Research Statement

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1. My research at a glance

My recent research has centered on the study of **cluster algebras**. These are commutative rings with a distinguished set of recursively-defined generators, called *cluster variables*. Introduced by Fomin and Zelevinsky in [FZ02], they occur in the coordinate rings of a remarkable number of important spaces–such as semisimple Lie groups, Grassmannians, and decorated Teichmüller spaces. Cluster algebras have appeared in applications as varied as discrete dynamical systems, representation theory, and scattering amplitudes in Yang-Mills theory.

An unexpected feature of the theory is that cluster algebras tend to be well-behaved or pathological, with a large gap in between. It is a significant problem in the field to find a 'missing axiom', some additional property which eliminates the horrible examples while containing the important ones. As a possible solution to this problem, I have introduced **locally acyclic cluster algebras** and developed their properties over a series of papers.

- Locally acyclic cluster algebras can be covered (in a geometric sense) by certain elementary cluster algebras [Mul13, Mul14].
- Locally acyclic cluster algebras are finitely generated [Mul13].
- The complex varieties associated to locally acyclic cluster algebras may be singular, but their singularities must be *canonical* [BMRS14]. There is also a combinatorial condition which implies smoothness [Mul13].
- Locally acyclic cluster algebras coincide with their *upper cluster algebra*, a closely related algebra which arises in geometric applications [Mul13, Mul14].

In parallel, I have proven that many important examples of cluster algebras are locally acyclic.

- A general linear group $GL(n)$ has a finite stratification into locally closed subvarieties whose coordinate rings are locally acyclic cluster algebras [MS14].
- Analogously, a Grassmannian $Gr(k, n)$ of k-planes in n-space has a finite stratification into locally closed subvarieties whose coordinate rings are locally acyclic cluster algebras [MS14].
- Given a triangulable marked surface, the lambda lengths on the associated *decorated Teichmüller space* generate a locally acyclic cluster algebra, except for two families of counterexamples [Mul13].

A number of techniques were developed in obtaining these results which are of independent interest, including connections with geometry in positive characteristic [BMRS14], flows in planar networks [MS14], and perfect matchings in graphs [MS].

Originally, I developed locally acyclic cluster algebras to study a non-commutative algebra coming from knot theory. Given a real surface with boundary and *marked points* on the boundary, I defined its **skein algebra** in [Mul12], generalizing an algebra of considerable interest in knot theory. When the surface is *triangulable* and has more than two marked points, a certain localization of the skein algebra is a quantum cluster algebra [Mul12].[1] The proof uses a non-commutative analog of locally acyclic cluster algebras.

In earlier research, I published results in invariant theory [MS11], integrable systems [Mul11], and D-modules on varieties [Mul10], the latter being my thesis work. While there are connections between this work and cluster algebras, I will not outline them here.

[1] A quantum cluster algebra is a special non-commutative deformation of a cluster algebra.

Since last spring, much of my research has been exploring new techniques coming from the work of Gross-Hacking-Keel-Kontsevich [GHKK]; specifically the conjectural basis of *theta functions* in a cluster algebra. I am currently preparing papers on the following.

- At a Snowbird workshop, a group of 8 people including myself proved that the theta basis of a rank 2 cluster algebra coincides with Sherman-Zelevinsky's 'greedy basis' [SZ04].
- I used the theory of theta functions to prove that the existence of a 'maximal green sequence' for a quiver is not a mutation invariant, disproving a widely-held expectation on the combinatorial side of cluster algebras.

One appealing aspect of this new theory is that theta basis is defined in terms of generating functions that count certain piecewise-linear curves inside a *tropical variety*. Hence, with a few lemmas, deep questions about cluster algebras can be reduced to the kinds of counting problems suitable for attack by a graduate student or gifted undergraduate.

2. A little bit about cluster algebras

A **cluster algebra** $\mathcal{A}$ is a commutative domain with a distinguished, possibly infinite set of generators called **cluster variables**. There are distinguished sets of cluster variables, called **clusters**, each of which is a transcendence basis of the fraction field $\mathrm{Frac}(\mathcal{A})$ of $\mathcal{A}$. Each cluster is also equipped with additional data (such as a quiver) that allows any cluster to be reconstructed from any other cluster by a sequence of **mutations**.

2.1. **The main examples.** The most important examples of cluster algebras occur in the rings of functions on many notable spaces. These include the original examples that lead to the definition, as well as many surprising examples that have emerged since then. Three of the most significant examples are provided below.

- A general linear group $GL(n)$ admits a stratification into *double Bruhat cells*, which roughly measure the failure of LU factorizations to exist. The coordinate ring of each such cell is a cluster algebra, by [BFZ05] and [GY13].

  The definition of cluster algebras were introduced to axiomatize patterns appearing the dual canonical basis of the coordinate ring of double Bruhat cells, making these the foundational examples of cluster algebras.
- A Grassmannian $Gr(k, n)$ admits a stratification into *positroid cells*, which roughly measure the failure of a k-dimensional subspace to be in generic position relative to certain coordinate subspaces. The coordinate ring of each such cell is a cluster algebra, by [Lec14] and [MS14].[2]
- The *decorated Teichmüller space* of a marked surface[3] parametrizes hyperbolic metrics on the surface, together with certain additional decorations at the marked points. Invariants of these metrics called *lambda lengths* define functions on the decorated Teichmüller space. When there are enough marked points to allow triangulations of the surface, the lambda lengths generate a cluster algebra inside the ring of functions [GSV05, FG06, FST08].

There are many other classes of examples, including generalizations of each of these three: double Bruhat cells in other semisimple Lie groups, open Richardson cells in partial flag varieties, and higher Teichmüller spaces.

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[2]Technically speaking, there is still a detail missing: the results of [Lec14] and [MS14] deal with two slightly different cluster algebras contained in the coordinate ring. I will gloss over this gap.

[3]Specifically, a real 2-dimensional manifold with boundary and finitely many 'marked points'.

2.2. **Upper cluster algebras.** The basic miracle of cluster algebras is the **Laurent phenomenon**: every element of $\mathcal{A}$ can be written as an integral Laurent polynomial in the cluster variables in any cluster. Equivalently, for any cluster $f_1, f_2, ..., f_n$,

$$\mathcal{A} \subset \mathbb{Z}[f_1^{\pm 1}, f_2^{\pm 1}, ..., f_n^{\pm 1}] \subset \mathrm{Frac}(\mathcal{A})$$

The set of all rational functions that satisfy the Laurent phenomenon form a subalgebra of Frac($\mathcal{A}$), called the **upper cluster algebra** $\mathcal{U}$ [BFZ05]. Concretely,

$$\mathcal{U} := \bigcap_{\text{clusters } f_1, f_2, ..., f_n} \mathbb{Z}[f_1^{\pm 1}, f_2^{\pm 1}, ..., f_n^{\pm 1}] \subset \mathrm{Frac}(\mathcal{A})$$

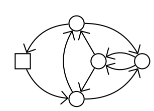
The Laurent phenomenon for $\mathcal{A}$ is equivalent to the inclusion $\mathcal{A} \subseteq \mathcal{U}$.

One can ask... when is $\mathcal{A} = \mathcal{U}$? This important question is linked to a curious dichotomy: cluster algebras tend to be well-behaved or pathological, with nothing in between. For example, rank 3 cluster algebras are either complete intersections generated by 6 elements, or are infinitely generated with non-Noetherian singularities [Mul13]. To date, the known cases where $\mathcal{A}$ is well-behaved coincide with the known cases when $\mathcal{A} = \mathcal{U}$ [BFZ05, Mul13, CLS14].

2.3. **Geometry.** Since the driving examples of cluster algebras naturally occur as coordinate rings, it is worth reinterpreting a cluster algebra $\mathcal{A}$ geometrically by considering the associated **cluster variety** $X(\mathcal{A}) := \mathrm{Spec}(\mathbb{C} \otimes \mathcal{A})$. This is the universal complex variety with $\mathcal{A}$ in its coordinate ring.[4] Each cluster determines an algebraic torus $(\mathbb{C}^\times)^n \subset X(\mathcal{A})$ parametrized by the cluster variables. The union of these tori defines a smooth, dense subscheme inside $X(\mathcal{A})$, and the ring of functions on this union is the upper cluster algebra $\mathcal{U}$. This union need not be all of $X(\mathcal{A})$; in fact, the interesting geometric behavior lies in its complement.

3. Locally acyclic cluster algebras

A cluster algebra can be defined by an *ice quiver* Q - a collection of circles and squares with arrows between them. The circles ○ denote *mutable* variables, and the squares □ denote *frozen* variables. A quiver is *acyclic* if there are no directed cycles of mutable variables.



3.1. **Local techniques.** Each mutable variable can be mutated repeatedly, while the frozen variables are static. Consequently, turning a mutable variable into a frozen one gives a new ice quiver $\mathsf{Q}^\dagger$ whose cluster algebra $\mathcal{A}(\mathsf{Q}^\dagger)$ is simpler than $\mathcal{A}(\mathsf{Q})$. It is then tempting to try to relate properties of $\mathcal{A}(\mathsf{Q}^\dagger)$ to properties of $\mathcal{A}(\mathsf{Q})$.

In [Mul13], I observed that under certain conditions, the algebra $\mathcal{A}(\mathsf{Q}^\dagger)$ is a localization of $\mathcal{A}(\mathsf{Q})$. Geometrically, the cluster variety $X(\mathcal{A}(\mathsf{Q}^\dagger))$ is an open subvariety in $X(\mathcal{A}(\mathsf{Q}))$; call such an open subset a *cluster chart*. If one can find a set of cluster charts $X(\mathcal{A}_i)$ which cover $X(\mathcal{A})$, then many algebraic properties of $\mathcal{A}$ can be checked on the $\mathcal{A}_i$.

An *acyclic cluster algebra* is a cluster algebra of the form $\mathcal{A}(\mathsf{Q})$, where Q is acyclic. Acyclic cluster algebras were shown to have many desirable properties in [BFZ05]. This led me to introduce the following definition.

Definition 3.1.1. A cluster algebra $\mathcal{A}$ is **locally acyclic** if $X(\mathcal{A})$ can be covered by cluster charts $X(\mathcal{A}_i)$, where each of the $\mathcal{A}_i$ is acyclic.

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[4] If the reader is comfortable with schemes, they should instead consider the integral scheme Spec($\mathcal{A}$).

3.2. **Properties.** A theme in algebraic geometry is that many algebraic properties can be verified by checking the corresponding geometric property locally. Using this idea, many properties of acyclic cluster algebras can be extended to locally acyclic cluster algebras.

Theorem. [Mul13, Mul14] *Let $\mathcal{A}$ be a locally acyclic cluster algebra. Then $\mathcal{A} = \mathcal{U}$, and $\mathcal{A}$ is finitely generated, integrally closed, and locally a complete intersection.*

This local perspective has produced original results even in the acyclic case. The *exchange matrix* $\mathsf{E}(\mathsf{Q})$ of Q is the matrix whose entries count the number of arrows in Q from a variable to a mutable variable. If $\mathsf{E}(\mathsf{Q})$ has full rank, we say $\mathcal{A}(\mathsf{Q})$ has *full rank*.

Theorem. [Mul13] *If $\mathcal{A}$ is locally acyclic and full rank, then $X(\mathcal{A})$ is smooth.*

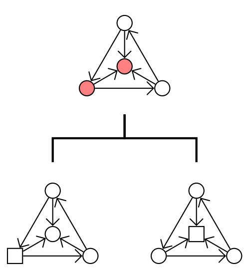
3.3. **Positive characteristic.** In joint work with Benito, Rajchgot and Smith [BMRS14], we further explored the geometry of locally acyclic cluster algebras by considering reductions to characteristic p; that is, algebras $\mathsf{k} \otimes_{\mathbb{Z}} \mathcal{A}$ where $\mathrm{char}(\mathsf{k}) = p > 0$. We prove that each upper cluster algebra has a canonical *splitting* of the Frobenius map which generates all other Frobenius splittings. In the case of locally acyclic cluster algebras, we furthermore demonstrate that the canonical splitting fixes no non-trivial ideals, which implies that $\mathsf{k} \otimes_{\mathbb{Z}} \mathcal{A}$ is *strongly F-regular*.

While an interesting result in its own right, this positive characteristic property has consequences for the complex geometry.

Theorem. [BMRS14] *If $\mathcal{A}$ is locally acyclic, then $X(\mathcal{A})$ has at worst canonical singularities.*

Canonical singularities are a class of well-behaved singularities which arise in the minimal model program; they are defined as singularities of $\mathbb{Q}$-Cartier normal varieties which can be resolved without negative discrepancies.

3.4. **First examples.** So what cluster algebras are locally acyclic, besides acyclic ones?



In [Mul13], I provide an explicit algorithm, called the **Banff algorithm**, for verifying that a given quiver defines a locally acyclic cluster algebra (an example is shown to the left). It takes place exclusively on the level of quivers, so it can be a quick and effective method to check a given example. Several examples of locally acyclic cluster algebras (which are not acyclic) were given in [Mul13].

A deeper class of examples comes from cluster algebras of *marked surfaces*. For this note, a *marked surface* Σ is a connected, oriented real surface with boundary, together with a finite set of *marked points* on the boundary $\partial\Sigma$.[5]

If $\partial\Sigma$ is not empty and each component of $\partial\Sigma$ has at least one marked point, Σ is *triangulable*. When Σ is triangulable, a cluster algebra $\mathcal{A}(\Sigma)$ was constructed in [GSV05]. These were the first large class of locally acyclic cluster algebras.

Theorem. [Mul13] *If Σ is triangulable with at least two marked points, then $\mathcal{A}(\Sigma)$ is a locally acyclic cluster algebra.*

[5]Generalizations which include interior marked points exist, but are more complicated [FST08].

3.5. **Subsequent examples.** In my three years at LSU, M. Yakimov and I ran a *Vertically Integrated Research (VIGRE)* seminar each semester on current research topics. In Spring of 2011, the seminar focused on locally acyclic cluster algebras, and specifically on the local acyclicity of other important examples of cluster algebras. The participants[6] produced new examples of locally acyclic cluster algebras and useful lemmas.

However, these results were subsumed by my later work with D. Speyer [MS14], which established that the cluster algebras associated to positroid cells in Grassmannians are locally acyclic, using an application of the Banff algorithm to affine permutations.

Theorem. [MS14] *The coordinate ring of a positroid cell is a locally acyclic cluster algebra.*

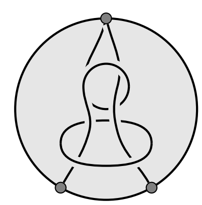
In fact, this resolves the issue for double Bruhat cells in $GL(n)$, because there is an isomorphism of cluster algebras between the coordinate ring of any double Bruhat cell in $GL(n)$ and the coordinate ring of a positroid.

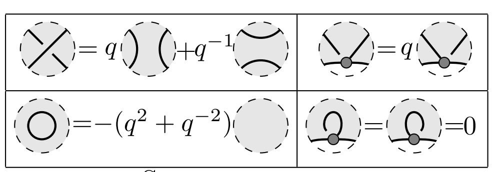
4. Skein algebras of marked surfaces

Locally acyclic cluster algebras have their origin in the study of a seemingly unrelated, non-commutative algebra originating in knot theory - the *skein algebra* of a surface.

4.1. **The skein algebra.** Recall that a **marked surface** Σ is a connected, oriented surface-with-boundary, together with a finite set of *marked points* on the boundary $\partial\Sigma$.

A *curve* on Σ is an immersion $C \to \Sigma$ of a curve-with-boundary C, such that any boundary ∂C is sent to the marked points. A *link diagram* L is a collection of curves on Σ such that any intersections are simple and transverse, together with an ordering of the strands at any intersection. Link diagrams generalize the projection of a knot in $\Sigma \times [0,1]$ onto Σ, where the orderings keep track of how the strands are passing over each other.





Skein relations

Let $\mathbb{Z}[q^{\pm 1}]$ be the ring of Laurent polynomials in a formal variable q. The **skein algebra** $Sk_q(\Sigma)$ of Σ is the $\mathbb{Z}[q^{\pm 1}]$-span link diagrams (up to isotopy), modulo the four classes of *skein relations* pictured to the left. The algebra $Sk_q(\Sigma)$ has an associative product given by superimposing link diagrams.

The skein algebra of an unmarked surface was introduced in [Tur89] and [Prz91] to generalize Kauffman's bracket for computing the Jones polynomial, and have since become a significant area of knot theory. My generalization to marked surfaces is new [Mul12], with motivation coming from cluster algebras and Teichmüller theory.

4.2. **Quantum cluster algebras.** Skein algebras can be related to *quantum cluster algebras*. Quantum cluster algebras are non-commutative deformations of cluster algebras, in which two cluster variables in the same cluster *quasi-commute*; that is, $x_i x_j = q^m x_j x_i$ for some $m \in \mathbb{Z}$. A remarkable amount of the theory generalizes to the quantum setting. There is a **quantum cluster algebra** $\mathcal{A}_q$ and a **quantum upper cluster algebra** $\mathcal{U}_q$.

If $\partial\Sigma$ is not empty and each component of $\partial\Sigma$ has at least one marked point, Σ is *triangulable*. In [Mul12], I construct algebras $\mathcal{A}_q(\Sigma)$ and $\mathcal{U}_q(\Sigma)$ for a triangulable marked surface, which naturally quantize the cluster algebras mentioned in the previous section.

[6] Roughly a dozen participants, containing a mix of graduate students, undergraduates and faculty.

A *boundary curve* in Σ is a curve which is isotopic to one contained in $\partial\Sigma$. The set of boundary curves generates a Ore set in $Sk_q(\Sigma)$; let $Sk_q^o(\Sigma)$ be the localization at this set.

Theorem. [Mul12] *For triangulable Σ, there are inclusions $\mathcal{A}_q(\Sigma) \subseteq Sk_q^o(\Sigma) \subseteq \mathcal{U}_q(\Sigma)$. If Σ has at least two marked points, these are equalities.*

The proof is via a non-commutative generalization of local acyclicity, which works even though the non-commutativity breaks any direct connection to geometry.

This theorem has implications for $\mathcal{A}_q(\Sigma)$ and $Sk_q(\Sigma)$. The cluster structure gives quasi-commuting families in $Sk_q(\Sigma)$, in which any other element can be written as a skew-Laurent polynomial. The diagrammatics for $\mathcal{A}_q(\Sigma)$ give a tool for rapid computation with elements which may be in distant clusters; this is prohibitively difficult in general.

5. The theta basis

5.1. **Scattering diagrams.** Over the last year, a revolutionary perspective has emerged from the work of Gross, Hacking, Keel and Kontsevich [GHKK] on *mirror symmetry*, a suite of conjectures which relate the complex geometry of a suitable space X to the symplectic geometry of a 'mirror dual space' $\check{X}$. By considering the 'tropical boundary' of the mirror dual of a cluster variety, they describe a conjectural basis of any cluster algebra.

Given a cluster algebra $\mathcal{A}$ with a distinguished choice of *initial* cluster, they associate a **scattering diagram**, consisting of a (typically infinite) collection of *walls* in $\mathbb{R}^n$. A wall is a codimension 1 polyhedral cone, equipped with some algebraic 'scattering data'.

A *broken line* in a scattering diagram is a piecewise linear map from $[0,\infty) \to \mathbb{R}^n$ which is non-linear only at the walls, and then only in certain ways allowed by the scattering data. Each lattice point $\alpha \in \mathbb{Z}^n \subset \mathbb{R}^n$ in a scattering diagram determines a **theta function** Θ_α given by formal sum over all broken lines with a fixed starting point and final derivative α. Products between theta functions can be also be computed by counting certain collections of broken lines.

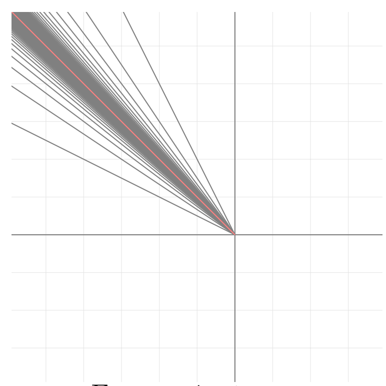


Figure 1. A scattering diagram

Conjecturally, these theta functions form a long-sought basis for the cluster algebra which contains the cluster variables and has good positivity properties. However, there are still many basic open questions, including a lack of explicit formulas in all but the most basic examples.

5.2. **Results on theta bases.** At an AMS workshop in Snowbird last summer, a group including myself [CCG$^+$] was able to prove that the theta basis of a 2 dimensional cluster algebra coincides with the *greedy basis*, a well-studied strongly positive basis first defined in [SZ04]. This provides the only non-trivial example where the theta basis is completely understood.

These new tools have significant potential to answer long-standing questions in the field. For example, a large amount of recent work on the combinatorics of quiver mutation has focused on *the existence of a maximal green sequence*, a property with a wide range of applications to representation theory, physics and Donaldson-Thomas theory. Using theta functions, I demonstrated that the existence of a maximal green sequence is not preserved by mutation [Mul], disproving a widely-held conjecture.

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