Twists for positroid cells

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Preliminaries: Grassmannians

Given a $k \times n$ matrix A of rank k, let

$$[A] := \text{span}(\text{rows of } A) \in Gr(k,n)$$

The map $A \to [A]$ defines an isomorphism

$$GL(k) \backslash \{k \times n \text{ matrices of rank } k\} \xrightarrow{\sim} Gr(k,n)$$

Definition (Matroid of a matrix)

The **matroid** of a $k \times n$ matrix is the set of subsets of columns which form a basis, written as k-subsets of $[n] := \{1, 2, ..., n\}$.

A matroid $\mathcal{M}$ defines a variety in the (k,n)-Grassmannian.

$$\Pi(\mathcal{M}) := \{[A] \in Gr(k,n) \mid (\text{the matroid of } A) \subseteq \mathcal{M}\}$$

Positroids and positroid cells

A **positroid** is the matroid of a 'totally positive matrix': a real-valued matrix whose maximal minors are all non-negative.

Definition (Positroid cell)

The **positroid cell** of a positroid $\mathcal{M}$ is

$$\Pi^\circ(\mathcal{M}) := \Pi(\mathcal{M}) - \bigcup_{\text{positroids } \mathcal{M}' \subsetneq \mathcal{M}} \Pi(\mathcal{M}')$$

These cells define a well-behaved stratification of $Gr(k, n)$.

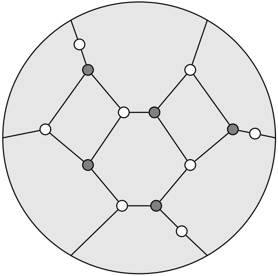
Example (The big positroid cell in $Gr(k, n)$)

For $\mathcal{M} = \{\text{all } k\text{-element subsets of } [n]\}$,

$$\Pi(\mathcal{M}) = Gr(k, n)$$

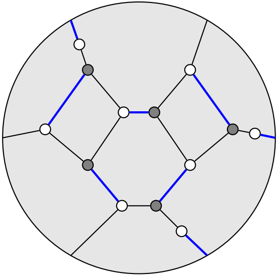
$$\Pi^\circ(\mathcal{M}) = \{\text{each set of } k \text{ cyc. cons. columns is a basis}\}$$

Preliminaries: matchings of graphs in a disc



Let G be a graph in the disc with a 2-coloring of its internal vertices. We assume each boundary vertex is adjacent to one white vertex, and no black or boundary vertices.

A **matching** of G is a subset of the edges for which every internal vertex is in exactly one edge. We assume a matching exists.



The positroid of a graph

Observation: Every matching of G must use

$$k := (\#\text{ of white vertices}) - (\#\text{ of black vertices})$$

boundary vertices.

If we index the boundary vertices clockwise by $[n]$, which k-element subsets of $[n]$ are the boundary of one or more matchings?

Theorem (Postnikov, Talaska, Postnikov-Speyer-Williams)

The k-element subsets of $[n]$ which are the boundary of some matching of G form a positroid. Every positroid occurs this way.

A graph G is **reduced** if it has the minimal number of faces among all graphs with the same positroid as G.

Two maps, both alike in dignity

In an unpublished preprint in 2003, Postnikov associates two maps to a reduced graph with positroid $\mathcal{M}$.

- A boundary measurement map

$$\mathbb{B} : \text{an algebraic torus} \longrightarrow \Pi^\circ(\mathcal{M})$$

- A cluster: a rational map

$$\mathbb{F} : \Pi^\circ(\mathcal{M}) \dashrightarrow \text{an algebraic torus}$$

Significance

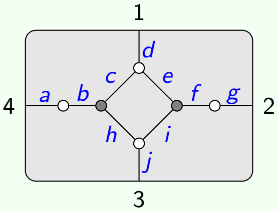
These maps have since proven fruitful in studying the combinatorics of positroids, the geometry of positroid cells, and applications to integrable systems and perturbative field theories.

Despite abundant applications, basic questions about these maps remained open for more than a decade; like, how are they related?

Generating functions of matchings

Given a k-element subset I of $[n]$, we can encode all the matchings with boundary I into a generating function D_I.

Example (Generating functions)



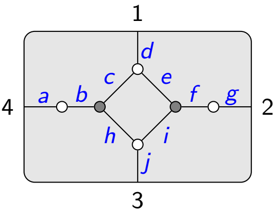
The $\binom{4}{2}$ generating functions are:

$$\begin{array}{cc} D_{12} = bdgi & D_{13} = bdfj \\ D_{14} = adfh & D_{23} = begj \\ D_{24} = acgi + aegh & D_{34} = acfj \end{array}$$

Clearly, a generating function D_I is identically zero if and only if I is not in the positroid of G.

Relations between generating functions

Remarkably, the generating functions satisfy the Plücker relations! As a consequence, there is a matrix-valued function...



$$\longrightarrow \begin{bmatrix} bd & \frac{bge}{f} & 0 & -ac \\ 0 & gi & fj & \frac{afh}{b} \end{bmatrix}$$

...whose Ith maximal minor equals the Ith generating function D_I.

$$\boxed{\begin{array}{cc} D_{12} = bdgi & D_{13} = bdfj \\ D_{14} = adfh & D_{23} = begj \\ D_{24} = acgi + aegh & D_{34} = acfj \end{array}} \qquad \boxed{\begin{array}{cc} \Delta_{12} = bdgi & \Delta_{13} = bdfj \\ \Delta_{14} = adfh & \Delta_{23} = begj \\ \Delta_{24} = acgi + aegh & \Delta_{34} = acfj \end{array}}$$

The boundary measurement map

Hence, we have a map

$$\mathbb{C}^{Edges(G)} \longrightarrow Gr(k,n)$$

This map is invariant under gauge transformations: simultaneously scaling the numbers at each edge incident to a fixed internal vertex.

Theorem (Postnikov, Talaska, M-Speyer)

For reduced G, the map $(\mathbb{C}^*)^{Edges(G)} \to Gr(k,n)$ factors through

$$\mathbb{B} : (\mathbb{C}^*)^{Edges(G)} / Gauge \longrightarrow \Pi^\circ(\mathcal{M})$$

where $\mathcal{M}$ is the positroid of G.

The map $\mathbb{B}$ is called the **boundary measurement map**.

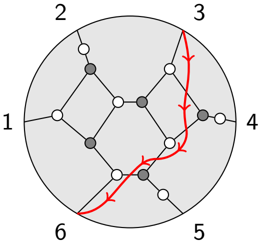
Conjecture (essentially Postnikov)

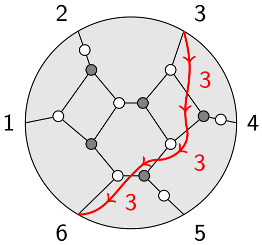
The map $\mathbb{B}$ is an open inclusion.

Strands in a reduced graph

A **strand** in reduced G is a path which...

- begins and ends at boundary vertices
- passes through the midpoints of edges
- alternates turning right around white vertices and left around black vertices



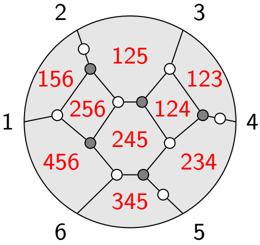


Index a strand by its source vertex, and label each face to the left of the strand by that label.

Face labels and the cluster structure

Repeating this for each strand, each face of G gets labeled by a subset of $[n]$.

Each face label is a k-element subset of $[n]$, which determines a Plücker coordinate on $Gr(k, n)$ and $\Pi^\circ(\mathcal{M})$.



Conjecture (essentially Postnikov)

The homogeneous coordinate ring of $\Pi^\circ(\mathcal{M})$ is a cluster algebra, and the Plücker coordinates of the faces of G form a cluster.

The conjecture implies the Plückers of the faces give a rational map

$$\mathbb{F} : \Pi^\circ(\mathcal{M}) \dashrightarrow (\mathbb{C}^*)^{Faces(G)}/Scaling$$

which is an isomorphism on its domain (the cluster torus).

Two conjectural tori associated to a reduced graph

We now see that a reduced graph G with positroid $\mathcal{M}$ determines two subvarieties in $\Pi^\circ(\mathcal{M})$, which are both conjecturally tori.

- The image of the boundary measurement map

$$\mathbb{B} : (\mathbb{C}^*)^{Edges(G)}/Gauge \longrightarrow \Pi^\circ(\mathcal{M})$$

- The domain of definition of the cluster of Plücker coordinates

$$\mathbb{F} : \Pi^\circ(\mathcal{M}) \dashrightarrow (\mathbb{C}^*)^{Faces(G)}/Scaling$$

Main question

What is the relation between these two subvarieties?

In a simple world, they'd coincide and we'd have an isomorphism

$$\mathbb{F} \circ \mathbb{B} : (\mathbb{C}^*)^{Edges(G)}/Gauge \xrightarrow{\sim} (\mathbb{C}^*)^{Faces(G)}/Scaling$$

The need for a twist

In the real world, we need a **twist** automorphism τ of $\Pi^\circ(\mathcal{M})$, which will fit into a composite isomorphism

$$\begin{tikzcd}
(\mathbb{C}^*)^{Edges(G)}/Gauge \arrow[dr, "\mathbb{B}"'] & & (\mathbb{C}^*)^{Faces(G)}/Scaling \\
& \Pi^\circ(\mathcal{M}) \arrow[r, "\tau"] & \Pi^\circ(\mathcal{M}) \arrow[u, dashed, "\mathbb{F}"']
\end{tikzcd}$$

and thus take the image of $\mathbb{B}$ to the domain of $\mathbb{F}$.

The twist of a matrix

Let A be a $k \times n$ matrix of rank k, and assume no zero columns. Denote the ith column of A by A_i, with cyclic indices: $A_{i+n} = A_i$.

Definition (The twist)

The **twist** $\tau(A)$ of A is the $k \times n$-matrix defined on columns by

$$\tau(A)_i \cdot A_i = 1$$

$$\tau(A)_i \cdot A_j = 0, \quad \text{if } A_j \text{ is not in the span of } \{A_i, A_{i+1}, ..., A_{j-1}\}$$

Hence, $\tau(A)_i$ is defined by its dot product with the 'first' basis of columns of A encountered starting at column i and moving right.

Example of a twist

Twisting a matrix

Consider the 3×4 matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first column $\tau(A)_1$ of the twist is a 3-vector v, such that...

$$v \cdot \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right] = 1, \quad v \cdot \left[\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right] = 0, \quad v \cdot \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right] \text{ is already fixed, and } \quad v \cdot \left[\begin{smallmatrix} 0 \\ 2 \\ 1 \end{smallmatrix}\right] = 0$$

We see that $v = \left[\begin{smallmatrix} 1 \\ -1 \\ 2 \end{smallmatrix}\right]$. In this way, we compute the twist matrix

$$\tau(A) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & -2 & 1 \end{bmatrix}$$

The twist on a positroid cell

Twisting matrices descends to a well-defined map of sets

$$Gr(k,n) \xrightarrow{\tau} Gr(k,n)$$

However, this map is not continuous; the defining equations jump when A_j deforms to a column in the span of $\{A_i, A_{i+1}, \ldots, A_{j-1}\}$.

Theorem (M-Speyer)

The domains of continuity of τ are precisely the positroid cells. The twist τ restricts to a regular automorphism of $\Pi^\circ(\mathcal{M})$.

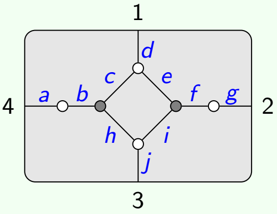
The inverse of τ is given by a virtually identical formula to τ, by reversing the order of the columns.

The induced map on tori

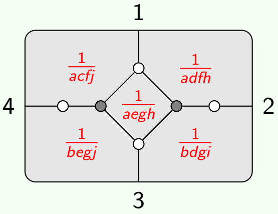
Let's consider our conjectural isomorphism of tori.

$$\mathbb{F} \circ \tau \circ \mathbb{B} : (\mathbb{C}^*)^{Edges(G)} / Gauge \longrightarrow (\mathbb{C}^*)^{Faces(G)} / Scaling$$

Example: the open cell of $Gr(2,4)$



$\xrightarrow{?}$



$$\begin{bmatrix} bd & \frac{beg}{f} & 0 & -ac \\ 0 & gi & fj & \frac{afh}{b} \end{bmatrix} \xrightarrow{\tau} \begin{bmatrix} \frac{1}{bd} & \frac{f}{beg} & \frac{h}{bcj} & 0 \\ -\frac{e}{dfi} & 0 & \frac{1}{fj} & \frac{b}{afh} \end{bmatrix}$$

$\mathbb{B}$ (from the left graph down to the left matrix), $\mathbb{F}$ (from the right matrix up to the right graph)

Minimal matchings

It looks like the entries are reciprocals of matchings! Which ones?

Proposition (Propp)

The set of matchings of G with fixed boundary I has a partial order, with a unique minimal and maximal element (if non-empty).

We verify our observation with the following lemma.

Lemma (M-Speyer)

If I is the label of a face in G, then

$$\Delta_I \circ \tau \circ \mathbb{B} = \frac{1}{\textit{product of edges in } M_I}$$

where M_I is the **minimal matching** *with boundary I.*

The isomorphism of model tori

We can collect these coordinates into a map

$$(\mathbb{C}^*)^{Edges(G)} \longrightarrow (\mathbb{C}^*)^{Faces(G)}$$

whose coordinate at a face labeled by I is the reciprocal of the product of the edges in the minimal matching with boundary I.

Lemma (M-Speyer)

The above map induces an isomorphism of algebraic tori

$$\mathbb{D} : (\mathbb{C}^*)^{Edges(G)}/Gauge \longrightarrow (\mathbb{C}^*)^{Faces(G)}/Scaling$$

In fact, the inverse $\mathbb{D}^{-1}$ can be induced from an explicit map

$$(\mathbb{C}^*)^{Faces(G)} \longrightarrow (\mathbb{C}^*)^{Edges(G)}$$

whose coordinate at an edge only uses the two adjacent faces.

Putting it all together

Theorem (M-Speyer)

For each reduced graph G, there is a commutative diagram

$$\begin{tikzcd}
(\mathbb{C}^*)^{Edges(G)}/Gauge \arrow[r, bend left, "\mathbb{D}"] \arrow[d, "\mathbb{B}"'] & (\mathbb{C}^*)^{Faces(G)}/Scaling \arrow[l, bend left, "\mathbb{D}^{-1}"] \\
\Pi^\circ(\mathcal{M}) \arrow[r, bend left, "\tau"] & \Pi^\circ(\mathcal{M}) \arrow[l, bend left, "\tau^{-1}"] \arrow[u, dashed, "\mathbb{F}"']
\end{tikzcd}$$

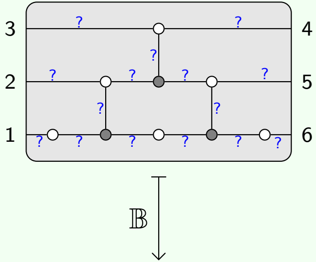
Corollaries:

- The image of $\mathbb{B}$ and the domain of $\mathbb{F}$ are open algebraic tori in $\Pi^\circ(\mathcal{M})$, and τ takes one to the other.
- The (rational) inverse of $\mathbb{B}$ is $\mathbb{D}^{-1} \circ \mathbb{F} \circ \tau$.
- The (regular) inverse of $\mathbb{F}$ is $\tau \circ \mathbb{B} \circ \mathbb{D}^{-1}$.

Application: Inverting the boundary measurement map

Let's invert $\mathbb{B}$ in a classic example!

Example: The unipotent cell in $GL(3)$, as a positroid cell



$$\begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix}$$

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$$\begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix} \xmapsto{\mathcal{T}} \begin{bmatrix} 0 & 0 & 1 & \frac{1}{a} & \frac{e}{bd-ce} & \frac{1}{c} \\ 0 & -1 & \frac{-b}{d} & \frac{-b}{ad} & \frac{-c}{be-cd} & 0 \\ 1 & \frac{e}{f} & \frac{be-cd}{df} & \frac{be-cd}{adf} & 0 & 0 \end{bmatrix}$$

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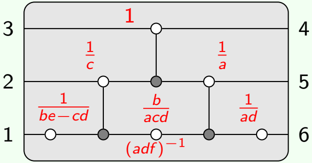

$$\Big\uparrow \mathbb{F}$$

$$\begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix} \xmapsto{\ \mathcal{T}\ } \begin{bmatrix} 0 & 0 & 1 & \frac{1}{a} & \frac{e}{bd-ce} & \frac{1}{c} \\ 0 & -1 & \frac{-b}{d} & \frac{-b}{ad} & \frac{-c}{be-cd} & 0 \\ 1 & \frac{e}{f} & \frac{be-cd}{df} & \frac{be-cd}{adf} & 0 & 0 \end{bmatrix}$$

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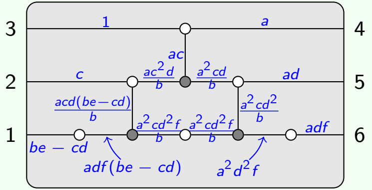
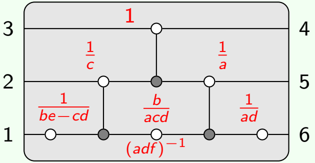


$\xleftarrow{\ \mathbb{D}^{-1}\ }$



$\uparrow \mathbb{F}$

$$\begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix} \xmapsto{\ \mathcal{T}\ } \begin{bmatrix} 0 & 0 & 1 & \frac{1}{a} & \frac{e}{bd-ce} & \frac{1}{c} \\ 0 & -1 & \frac{-b}{d} & \frac{-b}{ad} & \frac{-c}{be-cd} & 0 \\ 1 & \frac{e}{f} & \frac{be-cd}{df} & \frac{be-cd}{adf} & 0 & 0 \end{bmatrix}$$

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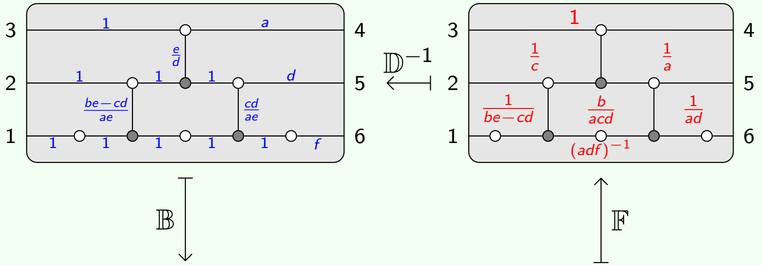
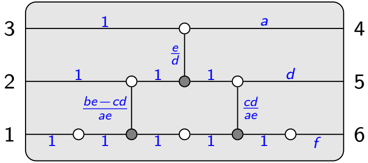


$$\begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix} \xmapsto{\mathcal{T}} \begin{bmatrix} 0 & 0 & 1 & \frac{1}{a} & \frac{e}{bd-ce} & \frac{1}{c} \\ 0 & -1 & \frac{-b}{d} & \frac{-b}{ad} & \frac{-c}{be-cd} & 0 \\ 1 & \frac{e}{f} & \frac{be-cd}{df} & \frac{be-cd}{adf} & 0 & 0 \end{bmatrix}$$

Relation to the Chamber Ansatz

So, we have the following boundary measurement map.



$$\overset{\mathbb{B}}{\longmapsto} \begin{bmatrix} 0 & 0 & 1 & a & b & c \\ 0 & -1 & 0 & 0 & d & e \\ 1 & 0 & 0 & 0 & 0 & f \end{bmatrix}$$

This is equivalent to a factorization into elementary matrices.

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} 1 & \frac{cd}{ae} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{e}{d} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{be-cd}{ae} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Our computation to find this factorization is identical to the *Chamber Ansatz* introduced by Berenstein-Fomin-Zelevinsky.

Application: Counting matchings

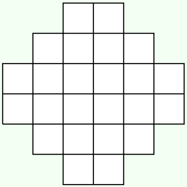
Corollary

Let G be a reduced graph with positroid $\mathcal{M}$. If A is a matrix with

- *the matroid of A is contained in $\mathcal{M}$, and*
- *for each face label I of G, the minor $\Delta_I(A) = 1$,*

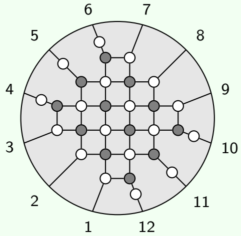
then $\Delta_J(\tau^{-1}(A))$ counts matchings with boundary J.

Example: Domino tilings of the Aztec diamond of order 3



Domino tilings of this shape...

...are the same as matchings of this graph, with boundary $\{4, 5, 6, 10, 11, 12\}$.



Application: Counting matchings

Example: (continued)

Here is an appropriate A and its inverse twist.

$$A = \begin{bmatrix} 1 & 6 & 18 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & 3 & 5 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 5 & 13 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 & 18 & 2 & 2 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 6 & 2 & 6 & 10 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 10 & 26 & 1 & 0 & 0 \end{bmatrix}$$

$$\tau^{-1}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & -2 & -2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 6 & 10 & 6 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & -6 & -18 & -26 & -10 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 5 & 13 & 18 & 6 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -3 & -5 & -6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

We compute that $\Delta_{\{4,5,6,10,11,12\}}(\tau^{-1}(A)) = 64.$ ✓

Finding A by brute force is probably not efficient, but verifying that a matrix has the necessary properties can be faster than counting.

Further directions

- Directions to generalize!
  - Positroid cells in $Gr(k,n) \to$ projected Richardson cells in partial flag varieties.
  - Reduced graphs in the disc $\to$ 'reduced graphs' in surfaces.
- If we label strands by their target instead of their source, we get a different cluster structure on the same algebra. How are they related?
- Conjecture: The twist is the decategorification of the shift functor in a categorification.
  Any cluster algebra with a Jacobi-finite potential has such a **shift** automorphism. Can this story can be extended?