^{-1} is of type AA for all sufficiently large k.

Let $g = a_1b_2\cdots b_n$ and $h = b'_1a'_2\cdots b'_m$ be reduced words for g and h. Then $(a_1\cdots b_n)^k$ is a reduced word for g^k. Now consider the word

$$(a_1\cdots b_n)^k b'^{-1}_m \cdots b'^{-1}_1$$

which reduces to a type AA alternating word. This means that every letter in $b_m'^{-1}\cdots b_1'^{-1}$ reduces, leaving

$$g^k h^{-1} \;=\; (a_1\cdots b_n)^{k-\ell-1}a_1b_2\cdots a_i c$$

for some i, some ℓ, and some $c\in C$. Hence

$$h \;=\; c^{-1}b_{i+1}\cdots b_n(a_1\cdots b_n)^{\ell},$$

and so

$$hgh^{-1} \;=\; c^{-1}b_{i+1}a_{i+2}\cdots b_na_1b_2\cdots a_ic.$$

Since $ghgh^{-1}\in C$, it follows easily now that

$$a_i^{-1}b_{i-1}^{-1}\cdots b_2^{-1}a_1^{-1}b_n^{-1}\cdots a_{i+2}^{-1}b_{i+1}^{-1}\in CgC.$$

If g is of type BA then the proof is entirely analogous, with A and B reversed. This completes the proof.

REFERENCES

[CF10] Danny Calegari and Koji Fujiwara, *Stable commutator length in word-hyperbolic groups*, Groups Geom. Dyn. **4** (2010), no. 1, 59–90. MR 2566301 (2011a:20109)

[CFL16] Matt Clay, Max Forester, and Joel Louwsma, *Stable commutator length in Baumslag-Solitar groups and quasimorphisms for tree actions*, Trans. Amer. Math. Soc. **368** (2016), no. 7, 4751–4785. MR 3456160

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