Lower bound cluster algebras: Presentations and properties

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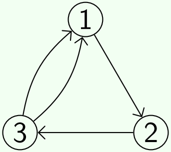
January 9, 2016

One-step cluster variables

We begin by fixing a quiver Q with vertices labeled $1, 2, \ldots, r$.



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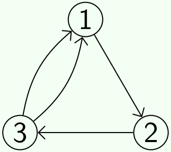

To each vertex i, we associate 3 rational functions in $x_1, x_2, \ldots, x_r$.

$$p_i^+ := \prod_{\text{arrows } j \leftarrow i} x_j, \qquad p_i^- := \prod_{\text{arrows } j \rightarrow i} x_j$$

$$x_i' := \frac{p_i^+ + p_i^-}{x_i} \qquad \text{(One-step cluster variable)}$$

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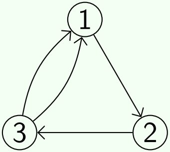

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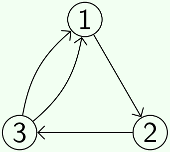

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Q: What are the relations among the $x_1, x_2, ..., x_r, x_1', x_2', ..., x_r'$?

Lower bound cluster algebras

This question is essentially the study of the following ring.

Definition: Lower bound cluster algebra [BFZ, 2005]

Given Q, the **lower bound cluster algebra** is the subring

$$L(Q) \subset \mathbb{Q}(x_1, x_2, ..., x_r)$$

generated by $x_1, x_2, ..., x_r, x_1', x_2', ..., x_r'$.

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Cluster algebra [Fomin-Zelevinsky, 2001]

More generally, any sequence of vertices can be used to defined a cluster variable. The **cluster algebra** $\mathcal{A}(Q)$ of Q is the subring generated by *all* cluster variables (often an infinite set).

For quivers without directed cycles, $L(Q) = \mathcal{A}(Q)$. In general, $L(Q)$ is more tractable than $\mathcal{A}(Q)$...but a bit less exciting.

Relations in $L(\mathrm{Q})$

Q: What are the relations among the $x_1, x_2, \ldots, x_r, x'_1, x'_2, \ldots, x'_r$?

More precisely, what is the kernel K of the following map?

$$\mathbb{Q}[x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_r] \to L(\mathrm{Q})$$

$$x_i \mapsto x_i, \quad y_i \mapsto x'_i$$

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Some relations (that is, elements of the kernel K) are easy!

Definition: Defining relation

For each x'_i, there is a defining equation

$$x_i x'_i = p_i^+ + p_i^-$$

Hence, for each vertex i, there is a **defining relation**

$$x_i y_i - p_i^+ - p_i^- \in K$$

The defining relations are not enough

A more interesting identity

$$\begin{aligned}
x_1'x_2'x_3' &= \frac{x_2 + x_3^2}{x_1} \cdot \frac{x_3 + x_1}{x_2} \cdot \frac{x_1^2 + x_2}{x_3} \\
&= \frac{x_1x_2x_3^2 + x_1^3x_3^2 + x_2x_3^3 + x_1^2x_3^3 + x_1x_2^2 + x_1^3x_2 + x_2^2x_3 + x_1^2x_2x_3}{x_1x_2x_3} \\
&= \frac{x_2 + x_3^2}{x_1} + \frac{x_1x_3^2 + x_1^2x_3}{x_2} + \frac{x_1^2 + x_2}{x_3} + x_1 + x_3 \\
&= x_1' + x_1x_3x_2' + x_3' + x_1 + x_3
\end{aligned}$$

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$$\begin{aligned} x_1'x_2'x_3' &= \frac{x_2+x_3^2}{x_1}\cdot\frac{x_3+x_1}{x_2}\cdot\frac{x_1^2+x_2}{x_3} \\ &= \frac{x_1x_2x_3^2+x_1^3x_3^2+x_2x_3^3+x_1^2x_3^3+x_1x_2^2+x_1^3x_2+x_2^2x_3+x_1^2x_2x_3}{x_1x_2x_3} \\ &= \frac{x_2+x_3^2}{x_1}+\frac{x_1x_3^2+x_1^2x_3}{x_2}+\frac{x_1^2+x_2}{x_3}+x_1+x_3 \\ &= x_1'+x_1x_3x_2'+x_3'+x_1+x_3 \end{aligned}$$

However, the corresponding element of K

$$y_1y_2y_3-(y_1+x_1x_3y_2+y_3+x_1+x_3)$$

is not in the ideal generated by the defining relations

$$x_1y_1-(x_2+x_3^2),\quad x_2y_2-(x_3+x_1),\quad x_3y_3-(x_1^2+x_2)$$

Cycle relations

Remarkably, this example generalizes! Given a **directed cycle**

$$v_1 \to v_2 \to \ldots \to v_k \to v_{k+1} = v_1$$

in Q, the product $x'_{v_1} x'_{v_2} \cdots x'_{v_k}$ satisfies an identity of the form

$$x'_{v_1} x'_{v_2} \cdots x'_{v_k} = \sum \text{monomials with} < k \text{ one-step cluster variables}$$

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Definition: Cycle relation

Given a directed cycle

$$v_1 \to v_2 \to \ldots \to v_k \to v_{k+1} = v_1$$

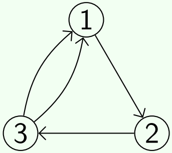
in Q, define the associated **cycle relation**

$$\left[\sum (-1)^{|S|} \left(\prod_{i \in S} \frac{p^+_{v_i} p^-_{v_{i+1}}}{x_{v_i} x_{v_{i+1}}} \right) \left(\prod_{i \notin S \cup (S+1)} y_{v_i} \right) \right] - \prod_{i=1}^{k} \frac{p^+_{v_i}}{x_{v_i}} - \prod_{i=1}^{k} \frac{p^-_{v_i}}{x_{v_i}}$$

where the sum is over $S \subset \{1, 2, \ldots, k\}$ without consecutive pairs.

Cycle relations (example)

$$\left[\sum(-1)^{|S|}\left(\prod_{i\in S}\frac{p^+_{v_i}p^-_{v_{i+1}}}{x_{v_i}x_{v_{i+1}}}\right)\left(\prod_{i\notin S\cup(S+1)} y_{v_i}\right)\right]-\prod_{i=1}^{k}\frac{p^+_{v_i}}{x_{v_i}}-\prod_{i=1}^{k}\frac{p^-_{v_i}}{x_{v_i}}$$



Cycle: $1\to 2\to 3\to 1$

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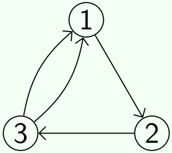
$$\left[\sum(-1)^{|S|}\left(\prod_{i\in S}\frac{p_{v_i}^{+}p_{v_{i+1}}^{-}}{x_{v_i}x_{v_{i+1}}}\right)\left(\prod_{i\notin S\cup(S+1)} y_{v_i}\right)\right]-\prod_{i=1}^{k}\frac{p_{v_i}^{+}}{x_{v_i}}-\prod_{i=1}^{k}\frac{p_{v_i}^{-}}{x_{v_i}}$$



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$S=\emptyset$

$y_1y_2y_3$

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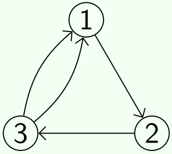


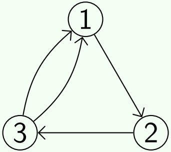
Cycle: $1 \to 2 \to 3 \to 1$

$S = \{1\}$

$$y_1 y_2 y_3 + \left(-\frac{x_2 x_1}{x_1 x_2} y_3\right)$$

Cycle relations (example)

$$\left[\sum(-1)^{|S|}\left(\prod_{i\in S}\frac{p^+_{v_i}p^-_{v_{i+1}}}{x_{v_i}x_{v_{i+1}}}\right)\left(\prod_{i\notin S\cup(S+1)} y_{v_i}\right)\right]-\prod_{i=1}^{k}\frac{p^+_{v_i}}{x_{v_i}}-\prod_{i=1}^{k}\frac{p^-_{v_i}}{x_{v_i}}$$



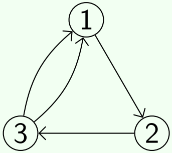
Cycle: $1 \to 2 \to 3 \to 1$

$S = \{2\}$

$$y_1y_2y_3 - y_3 + \left(-\frac{x_3x_2}{x_2x_3}y_1\right)$$

Cycle relations (example)

$$\left[\sum(-1)^{|S|}\left(\prod_{i\in S}\frac{p^+_{v_i}p^-_{v_{i+1}}}{x_{v_i}x_{v_{i+1}}}\right)\left(\prod_{i\notin S\cup(S+1)} y_{v_i}\right)\right] - \prod_{i=1}^{k}\frac{p^+_{v_i}}{x_{v_i}} - \prod_{i=1}^{k}\frac{p^-_{v_i}}{x_{v_i}}$$



Cycle: $1 \to 2 \to 3 \to 1$

$S = \{3\}$ $\qquad y_1y_2y_3 - y_3 - y_1 + \left(-\frac{x_1^2x_3^2}{x_3x_1}y_2\right)$

Cycle relations (example)

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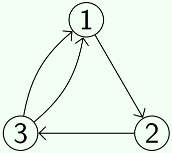

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$$y_1y_2y_3 - y_3 - y_1 - x_1x_3y_2 + \left(-\frac{x_1^2x_2x_3}{x_1x_2x_3} - \frac{x_1x_2x_3^2}{x_1x_2x_3}\right)$$

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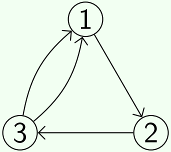

Cycle: $1 \to 2 \to 3 \to 1$

$$y_1y_2y_3 - y_3 - y_1 - x_1x_3y_2 - x_1 - x_3$$

The defining and cycle relations are enough!

Theorem [M-Rajchgot-Zykoski, 2015]

The lower bound cluster algebra $L(Q)$ is isomorphic to

$$\mathbb{Z}[x_1, x_2, ..., x_r, y_1, y_2, ..., y_r]/K$$

where K is the ideal generated by the defining relations and the cycle relations.

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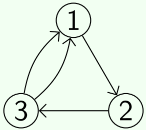
For Q on the right, $L(\mathrm{Q})$ is the quotient of

$$\mathbb{Z}[x_1, x_2, x_3, y_1, y_2, y_3]$$

by the ideal generated by the four relations



$$x_1 y_1 - (x_2 + x_3^2), \quad x_2 y_2 - (x_3 + x_1), \quad x_3 y_3 - (x_1^2 + x_2)$$

$$y_1 y_2 y_3 - (y_1 + x_1 x_3 y_2 + y_3 + x_1 + x_3)$$

Reducing expressions in $L(Q)$

Observe that each relation is a kind of **reduction rule**:

$$x_i x_i' = \text{a binomial in } x_1, x_2, \ldots, x_r$$

$$x'_{v_1} x'_{v_2} \cdots x'_{v_k} = \sum \text{monomials with} < k \text{ one-step cluster variables}$$

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Every element of $L(\mathrm{Q})$ can be written **uniquely** as a polynomial in $x_1, x_2, \ldots, x_r, x_1', x_2', \ldots, x_r'$ that cannot be reduced by these rules.

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Theorem [M-Rajchgot-Zykowski, 2015]

The defining relations and cycle relations collectively form a **Gröbner basis** for K, with respect to any term order in which the y-variables are much more expensive than the x-variables.

The irreducible monomials can be encoded in a simplicial complex.

Defn: The Stanley-Reisner complex

The **Stanley-Reisner complex** of K is the simplicial complex with

- a vertex for each generator $x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_r$, and
- a simplex for each subset of the the generators whose product cannot be reduced (doesn't contain any $x_i y_i$ or $y_{v_1} y_{v_2} \cdots y_{v_k}$).

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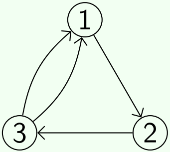

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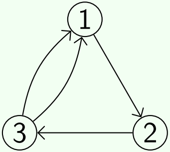

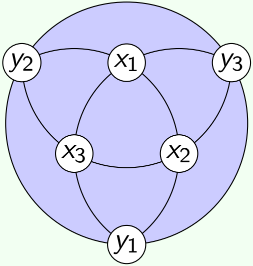

As the example suggests, these complexes are well-behaved.

Theorem [M-Rajchgot-Zykowski, 2015]

Let Δ be the Stanley-Reisner complex of K.

- If Q has no directed cycles, Δ is a simplicial $(r-1)$-sphere.
- Otherwise, Δ is a simplicial $(r-1)$-ball.

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This topological fact has algebraic consequences!

Theorem [M-Rajchgot-Zykowski, 2015]

For all quivers Q, $L(\mathrm{Q})$ is Cohen-Macaulay and normal.

Cohen-Macaulayness is a direct corollary, and normality requires considering a finite set of height 1 primes.

What next?

The bigger fish

Extending these results to general cluster algebras.

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Extending these results to general cluster algebras.

This is not so crazy! While a cluster algebra is defined by an infinite generating set, often a finite subset suffices.

Extending our work to larger finite sets of cluster variables would yield analogous results for every finitely-generated cluster algebra.