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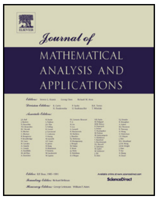

Lipschitz factorization through subsets of Hilbert space ☆

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ABSTRACT

The Euclidean distortion of a metric space, a measure of how well the metric space can be embedded into a Hilbert space, is currently an active interdisciplinary research topic. We study the corresponding notion for mappings instead of spaces, which is that of Lipschitz factorization through subsets of Hilbert space. The main theorems are two characterizations of when a mapping admits such a factorization, both of them inspired by results dealing with linear factorizations through Hilbert space. The first is a nonlinear version of a classical theorem of Kwapień in terms of "dominated" sequences of vectors, whereas the second is a duality result by means of a tensor-product approach.

1. Introduction

The matter of embedding a metric space into a "nice" Banach space is currently the subject of much interest, due to its connection to different areas like geometric group theory and theoretical computer science; a few examples of this are [20,3,13,2]. At least on the computer science side of things, the basic idea is easy to understand. To paraphrase Matoušek [14, Chap. 15]: a metric on a finite set of points is just a list of numbers that satisfy the triangle inequality, so it is hard to see any structure in that. If we could represent our given metric space as a subset of a (finite-dimensional) Hilbert space, there would be several immediate advantages. Most importantly, this representation would allow us to see much more clearly the structure of the metric space (clusters, isolated points, etc.) because in a Hilbert space we have access to geometric tools that are not available in a general metric space (think, for example, of partitioning the set into two pieces by using a hyperplane). If we insist on having an embedding of the metric space into Hilbert space that preserves the distances exactly (a so-called isometric embedding), we will only be able to do it for some very special metric spaces. It is not hard to find examples of small metric spaces (even with just four points) that cannot be represented by subsets of a Hilbert space! Such spaces have long been understood,

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characterizations of metric spaces that admit an isometric embedding into Hilbert space go back to at least the 1930's [17]. But when it comes to applications, insisting on preserving the distances exactly is not really important or practical. For example, if our data comes from measurements from the real world then our information is already only approximate. In addition plenty of computational problems are impossible to do exactly in a reasonable amount of time, so oftentimes the algorithms used in practice only give approximate answers. Thus, for most practical purposes it is more than enough to ask for an approximate embedding of a metric space into Hilbert space. Several notions of approximate embeddings appear in the literature (uniform, coarse, etc.), but in the case of finite metric spaces the most useful is that of a bi-Lipschitz embedding. Given an injective map $f : X \to Y$ between metric spaces, the *distortion* of f is the quantity $\mathrm{Lip}(f) \cdot \mathrm{Lip}(f^{-1})$. It is a quantitative measure of similarity between two metric spaces: X, and the "copy" of X that lives in Y given by $f(X)$. The infimum of the distortions of all the possible injective maps from X to Y is called the Y-distortion of X, or the Euclidean distortion of X in the special case when Y is a Hilbert space.

Let us see how this idea can naturally be generalized to maps instead of spaces. The Euclidean distortion of a metric space X is the infimum of $\mathrm{Lip}(f) \cdot \mathrm{Lip}(f^{-1})$ taken over all diagrams of the form

$$\begin{tikzcd}
 & H & \\
 & Z \arrow[u, hook, "j"'] \arrow[dr, "f^{-1}"] & \\
X \arrow[ur, "f"] \arrow[rr, "Id_X"'] & & X
\end{tikzcd}$$

where H is a Hilbert space, j is the inclusion map and Id_X is the identity map of X. Analogously, for any Lipschitz map $T : X \to Y$ between metric spaces, we can consider the infimum of $\mathrm{Lip}(R) \cdot \mathrm{Lip}(S)$ taken over all factorizations of the form

$$\begin{tikzcd}
 & H & \\
 & Z \arrow[u, hook, "j"'] \arrow[dr, "S"] & \\
X \arrow[ur, "R"] \arrow[rr, "T"'] & & Y
\end{tikzcd} \tag{1.1}$$

We will denote such infimum by $\gamma_2^{\mathrm{Lip}}(T)$, inspired by the notation of a similar situation in Banach space theory: given a linear map $T : E \to F$ between Banach spaces, $\gamma_2(T)$ is the infimum of $\|R\| \cdot \|S\|$ where $R : E \to H$ and $S : H \to F$ are linear maps such that $T = S \circ R$ and H is a Hilbert space. Note that in the linear case one can get factorizations through the whole Hilbert space and not just a subspace of it, and that is because every subspace of a Hilbert space is norm-one complemented in it. Since not every subset of a Hilbert space is a Lipschitz retract of it (that is, the image of an idempotent Lipschitz map), factoring through a subset or through the whole Hilbert space are different concepts in the Lipschitz category.

It should be noted that the factorizations in (1.1), and in fact more general factorizations through subsets of L_p spaces, have been studied by Johnson, Maurey and Schechtman [10]. Among other important results they proved that when $T : E \to F$ is a linear map between Banach spaces, $\gamma_2(T) = \gamma_2^{\mathrm{Lip}}(T)$ [10, Thm. 2].

The main results of this paper are two characterizations of when a Lipschitz map admits a factorization through a subset of a Hilbert space, as in (1.1). Both of them are inspired by results in Banach space theory that deal with linear factorizations through Hilbert space. The first (Theorem 3.3) is a nonlinear version of a classical theorem of Kwapień in terms of “dominated” sequences of vectors. The second (Theorem 4.5) is a duality result by means of a tensor-product approach, using the ideas of [4] to construct a “tensor product” between a metric space and a Banach space.

2. Notation and preliminaries

X, Y, Z will always denote metric spaces, whereas E, F, G will denote real Banach spaces. We use the convention of having *pointed* metric spaces, i.e. with a designated special point always denoted by 0. As customary, B_E denotes the closed unit ball of E and E^* its linear dual. $\mathrm{Lip}_0(X,E)$ denotes the Banach space of Lipschitz functions $T : X \to E$ such that $T(0) = 0$, with addition defined pointwise and the Lipschitz constant $\mathrm{Lip}(T)$ as the norm of T. We use the shorthand $X^\# := \mathrm{Lip}_0(X,\mathbb{R})$.

Let us recall the definition and basic properties of the space of Arens and Eells [1]. We follow the presentation in [18]. A *molecule* on a metric space X is a finitely supported function $m : X \to \mathbb{R}$ such that $\sum_{x\in X} m(x) = 0$. For $x, x' \in X$ we denote by $m_{xx'}$ the molecule $\chi_{\{x\}} - \chi_{\{x'\}}$. The simplest molecules, i.e. those of the form $am_{xx'}$ with $x, x' \in X$ and a a real number are called *atoms*. It is easy to show that every molecule can be expressed as a sum of atoms (for instance, by induction on the cardinality of the support of the molecule). The Arens–Eells space of X, denoted $\mathscr{F}(X)$, is the completion of the space of molecules with the norm

$$\|m\|_{\mathscr{F}} := \inf\left\{\sum_{j=1}^{n} |a_j| d\big(x_j, x_j'\big) : m = \sum_{j=1}^{n} a_j m_{x_j x_j'}\right\}. \tag{2.1}$$

The fundamental properties of the Arens–Eells space are summarized in the following theorem [1], [18, pp. 39–41].

Theorem 2.1.

- *(i)* *$\|\cdot\|_{\mathscr{F}}$ is a norm on the vector space of molecules on X.*
- *(ii)* *The dual of $\mathscr{F}(X)$ is (canonically) isometrically isomorphic to $X^\#$. Moreover, on bounded subsets of $X^\#$ the weak* topology coincides with the topology of pointwise convergence.*
- *(iii)* *The map $\iota : x \mapsto m_{x0}$ is an isometric embedding of X into $\mathscr{F}(X)$. Moreover, for any Banach space E and any Lipschitz map $T : X \to E$ with $T(0) = 0$ there is a unique linear map $\hat{T} : \mathscr{F}(X) \to E$ such that $\hat{T} \circ \iota = T$. Furthermore, $\|\hat{T}\| = \mathrm{Lip}(T)$.*

Because of the universal property (iii), the space $\mathscr{F}(X)$ is also called the free Lipschitz space of X, or simply the free space of X. These spaces have been recently used as tools in nonlinear Banach space theory, see [7,11] and the survey [8].

In the spirit of Arens and Eells’ original formulation, for a metric space X and a Banach space E we defined [4] an *E-valued molecule* on X to be a finitely supported function $m : X \to E$ such that $\sum_{x\in X} m(x) = 0$. The vector space of all E-valued molecules on X is denoted by $\mathcal{M}(X,E)$. An *E-valued atom* is a function of the form $vm_{xx'}$ with $v \in E$, $x, x' \in X$. Atoms are the building blocks of the space of molecules in the same sense that elementary tensors are the building blocks of the tensor product: every molecule is a sum of atoms. The analogy goes well beyond just that, a tensor-product inspired approach using these spaces of Banach-space valued molecules was used in [4] to solve the problem of duality for Lipschitz p-summing maps [6, Prob. 3].

Define a pairing $\langle\cdot,\cdot\rangle$ of $\mathrm{Lip}_0(X,E^*)$ and $\mathcal{M}(X,E)$ by

$$\langle T,m\rangle=\sum_{x\in X}\bigl\langle T(x),m(x)\bigr\rangle \quad m\in\mathcal{M}(X,E),\ T\in\mathrm{Lip}_0\bigl(X,E^*\bigr).$$

Note that this sum makes sense because m is finitely supported, and clearly $\langle\cdot,\cdot\rangle$ is bilinear. For an atom $m=vm_{x'y'}$ and $T\in\mathrm{Lip}_0(X,E^*)$,

$$\begin{aligned}\langle T,m\rangle&=\sum_{x\in X}\bigl\langle T(x),vm_{x'y'}(x)\bigr\rangle\\&=\bigl\langle T\bigl(x'\bigr),vm_{x'y'}\bigl(x'\bigr)\bigr\rangle+\bigl\langle T\bigl(y'\bigr),vm_{x'y'}\bigl(y'\bigr)\bigr\rangle=\bigl\langle T\bigl(x'\bigr)-T\bigl(y'\bigr),v\bigr\rangle.\end{aligned}$$

Therefore, for a general molecule $m=\sum_j v_j m_{x_j x'_j}$,

$$\langle T,m\rangle=\sum_j\bigl\langle Tx_j-Tx'_j,v_j\bigr\rangle. \tag{2.2}$$

We will be interested in norms on spaces of molecules that are consistent with the metric structure already present in the metric space and the Banach space involved. Borrowing from the terminology for tensor products of normed spaces, we say that a norm $\|\cdot\|$ on the space $\mathcal{M}(X,E)$ of E-valued molecules on a metric space X is *reasonable* if

(i) $\|vm_{xx'}\|\le\|v\|_E d(x,x')$ for all $x,x'\in X$, $v\in E$.
(ii) $|\langle f,v^*\circ m\rangle|\le\|v^*\|_{E^*}\,\mathrm{Lip}(f)\|m\|$ for all $v^*\in E^*$, $m\in\mathcal{M}(X,E)$ and $f\in X^\#$.

3. A nonlinear version of a theorem of Kwapień

Linear operators admitting a factorization through Hilbert space have a long history in functional analysis, going back at least to Grothendieck's Résumé [9]. Two excellent references for the subject are [5, Chap. 7] and [16, Chap. 2]. Our main result in this section is a Lipschitz version of a theorem of Kwapień [12] that characterizes the property of being factorizable through a Hilbert space, although our approach is closer to that of [16, Chap. 2]. In the aforementioned theorem, having a factorization through Hilbert space is characterized via an inequality that can be interpreted as saying that the operator behaves well with respect to a certain "domination" relationship between sequences of vectors in the domain. Our first step is to state an analogous definition of when a sequence of pairs of points in a metric space "dominates" another one. Compare to [16, p. 22].

Definition 3.1. Let $x_j,x'_j,y_i,y'_i\in X$ and $\mu_j,\lambda_i\in\mathbb{R}$. We write

$$\bigl(\lambda_i,y_i,y'_i\bigr)_{i=1}^n\prec\bigl(\mu_j,x_j,x'_j\bigr)_{j=1}^m$$

if for all $f\in X^\#$,

$$\sum_{i=1}^n\lambda_i^2\bigl|f(y_i)-f\bigl(y'_i\bigr)\bigr|^2\le\sum_{j=1}^m\mu_j^2\bigl|f(x_j)-f\bigl(x'_j\bigr)\bigr|^2.$$

In similarity with the linear case, there is an alternate characterization of this "dominance" relation that involves contractions between finite-dimensional Hilbert spaces. This corresponds to [16, Prop. 2.2].

Lemma 3.2. *Let* $x_j, x_j', y_i, y_i' \in X$ *and* $\mu_j, \lambda_i \in \mathbb{R}$. *Then* $(\lambda_i, y_i, y_i')_{i=1}^n \prec (\mu_j, x_j, x_j')_{j=1}^m$ *if and only if there exists a matrix* (a_{ij}) *such that*

$$\sum_{i=1}^n \left| \sum_{j=1}^m a_{ij} t_j \right|^2 \le \sum_{j=1}^m |t_j|^2 \quad \textit{for all } (t_j)_{j=1}^m \in \mathbb{R}^m \tag{3.1}$$

and for $1 \le i \le n$,

$$\lambda_i m_{y_i y_i'} = \sum_{j=1}^m a_{ij} \mu_j m_{x_j x_j'}.$$

Proof. Suppose that $(\lambda_i, y_i, y_i')_{i=1}^n \prec (\mu_j, x_j, x_j')_{j=1}^m$. Let

$$S = \{ (\mu_j [f(x_j) - f(x_j')])_{j=1}^m : f \in X^\# \} \subseteq \ell_2^m.$$

Define a linear operator $A : S \to \ell_2^n$ by

$$A((\mu_j [f(x_j) - f(x_j')])_{j=1}^m) = (\lambda_i [f(y_i) - f(y_i')])_{i=1}^n.$$

Note that the condition $(\lambda_i, y_i, y_i')_{i=1}^n \prec (\mu_j, x_j, x_j')_{j=1}^m$ implies that A is well-defined, whereas (3.1) implies that $\|A\| \le 1$. By the Hahn–Banach theorem, we can extend A to an operator $\tilde{A} : \ell_2^m \to \ell_2^n$ with $\|\tilde{A}\| \le 1$. The matrix representation (a_{ij}) of $\tilde{A}$ with respect to the canonical bases of ℓ_2^m and ℓ_2^n clearly satisfies (3.1), and by definition we have for all $f \in X^\#$ and $1 \le i \le n$

$$\lambda_i [f(y_i) - f(y_i')] = \sum_{j=1}^m a_{ij} \mu_j [f(x_j) - f(x_j')].$$

By the duality between $\mathscr{F}(X)$ and $X^\#$ (Theorem 2.1), this means precisely that

$$\lambda_i m_{y_i y_i'} = \sum_{j=1}^m a_{ij} \mu_j m_{x_j x_j'}.$$

For the converse, we just reverse the preceding argument. □

We now proceed to prove the promised characterization of Lipschitz maps that factor through a subset of a Hilbert space (compare to [16, Thm. 2.4]).

Theorem 3.3. *Let* X *and* Z *be metric spaces, and* $C > 0$. *For a map* $T : X \to Z$ *the following are equivalent:*

(i) There exist a Hilbert space H *and Lipschitz maps* $R : X \to H$, $S : R(X) \to Z$ *such that* $T = SR$ *and* $\mathrm{Lip}(R) \cdot \mathrm{Lip}(S) \le C$.

(ii) Whenever $x_j, x_j', y_i, y_i' \in X$ *and* $\mu_j, \lambda_i \in \mathbb{R}$ *satisfy* $(\lambda_i, y_i, y_i')_{i=1}^n \prec (\mu_j, x_j, x_j')_{j=1}^m$, *we have that*

$$\sum_{i=1}^n \lambda_i^2 d_Z(Ty_i, Ty_i')^2 \le C^2 \sum_{j=1}^m \mu_j^2 d_X(x_j, x_j')^2.$$

Note that, in this case, $\gamma_2^{\mathrm{Lip}}(T)$ *is the infimum of such constants* C.

Proof. *(i)* $\Rightarrow$ *(ii)* Suppose we have such a factorization. By rescaling the Hilbert space, we may assume that $\operatorname{Lip}(S) = 1$ and $\operatorname{Lip}(R) \le C$. Let $x_j, x'_j, y_i, y'_i \in X$ and $\lambda_i, \mu_j \in \mathbb{R}$ be such that $(\lambda_i, y_i, y'_i)_{i=1}^n \prec (\mu_j, x_j, x'_j)_{j=1}^m$. Note that for any $v \in H$ the function $x \mapsto \langle Rx, v\rangle$ is in $X^\#$, so

$$\sum_{i=1}^n \lambda_i^2 \bigl|\bigl\langle Ry_i - Ry'_i, v\bigr\rangle\bigr|^2 \le \sum_{j=1}^m \mu_j^2 \bigl|\bigl\langle Rx_j - Rx'_j, v\bigr\rangle\bigr|^2 \quad \text{for all } v \in H.$$

Let $(v_\alpha)_{\alpha\in A}$ be an orthonormal basis for H. Since $\|v\|^2 = \sum_{\alpha\in A} |\langle v, v_\alpha\rangle|^2$ for any $v \in H$, we conclude that

$$\sum_{i=1}^n \lambda_i^2 \bigl\|Ry_i - Ry'_i\bigr\|^2 \le \sum_{j=1}^m \mu_j^2 \bigl\|Rx_j - Rx'_j\bigr\|^2.$$

Hence,

$$\begin{aligned}
\sum_{i=1}^n \lambda_i^2 d_Z\bigl(Ty_i, Ty'_i\bigr)^2 &= \sum_{i=1}^n \lambda_i^2 d_Z\bigl(SRy_i, SRy'_i\bigr)^2 \le \sum_{i=1}^n \lambda_i^2 \operatorname{Lip}(S)^2 \bigl\|Ry_i - Ry'_i\bigr\|^2 \\
&= \sum_{i=1}^n \lambda_i^2 \bigl\|Ry_i - Ry'_i\bigr\|^2 \le \sum_{j=1}^m \mu_j^2 \bigl\|Rx_j - Rx'_j\bigr\|^2 \\
&\le \sum_{j=1}^m \mu_j^2 \operatorname{Lip}(R)^2 d_X\bigl(x_j, x'_j\bigr)^2 \le C^2 \sum_{j=1}^m \mu_j^2 d_X\bigl(x_j, x'_j\bigr)^2.
\end{aligned}$$

(ii) $\Rightarrow$ *(i)* For each $x \in X$, denote by δ_x its corresponding evaluation function in $C(B_{X^\#})$. Consider the following subsets of $C(B_{X^\#})$:

$$\begin{aligned}
K_1 &:= \left\{ \sum_{i=1}^n \lambda_i^2 |\delta_{y_i} - \delta_{y'_i}|^2 : \lambda_i \in \mathbb{R},\ \sum_{i=1}^n \lambda_i^2 d_Z\bigl(Ty_i, Ty'_i\bigr)^2 \ge 1 \right\} \\
K_2 &:= \left\{ \sum_{j=1}^m \mu_j^2 |\delta_{x_j} - \delta_{x'_j}|^2 : \mu_j \in \mathbb{R},\ \sum_{j=1}^m \mu_j^2 d_X\bigl(x_j, x'_j\bigr)^2 \le 1 \right\}.
\end{aligned}$$

Clearly, both K_1 and K_2 are convex. Set

$$K = \bigcup_{\rho > C} (\rho K_1 - K_2).$$

Note that K is also convex: Let $h \in \rho_1 K_1 - K_2$, $h' \in \rho' K_1 - K_2$ with $\rho' > \rho$. Then $h = \rho h_1 - h_2$, $h' = \rho' h'_1 - h_2$ with $h_r, h'_r \in K_r$, $r = 1, 2$. Note that $\rho' h'_1 = \rho(\rho'/\rho) h'_1$ and $\rho'/\rho > 1$, so in fact $\rho' h'_1 \in \rho K_1$ (since $\tilde{h} \in K_1$, $\eta \ge 1$ imply $\eta \tilde{h} \in K_1$). Therefore, we in fact have $h, h' \in \rho K_1 - K_2$ from where, using the convexity of K_1 and K_2, it is obvious that $\omega h + (1 - \omega) h \in \rho K_1 - K_2 \subset K$ for any $\omega \in [0, 1]$. Moreover, the condition *(ii)* implies that every function $h \in K$ has a maximum ≥ 0 on $B_{X^\#}$. Otherwise, we would have $\rho > C$, $x_j, x'_j, y_i, y'_i \in X$, $\lambda_i, \mu_j \in \mathbb{R}$ such that

$$\sum_{i=1}^n \lambda_i^2 d_Z\bigl(Ty_i, Ty'_i\bigr)^2 \ge 1 \ge \sum_{j=1}^m \mu_j^2 d_X\bigl(x_j, x'_j\bigr)^2$$

and for all $f \in B_{X^\#}$,

$$\rho^2 \sum_{i=1}^n \lambda_i^2 \bigl|f(y_i) - f\bigl(y'_i\bigr)\bigr|^2 - \sum_{j=1}^m \mu_j^2 \bigl|f(x_j) - f(x_j)\bigr|^2 \le 0.$$

But then

$$\sum_{i=1}^{n}\rho^2\lambda_i^2\big|f(y_i)-f\big(y_i'\big)\big|^2\le\sum_{j=1}^{m}\mu_j^2\big|f(x_j)-f(x_j)\big|^2\quad\text{for all } f\in X^{\#}$$

despite the fact that

$$\sum_{i=1}^{n}\rho^2\lambda_i^2 d_Z\big(Ty_i,Ty_i'\big)^2>C^2\sum_{j=1}^{m}\mu_j^2 d_X\big(x_j,x_j'\big)^2,$$

in plain contradiction with (ii).

Therefore, K is disjoint from the open cone N of negative functions in $C(B_{X^{\#}})$. By the Hahn–Banach and Riesz representation theorems, there exists a signed Borel measure ν that separates N and K, i.e. there exists a real number α such that for all $h\in K$, $g\in N$ $\int h d\nu\ge\alpha\ge g d\nu$. Since N is closed under multiplication by positive constants, $\alpha\ge 0$. Then $\int g d\nu\le 0$ for all $g\in N$, so ν is a positive measure such that $\int h d\nu\ge 0$ for all $h\in K$. Define $R:X\to L_2(\nu)$ by $R(x)=\delta_x$ and $S:R(X)\subset L_2(\nu)\to E$ by $S(\delta_x)=Tx$. Note that $T=SR$ and multiplying ν by an appropriate positive constant we may assume that $\operatorname{Lip}(R)=C$.

Let $x,x',y,y'\in X$ be such that $x\ne x'$ and $Ty\ne Ty'$. From the definition of ν we have

$$C^2\frac{1}{d_Z(Ty,Ty')^2}\int_{B_X^{\#}}\big|f(y)-f\big(y'\big)\big|^2 d\nu(f)\ge\frac{1}{d_X(x,x')^2}\int_{B_X^{\#}}\big|f(x)-f\big(x'\big)\big|^2 d\nu(f)$$

or equivalently

$$\frac{C}{d_Z(Ty-Ty')}\big\|Ry-Ry'\big\|_{L_2(\nu)}\ge\frac{1}{d_X(x,x')}\big\|Rx-Rx'\big\|_{L_2(\nu)}.$$

Choosing $x,x'\in X$ so that $\|Rx-Rx'\|_{L_2(\nu)}/d_X(x,x')$ is arbitrarily close to $\operatorname{Lip}(R)=C$, we conclude

$$\|\delta_y-\delta_{y'}\|_{L_2(\nu)}=\big\|Ry-Ry'\big\|_{L_2(\nu)}\ge d_Z\big(Ty,Ty'\big)=d_Z\big(S(\delta_y),S(\delta_{y'})\big).$$

Therefore $\operatorname{Lip}(S)\le 1$, so we have condition (i). □

Observe that, as in the linear case, the measure that appears in the preceding proof is not necessarily a probability measure. Moreover, let us note that using the by now classical argument due to Farmer, Johnson, Mendel and Schechtman appearing in [6], in condition (ii) of Theorem 3.3 it suffices to consider the case where all λ_i and μ_j are equal to 1. As an easy consequence of the theorem, we get that γ_2^{Lip} is finitely determined (compare to [16, Thm. 2.3]).

Corollary 3.4. *For a Lipschitz map $T:X\to Y$ between metric spaces and a constant $C>0$, $\gamma_2^{\mathrm{Lip}}(T:X\to Y)\le C$ if and only if for every finite subset $Z\subset X$, $\gamma_2^{\mathrm{Lip}}(T|_Z:Z\to Y)\le C$.*

This is very well known in the case of isometric embeddings of metric spaces into Hilbert space (see, for example, [19, Lemma 2.3]), and more generally it is known that the embeddability of locally finite metric spaces into Banach spaces is finitely determined [15, Thm. 2.6] (we thank Mikhail Ostrovskii for pointing out the latter reference). A classical way of proving the linear result corresponding to Corollary 3.4 is to use ultraproduct techniques, as it is done in [16, Thm. 2.3]. The proof of [15, Thm. 2.6] also makes use of ultraproducts, but our proof avoids such sophisticated tools.

Another immediate corollary is the following (compare to [16, Cor. 2.7]):

Corollary 3.5. *Let $T_i : X \to Y$ be a net of Lipschitz maps between the metric spaces X and Y which converges pointwise to a map $T : X \to Y$. If $\sup_i \gamma_2^{\mathrm{Lip}}(T_i) < \infty$, then T factors through a subset of a Hilbert space and moreover $\gamma_2^{\mathrm{Lip}}(T) \leq \sup_i \gamma_2^{\mathrm{Lip}}(T_i)$.*

An easy modification of the arguments in this section gives analogous results for factorizations through a subset of an $L_p(\mu)$ space, $1 \leq p < \infty$, but we have chosen to present only the case $p = 2$ for the sake of clarity.

4. Duality

Given pointed metric spaces X and Y, let us denote by $\Gamma_2^{\mathrm{Lip}}(X,Y)$ the set of all maps $T : X \to Y$ such that $\gamma_2^{\mathrm{Lip}}(T) < \infty$ and $T(0) = 0$. We will be particularly interested in the case when the codomain Y is a Banach space, because in that case $\Gamma_2^{\mathrm{Lip}}(X,Y)$ turns out to be a Banach space itself.

Proposition 4.1. *Let X be a pointed metric space and E be a Banach space. Then $(\Gamma_2^{\mathrm{Lip}}(X,E), \gamma_2^{\mathrm{Lip}}(\cdot))$ is a Banach space.*

Proof. Let $T_1, T_2 \in \Gamma_2^{\mathrm{Lip}}(X,E)$. Quite clearly, $(T_1 + T_2)(0) = 0$. Now let $x_j, x'_j, y_i, y'_i \in X$ and $\mu_j, \lambda_i \in \mathbb{R}$ satisfy $(\lambda_i, y_i, y'_i)_{i=1}^n \prec (\mu_j, x_j, x'_j)_{j=1}^m$. By Theorem 3.3,

$$\left(\sum_{i=1}^n \lambda_i^2 \left\|T_1 y_i - T_1 y'_i\right\|^2\right)^{1/2} \leq \gamma_2^{\mathrm{Lip}}(T_1)\left(\sum_{j=1}^m \mu_j^2 d_X\left(x_j, x'_j\right)^2\right)^{1/2}, \quad \text{and}$$

$$\left(\sum_{i=1}^n \lambda_i^2 \left\|T_2 y_i - T_2 y'_i\right\|^2\right)^{1/2} \leq \gamma_2^{\mathrm{Lip}}(T_2)\left(\sum_{j=1}^m \mu_j^2 d_X\left(x_j, x'_j\right)^2\right)^{1/2}.$$

Using the triangle inequality for both the norms on E and ℓ_2, together with the two inequalities above, we have

$$\begin{aligned}
&\left(\sum_{i=1}^n \lambda_i^2 \left\|(T_1+T_2) y_i - (T_1+T_2) y'_i\right\|^2\right)^{1/2} \\
&\quad \leq \left(\sum_{i=1}^n \lambda_i^2 \left(\left\|T_1 y_i - T_1 y'_i\right\| + \left\|T_2 y_i - T_2 y'_i\right\|\right)^2\right)^{1/2} \\
&\quad \leq \left(\sum_{i=1}^n \lambda_i^2 \left\|T_1 y_i - T_1 y'_i\right\|^2\right)^{1/2} + \left(\sum_{i=1}^n \lambda_i^2 \left\|T_2 y_i - T_2 y'_i\right\|^2\right)^{1/2} \\
&\quad \leq \left(\gamma_2^{\mathrm{Lip}}(T_1) + \gamma_2^{\mathrm{Lip}}(T_2)\right)\left(\sum_{j=1}^m \mu_j^2 d_X\left(x_j, x'_j\right)^2\right)^{1/2}
\end{aligned}$$

Another appeal to Theorem 3.3 tells us that $T_1 + T_2$ is in $\Gamma_2^{\mathrm{Lip}}(X,E)$, and moreover $\gamma_2^{\mathrm{Lip}}(T_1 + T_2) \leq \gamma_2^{\mathrm{Lip}}(T_1) + \gamma_2^{\mathrm{Lip}}(T_2)$. It is clear that for a constant λ and $T \in \Gamma_2^{\mathrm{Lip}}(X,E)$ we have $\gamma_2^{\mathrm{Lip}}(\lambda T) = |\lambda| \gamma_2^{\mathrm{Lip}}(T)$. Moreover, $\gamma_2^{\mathrm{Lip}}(T) = 0$ implies that $\mathrm{Lip}(T) = 0$, so we conclude $T = 0$ (this is where it is important to have the condition $T(0) = 0$). Thus, $(\Gamma_2^{\mathrm{Lip}}(X,E), \gamma_2^{\mathrm{Lip}}(\cdot))$ is a normed space.

To prove that $(\Gamma_2^{\mathrm{Lip}}(X,E),\gamma_2^{\mathrm{Lip}}(\cdot))$ is complete, it suffices to prove that every absolutely convergent sequence is convergent. So let $(T_n)_{n=1}^{\infty}$ be a sequence in $\Gamma_2^{\mathrm{Lip}}(X,E)$ such that $\sum_{n=1}^{\infty}\gamma_2^{\mathrm{Lip}}(T_n)<\infty$. Since $\mathrm{Lip}(\cdot)\le\gamma_2^{\mathrm{Lip}}(\cdot)$, the series $\sum_{n=1}^{\infty}T_n$ converges in $\mathrm{Lip}_0(X,E)$ to a function $T\in\mathrm{Lip}_0(X,E)$. If we can prove that $\gamma_2^{\mathrm{Lip}}(T)\le\sum_{n=1}^{\infty}\gamma_2^{\mathrm{Lip}}(T_n)$ we will be done, since then we can easily get that T is the γ_2^{Lip}-limit of the series $\sum_{n=1}^{\infty}T_n$. But that follows easily from Corollary 3.5 applied to the partial sums $S_n=\sum_{m=1}^{n}T_n$. □

Thus, it would be interesting to figure out what Banach space is in duality with $\Gamma_2^{\mathrm{Lip}}(X,E)$. Our main result in this section will answer this question, borrowing from the classical tensor-product approach (see [12], [16, Sec. 2.b], [5, Chap. 7]). In order to do that we will use the setting of Banach-space valued molecules on a metric space from [4] (see Section 2). Let us now give the definition of the norm that is in duality with γ_2^{Lip}.

Definition 4.2. Let m be an E-valued molecule on X. Define

$$\begin{aligned}\|m\|_* = \inf\Bigg\{ &\left(\sum_{i=1}^{n}\|v_i\|^2\right)^{1/2}\left(\sum_{j=1}^{m}\mu_j^2 d(x_j,x_j')^2\right)^{1/2}:\\ &x_j,x_j',y_i,y_i'\in X,\ \lambda_i,\mu_j\in\mathbb{R},\ v_i\in E,\\ &m=\sum_{i=1}^{n}\lambda_i v_i m_{y_i y_i'} \text{ and } (\lambda_i,y_i,y_i')_{i=1}^{n}\prec(\mu_j,x_j,x_j')_{j=1}^{m}\Bigg\}\end{aligned}$$

One could be tempted to take a slightly different condition more closely related to the one in the linear case, namely

$$m=\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}v_i\mu_j m_{x_j x_j'} \quad \text{with } (a_{ij}) \text{ satisfying (3.1)},$$

which is equivalent to having for all $f\in X^{\#}$ and $v^*\in E^*$ the inequality

$$\left|\langle v^*\circ m,f,\times\rangle\right|\le\left(\sum_{i=1}^{n}\left|v^*(v_i)\right|^2\right)^{1/2}\left(\sum_{j=1}^{m}\mu_j^2\left|f(x_j)-f(x_j')\right|^2\right)^{1/2}.$$

Unfortunately that is not the right choice, it turns out that we also need each of the sums $\sum_{j=1}^{m}a_{ij}\mu_j m_{x_j x_j'}$ to be an elementary molecule.

Lemma 4.3. *$\|\cdot\|_*$ is a norm on $\mathcal{M}(X,E)$.*

Proof. It is clear that for any molecule $m\in\mathcal{M}(X,E)$ and any scalar λ, $\|m\|_*\ge 0$ and $\|\lambda m\|_*=|\lambda|\|m\|_*$.

Let $m_1,m_2\in\mathcal{M}(X,E)$ and $\varepsilon>0$. Choose a representation $m_1=\sum_{i=1}^{n}\lambda_i v_i m_{y_i y_i'}$ and $(\mu_j,x_j,x_j')_{j=1}^{m}\succ(\lambda_i,y_i,y_i')_{i=1}^{n}$ such that

$$\left(\sum_{i=1}^{n}\|v_i\|^2\right)^{1/2}\left(\sum_{j=1}^{m}\mu_j^2 d(x_j,x_j')^2\right)^{1/2}\le\|m_1\|_*+\varepsilon.$$

By absorbing a constant into the v_i's, we may assume that

$$\left(\sum_{i=1}^{n}\|v_i\|^2\right)^{1/2}\le\left(\|m_1\|_*+\varepsilon\right)^{1/2} \quad\text{and}\quad \left(\sum_{j=1}^{m}\mu_j^2 d(x_j,x_j')^2\right)^{1/2}\le\left(\|m_1\|_*+\varepsilon\right)^{1/2}.$$

Similarly, choose a representation $m_2 = \sum_{i=n+1}^{n+k} \lambda_i v_i m_{y_i y_i'}$ and $(\mu_j, x_j, x_j')_{j=m+1}^{m+l} \succ (\lambda_i, y_i, y_i')_{i=n+1}^{n+k}$ such that

$$\left(\sum_{i=n+1}^{n+k} \|v_i\|^2\right)^{1/2} \leq \left(\|m_2\|_* + \varepsilon\right)^{1/2} \quad \text{and} \quad \left(\sum_{j=m+1}^{m+l} \mu_j^2 d\left(x_j, x_j'\right)^2\right)^{1/2} \leq \left(\|m_2\|_* + \varepsilon\right)^{1/2}.$$

Then $m_1 + m_2 = \sum_{i=1}^{n+k} \lambda_i v_i m_{y_i y_i'}$, $(\mu_j, x_j, x_j')_{j=1}^{m+l} \succ (\lambda_i, y_i, y_i')_{i=1}^{n+k}$ and

$$\left(\sum_{i=1}^{n+k} \|v_i\|^2\right)^{1/2} \left(\sum_{j=1}^{m+l} \mu_j^2 d\left(x_j, x_j'\right)^2\right)^{1/2} \leq \|m_1\|_* + \|m_2\|_* + 2\varepsilon$$

so $\|m_1 + m_2\|_* \leq \|m_1\|_* + \|m_2\|_* + 2\varepsilon$, and by letting $\varepsilon \downarrow 0$ we have the triangle inequality for $\|\cdot\|_*$.

Let $T \in \mathrm{Lip}_0(X, E^*)$ be a map that admits a representation as a finite sum of the form $\sum_k v_k^* f_k$ with $(v_k^*)_k \subset E^*$, $(f_k)_k \subset X^\#$ (i.e. such that the linearization $\hat{T} : \mathscr{F}(X) \to E^*$ has finite rank). For such a T, set

$$\theta(T) = \inf\left\{\sum_k \left\|v_k^*\right\| \mathrm{Lip}(f_k)\right\}$$

where the infimum is taken over all representations as above. Now, given $m = \sum_{i=1}^n \lambda_i v_i m_{y_i y_i'} \in \mathcal{M}(X, E)$, and assume $(\lambda_i, y_i, y_i')_{i=1}^n \prec (\mu_j, x_j, x_j')_{j=1}^m$. From Lemma 3.2, there exists a matrix (a_{ij}) satisfying (3.1) and such that for $1 \leq i \leq n$, $\lambda_i m_{y_i y_i'} = \sum_{j=1}^m a_{ij} \mu_j m_{x_j x_j'}$. We then have from the pairing formula (2.2), the Cauchy–Schwarz inequality and the property (3.1) of the matrix (a_{ij}),

$$\begin{aligned}
\left|\langle T, m\rangle\right| &= \left|\sum_{i,j,k} v_k^*(v_i) a_{ij} \mu_j \left[f_k(x_j) - f_k\left(x_j'\right)\right]\right| \\
&\leq \sum_k \sum_i \left|v_k^*(v_i) \sum_j a_{ij} \mu_j \left[f_k(x_j) - f_k\left(x_j'\right)\right]\right| \\
&\leq \sum_k \left(\sum_i \left|v_k^*(v_i)\right|^2\right)^{1/2} \left(\sum_i \left|\sum_j a_{ij} \mu_j \left[f_k(x_j) - f_k\left(x_j'\right)\right]\right|^2\right)^{1/2} \\
&\leq \sum_k \left\|v_k^*\right\| \left(\sum_i \|v_i\|^2\right)^{1/2} \left(\sum_j \mu_j^2 \left|f_k(x_j) - f_k\left(x_j'\right)\right|^2\right)^{1/2} \\
&\leq \sum_k \left\|v_k^*\right\| \mathrm{Lip}(f_k) \left(\sum_i \|v_i\|^2\right)^{1/2} \left(\sum_j \mu_j^2 d\left(x_j, x_j'\right)^2\right)^{1/2}.
\end{aligned}$$

Taking the infimum over all representations of both T and m, we deduce $|\langle T, m\rangle| \leq \|m\|_\pi \theta(T)$. In particular, this applies to maps T of the form $v^* \circ f$ with $v^* \in E^*$ and $f \in X^\#$, so if m is such that $\|m\|_* = 0$ then we have, using the pairing formula (2.2),

$$0 = \left\langle v^* \circ f, m\right\rangle = \sum_j v^*(v_j)\left[f(x_j) - f\left(x_j'\right)\right] \quad \text{for all } v^* \in E^*,\ f \in X^\#.$$

By the duality between $\mathscr{F}(X)$ and $X^\#$ (see Theorem 2.1), this means that the real-valued molecule $v^* \circ m$ is equal to 0 for all $v^* \in E^*$ and consequently $m = 0$. □

Moreover, let us now show that this norm is a reasonable one.

Proposition 4.4. *The norm $\| \cdot \|_*$ is a reasonable norm.*

Proof. The obvious representation of an atom shows that $\|vm_{xx'}\|_* \le \|v\| d(x,x')$. Now, suppose $m = \sum_{i=1}^n \lambda_i v_i m_{y_i y_i'}$ and $(\lambda_i, y_i, y_i')_{i=1}^n \prec (\mu_j, x_j, x_j')_{j=1}^m$. Then

$$\begin{aligned}
\left|\langle v^* \circ m, f\rangle\right| &\le \|v^*\| \sum_{i=1}^n \|v_i\| \cdot |\lambda_i| \cdot \left|f(y_i) - f(y_i')\right| \\
&\le \|v^*\| \left(\sum_{i=1}^n \|v_i\|^2\right)^{1/2} \left(\sum_{i=1}^n \lambda_i^2 \left|f(y_i) - f(y_i')\right|^2\right)^{1/2} \\
&\le \|v^*\| \left(\sum_{i=1}^n \|v_i\|^2\right)^{1/2} \left(\sum_{j=1}^m \mu_j^2 \left|f(x_j) - f(x_j')\right|^2\right)^{1/2} \\
&\le \|v^*\| \left(\sum_{i=1}^n \|v_i\|^2\right)^{1/2} \operatorname{Lip}(f) \left(\sum_{j=1}^m \mu_j^2 d(x_j, x_j')^2\right)^{1/2}
\end{aligned}$$

so taking the infimum over all representations of m we obtain the desired inequality: $|\langle v^* \circ m, f\rangle| \le \|v^*\| \operatorname{Lip}(f) \|m\|_*$. □

The following theorem is the main result of this section. Compare to [16, Thm. 2.8].

Theorem 4.5. *Let $T : X \to E^*$ and $C > 0$. The following are equivalent:*

(i) $\gamma_2^{\mathrm{Lip}}(T) \le C$.
(ii) $|\langle T, m\rangle| \le C\|m\|_*$ *for all* $m \in \mathcal{M}(X, E)$.

Proof. *(i)* ⇒ *(ii)* Suppose that $\gamma_2^{\mathrm{Lip}}(T) \le C$. Let $m \in \mathcal{M}(X, E)$. Let $x_j, x_j', y_i, y_i' \in X$, $\lambda_i, \mu_j \in \mathbb{R}$ and $v_i \in E$ such that $m = \sum_{i=1}^n \lambda_i v_i m_{y_i y_i'}$ and $(\lambda_i, y_i, y_i')_{i=1}^n \prec (\mu_j, x_j, x_j')_{j=1}^m$. Then, using Theorem 3.3,

$$\begin{aligned}
\left|\langle T, m\rangle\right| &\le \sum_{i=1}^n \lambda_i \left|\langle Ty_i - Ty_i', v_i\rangle\right| \\
&\le \left(\sum_{i=1}^n \|v_i\|^2\right)^{1/2} \left(\sum_{i=1}^n \lambda_i^2 \left\|Ty_i - Ty_i'\right\|^2\right)^{1/2} \\
&\le \left(\sum_{i=1}^n \|v_i\|^2\right)^{1/2} \left(\sum_{j=1}^m \mu_j^2 d(x_j, x_j')^2\right)^{1/2}
\end{aligned}$$

and therefore $|\langle T, m\rangle| \le C\|m\|_*$ for all $m \in \mathcal{M}(X, E)$.

(ii) ⇒ *(i)* Assume condition *(ii)*. Suppose $x_j, x_j', y_i, y_i \in X$ and $\mu_j, \lambda_i \in \mathbb{R}$ satisfy $(\lambda_i, y_i, y_i')_{i=1}^n \prec (\mu_j, x_j, x_j')_{j=1}^m$. Then, by Lemma 3.2, there exists a matrix (a_{ij}) satisfying (3.1) such that for $1 \le i \le n$,

$$\lambda_i m_{y_i y_i'} = \sum_{j=1}^m a_{ij} \mu_j m_{x_j x_j'}.$$

Fix $\varepsilon > 0$. For each $1 \le i \le n$, choose $v_i \in E^*$ with $\|v_i\| \le 1 + \varepsilon$ and $\langle Ty_i - Ty_i', v_i\rangle = \|Ty_i - Ty_i'\|$. Let $\alpha_i \in \mathbb{R}$ be such that $\sum_{i=1}^n \alpha_i^2 = 1$ and

$$\sum_{i=1}^{n}\alpha_i\lambda_i\left\|Ty_i-Ty_i'\right\|=\left(\sum_{i=1}^{n}\lambda_i^2\left\|Ty_i-Ty_i'\right\|^2\right)^{1/2}.$$

Define a molecule by

$$m=\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}\alpha_i v_i\mu_j m_{x_jx_j'}=\sum_{i=1}^{n}\alpha_i\lambda_i v_i m_{y_iy_i'}.$$

Then, by condition (ii),

$$\begin{aligned}\left(\sum_{i=1}^{n}\lambda_i^2\left\|Ty_i-Ty_i'\right\|^2\right)^{1/2}&=\sum_{i=1}^{n}\alpha_i\lambda_i\left\|Ty_i-Ty_i'\right\|\\&=\sum_{i=1}^{n}\alpha_i\lambda_i\left\langle Ty_i-Ty_i',v_i\right\rangle=\langle T,m\rangle\\&\le C\left(\sum_{i=1}^{n}\alpha_i^2\|v_i\|^2\right)^{1/2}\left(\sum_{j=1}^{m}\mu_j^2 d\left(x_jx_j'\right)^2\right)^{1/2}\\&\le C(1+\varepsilon)\left(\sum_{i=1}^{n}\alpha_i^2\right)^{1/2}\left(\sum_{j=1}^{m}\mu_j^2 d\left(x_jx_j'\right)^2\right)^{1/2}\\&=C(1+\varepsilon)\left(\sum_{j=1}^{m}\mu_j^2 d\left(x_jx_j'\right)^2\right)^{1/2}.\end{aligned}$$

Letting $\varepsilon\downarrow 0$

$$\sum_{i=1}^{n}\lambda_i^2\left\|Ty_i-Ty_i'\right\|^2\le C^2\sum_{j=1}^{m}\mu_j^2 d\left(x_j,x_j'\right)^2,$$

so by Theorem 3.3, $\gamma_2^{\mathrm{Lip}}(T)\le C$. □

Let us finish the section by noting that Theorem 4.5 immediately implies the promised result:

Corollary 4.6. *Given a metric space X and a Banach space E, the dual space of $(\mathcal{M}(X,E),\|\cdot\|_*)$ can be isometrically isomorphically identified with $\Gamma_2^{\mathrm{Lip}}(X,E^*)$ via the pairing (2.2).*

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