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LIPSCHITZ (q,p)-MIXING OPERATORS

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ABSTRACT. Several useful results in the theory of p-summing operators, such as Pietsch's composition theorem and Grothendieck's theorem, share a common form: for certain values q and p, there is an operator such that whenever it is followed by a q-summing operator, the composition is p-summing. This is precisely the concept of (q,p)-mixing operators, defined and studied by A. Pietsch. On the other hand, J. Farmer and W. B. Johnson recently introduced the notion of a Lipschitz p-summing operator, a nonlinear generalization of p-summing operators. In this paper, a corresponding nonlinear concept of Lipschitz (q,p)-mixing operators is introduced, and several characterizations of it are proved. An interpolation-style theorem relating different Lipschitz (q,p)-mixing constants is obtained, and it is used to show reversed inequalities between Lipschitz p-summing norms.

1. INTRODUCTION

The theory of p-summing operators plays a very important role in modern Banach space theory, not only for its instrinsic beauty but also for its far-reaching applications among a wide spectrum of subjects such as Banach space geometry, harmonic analysis, approximation theory, operator theory and others. When working with p-summing operators, it is not unusual to come across an operator T with the property that $S \circ T$ is p-summing whenever S is q-summing. One example of such a situation appears in A. Pietsch's composition theorem, a very useful tool already present in his seminal paper [Pie67]: whenever $p, q, r \in [1, \infty]$ satisfy $1/p = 1/q + 1/r$, the composition of a q-summing operator followed by an r-summing operator is p-summing. Another example with T being the identity on an L_1 space is provided by a celebrated theorem of A. Grothendieck, stating that every continuous linear operator from L_1 into Hilbert space is 1-summing; therefore, any 2-summing operator with an L_1 space as domain is 1-summing. More generally, by a theorem of B. Maurey, any 2-summing operator defined on a cotype 2 space is 1-summing. Similarly, any continuous linear operator from a $C(K)$ space into a cotype 2 space is 2-summing.

Inspired by ideas of Maurey [Mau74], Pietsch [Pie80, Chap. 20] systematically studied the situation described in the previous paragraph and called such operators (q,p)-mixing. Another exposition of the subject, with a more "tensorial" point of view, can be found in [DF93, Sec. 32]. On the other hand, J. Farmer and W. B. Johnson [FJ09] recently introduced the concept of a Lipschitz p-summing operator between metric spaces. They proved that this is a true extension of the linear

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concept and obtained a nonlinear counterpart of one of the cornerstones of the theory of (linear) p-summing operators: Pietsch's celebrated domination/factorization theorem.

In the present paper, the corresponding concept of Lipschitz (q,p)-mixing operators is defined and studied. We start by recalling the necessary theory of Lipschitz p-summing operators and then introduce the main definition. Afterwards three different characterizations of Lipschitz (q,p)-mixing operators are presented. The first one is an integral inequality along the lines of Pietsch's domination theorem, while the second one corresponds to his (q,p)-mixed sequences. The third one relies on the recently developed [CD] duality theory for Lipschitz p-summing operators. Finally these characterizations are used to prove relationships between (q,p)-mixing constants and s-summing norms in various situations, in particular obtaining reversed inequalities for Lipschitz p-summing norms.

2. Notation and preliminaries

The letters X, Y, Z will denote metric spaces, whereas E, F, G will denote Banach spaces. All metric spaces under consideration will be *pointed*; i.e. each one has a special point designated by 0. For a mapping T between metric spaces, $\mathrm{Lip}(T)$ denotes its Lipschitz constant. Given a metric space X, the Banach space of real-valued Lipschitz functions defined on X that send 0 to 0 with the Lipschitz norm $\mathrm{Lip}(\cdot)$ will be denoted by $X^{\#}$. As is customary, B_E denotes the closed unit ball of a Banach space E. The letters p,q,r,s will designate elements of $[1,\infty]$, and p' denotes the exponent conjugate to p (i.e. the one that satisfies $1/p+1/p'=1$).

The remainder of this section is all from [FJ09]. Recall that for $1 \le p < \infty$ a linear operator $T : E \to F$ is called *p-summing* if there is a nonnegative constant C such that for any vectors v_j in E, the inequality

$$\sum_j \|Tv_j\|^p \le C^p \sup_{v^* \in B_{E^*}} \sum_j |v^*(v_j)|^p$$

holds. In this case, the *p-summing norm* $\pi_p(T)$ of T is the infimum of such constants C. Inspired by this useful concept, Farmer and Johnson defined the *Lipschitz p-summing norm* π_p^L of a (not necessarily linear) mapping $T : X \to Y$ as the smallest nonnegative constant C such that for any x_j, x_j' in X and any positive reals a_j,

$$\sum_j a_j d(Tx_j, Tx_j')^p \le C^p \sup_{f \in B_{X^\#}} \sum_j a_j \big|f(x_j) - f(x_j')\big|^p.$$

This definition remains unchanged if we consider only the case $a_j = 1$, a very useful observation in [FJ09] also credited to M. Mendel and G. Schechtman. The set of all Lipschitz p-summing maps from X to Y is denoted by $\Pi_p^L(X,Y)$. Note that the condition that would naturally correspond to being Lipschitz ∞-summing is just the Lipschitz condition, and we adopt this convention for notational convenience.

It is clear from the definition that the Lipschitz p-summing norm of a mapping is equal to the supremum of the Lipschitz p-summing norms of all the restrictions of said mapping to finite subsets of its domain. Also directly from the definition, it is clear that the Lipschitz p-summing norm has the ideal property: $\pi_p^L(A \circ T \circ B) \le \mathrm{Lip}(A) \cdot \pi_p^L(T) \cdot \mathrm{Lip}(B)$ whenever the composition makes sense. We next state the domination/factorization theorem for Lipschitz p-summing operators

[FJ09, Thm. 1], a particular case of the general Pietsch-type domination theorems considered in [BPR10].

Theorem 2.1. *For a mapping $T : X \to Y$ and a constant $C \geq 0$, the following are equivalent:*

(a) $\pi_p^L(T) \leq C$.

(b) There is a probability μ on $B_{X^\#}$ such that for any $x, x' \in X$,

$$d(Tx, Tx') \leq C\left[\int_{B_{X^\#}} \left|f(x) - f(x')\right|^p d\mu(f)\right]^{1/p}.$$

(c) For some (or any) isometric embedding J of Y into a 1-injective space Z, there is a factorization

$$\begin{tikzcd} L_\infty(\mu) \arrow[r, "I_{\infty,p}"] & L_p(\mu) \arrow[d, "B"] \\ X \arrow[u, "A"] \arrow[r, "T"] & Y \arrow[r, "J"] & Z \end{tikzcd}$$

with μ a probability and $\mathrm{Lip}(A) \cdot \mathrm{Lip}(B) \leq C$.

The domination theorem immediately implies the monotonicity of the Lipschitz p-summing norms, that is, $\pi_p^L(T) \geq \pi_q^L(T)$ whenever $p \leq q$.

It is important to stress that the concept of a Lipschitz p-summing operator is a true generalization of that of a (linear) p-summing operator: for a bounded linear operator T between Banach spaces, T is Lipschitz p-summing if and only if it is (linearly) p-summing, and moreover $\pi_p(T) = \pi_p^L(T)$ [FJ09, Thm. 2].

3. Definition and elementary properties

Let $1 \leq p, q \leq \infty$. An operator $T : X \to Y$ is said to be *Lipschitz (q,p)-mixing with constant* K if for any metric space Z and any Lipschitz q-summing operator $S : Y \to Z$, the composition $S \circ T$ is a Lipschitz p-summing operator and $\pi_p^L(S \circ T) \leq K\pi_q^L(S)$. The smallest such K will be denoted by $\mathfrak{m}_{q,p}^L(T)$.

A first example of such an operator already appears in [FJ09], where a nonlinear Grothendieck inequality is proved. Namely, any Lipschitz map T from a metric tree X into a Hilbert space is Lipschitz 1-summing and in fact $\pi_1^L(T) \leq K_G \,\mathrm{Lip}(T)$, where K_G is Grothendieck's constant. This result, together with the factorization theorem, Theorem 2.1, implies that the identity on X is Lipschitz $(2,1)$-mixing with constant at most K_G. D. Chen and B. Zheng [CZ] gave another proof of this nonlinear Grothendieck inequality, showing that $\mathfrak{m}_{2,1}^L(id_X) \leq A_1^{-1}$ where A_1 is the constant in Khintchine's inequality.

Note that in order to determine if a mapping $T : X \to Y$ is Lipschitz (q,p)-mixing, it suffices to consider its compositions with mappings from Y to ℓ_q (or any other infinite-dimensional L_q space, in fact). First, we may assume without loss of generality that X and Y are finite metric spaces. Now suppose that

$$(\star) \qquad \pi_p^L(R \circ T) \leq C\pi_q^L(R) \quad \text{for any} \quad R : Y \to \ell_q,$$

and let $S : Y \to Z$ be a Lipschitz q-summing map. Let $J : Z \to W$ be an isometric embedding of Z into a 1-injective space W. By the factorization theorem for Lipschitz q-summing operators, we can find a factorization

$$\begin{tikzcd} L_\infty(\mu) \arrow[r, "I_{\infty,q}"] & & L_q(\mu) \arrow[d, "B"] \\ Y \arrow[u, "A"] \arrow[r, "S"] & Z \arrow[r, "J"] & W \end{tikzcd}$$

with $\mathrm{Lip}(A) \cdot \mathrm{Lip}(B) = \pi_q^L(S)$. Since Y is a finite set, the range of $I_{\infty,q} \circ A$ is a finite subset of $L_q(\mu)$ and therefore is almost isometric to a subset of ℓ_q. Thus, for the purposes of computing Lipschitz summing norms, we may assume that $I_{\infty,q} \circ A$ is a map from Y into ℓ_q, so condition ($\star$) applies and therefore $\pi_p^L(I_{\infty,q} \circ A \circ T) \leq C\pi_q^L(I_{\infty,q} \circ A)$. The ideal property for Lipschitz q-summing operators implies that $\pi_q^L(I_{\infty,q} \circ A) \leq \mathrm{Lip}(A) \cdot \pi_q^L(I_{\infty,q}) \leq \mathrm{Lip}(A) \cdot 1$, whereas the ideal property for Lipschitz p-summing operators gives us

$$\begin{aligned} \pi_p^L(J \circ S \circ T) &= \pi_p^L(B \circ I_{\infty,q} \circ A \circ T) \\ &\leq \mathrm{Lip}(B) \cdot \pi_q^L(I_{\infty,q} \circ A \circ T) \leq \mathrm{Lip}(B) \cdot C \cdot \mathrm{Lip}(A) = C\pi_q^L(S). \end{aligned}$$

But since J is an isometric embedding $J \circ S \circ T$ and $S \circ T$ have the same Lipschitz p-summing norm, so we conclude that $\pi_p^L(S \circ T) \leq C\pi_q^L(S)$, i.e. that T is Lipschitz (q,p)-mixing with constant C.

The ideal property for Lipschitz p-summing operators implies that for any operator T, $\mathfrak{m}_{q,p}^L(T) = \mathrm{Lip}(T)$ whenever $q \leq p$ and $\mathfrak{m}_{\infty,p}^L(T) = \pi_p^L(T)$, so only the case $1 \leq p < q < \infty$ gives something new. Moreover, Lipschitz (q,p)-mixing operators also satisfy the ideal property and $\mathfrak{m}_{q,p}^L(A \circ T \circ B) \leq \mathrm{Lip}(A) \cdot \mathfrak{m}_{q,p}^L(T) \cdot \mathrm{Lip}(B)$ whenever the composition makes sense.

Just from the definition, we obtain a trivial composition formula for Lipschitz (q,p)-mixing operators: regardless of the values of p, q and r in $[1,\infty]$, the composition of a Lipschitz (p,r)-mixing operator T followed by a Lipschitz (q,p)-mixing operator S is Lipschitz (q,r)-mixing and moreover $\mathfrak{m}_{q,r}^L(ST) \leq \mathfrak{m}_{q,p}^L(S) \cdot \mathfrak{m}_{p,r}^L(T)$.

Additionally, the monotonicity of the Lipschitz p-summing norms implies a monotonicity condition for the Lipschitz (q,p)-mixing constants: whenever $p_1 \leq p_2$ and $q_2 \leq q_1$, $\mathfrak{m}_{q_2,p_2}^L(T) \leq \mathfrak{m}_{q_1,p_1}^L(T)$ for any T.

4. Characterizations

In this section three different characterizations of Lipschitz (q,p)-mixing operators are presented, all of them somewhat inspired by analogous results in the linear theory.

4.1. **Domination.** The first characterization is close in spirit to the characterization of Lipschitz p-summing operators via a dominating measure [FJ09]. Compare with [DF93, Prop. 32.4].

Theorem 4.1. *Let $1 \leq p \leq q \leq \infty$, $T : X \to Y$ Lipschitz and $C \geq 0$. The following are equivalent:*

(a) T is Lipschitz (q,p)-mixing with $\mathfrak{m}_{q,p}^L(T) \leq C$.

(b) For any probability measure μ on $B_{Y^\#}$ there exists a probability measure ν on $B_{X^\#}$ such that for all $x, x' \in X$,

$$\left[\int_{B_{Y^\#}} |g(Tx) - g(Tx')|^q \, d\mu(g)\right]^{1/q} \leq C \left[\int_{B_{X^\#}} |f(x) - f(x')|^p \, d\nu(f)\right]^{1/p}.$$

(c) For any $x_1, \ldots, x_m, x'_1, \ldots, x'_m \in X$ and $g_1, \ldots, g_n \in Y^\#$,

$$\left[\sum_{j=1}^m \left[\sum_{k=1}^n \left|g_k(Tx_j) - g_k(Tx'_j)\right|^q\right]^{p/q}\right]^{1/p} \leq C \left[\sum_{k=1}^n \operatorname{Lip}(g_k)^q\right]^{1/q} \cdot \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^m \left|f(x_j) - f(x'_j)\right|^p\right]^{1/p}.$$

(d) For any $x_1, \ldots, x_m, x'_1, \ldots, x'_m \in X$ and any probability measure μ on $B_{Y^\#}$,

$$\left[\sum_{j=1}^m \left(\int_{B_{Y^\#}} \left|g(Tx_j) - g(Tx'_j)\right|^q d\mu(g)\right)^{p/q}\right]^{1/p} \leq C \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^m \left|f(x_j) - f(x'_j)\right|^p\right]^{1/p}. \tag{4.1}$$

In this case, $\mathfrak{m}^L_{q,p}(T)$ is equal to the infimum of such constants C in either (b), (c) or (d).

Proof. The case $q = \infty$ reduces to the Domination Theorem for Lipschitz p-summing operators (Thm. 2.1), so we will assume $1 \leq p \leq q < \infty$.

$(a) \Rightarrow (b)$: Suppose that $T : X \to Y$ is Lipschitz (q,p)-mixing, and let μ be a probability measure on $B_{Y^\#}$. By restricting to Y the canonical inclusion $C(B_{Y^\#}) \hookrightarrow L_q(\mu)$, we get a Lipschitz q-summing operator $j_\mu : Y \to L_q(\mu)$ with Lipschitz q-summing norm at most 1. Hence, since T is Lipschitz (q,p)-mixing, the composition $j_\mu \circ T : X \to L_q(\mu)$ is Lipschitz p-summing. By the Pietsch domination theorem for Lipschitz p-summing operators (Thm. 2.1), there is a probability measure ν on $B_{X^\#}$ such that for all $x, x' \in X$,

$$\|j_\mu(Tx) - j_\mu(Tx')\|_{L_q(\mu)} \leq \pi^L_p(j_\mu \circ T) \left[\int_{B_{X^\#}} \left|f(x) - f(x')\right|^p d\nu(f)\right]^{1/p},$$

i.e.

$$\left[\int_{B_{Y^\#}} \left|g(Tx) - g(Tx')\right|^q d\mu(g)\right]^{1/q} \leq \pi^L_p(j_\mu \circ T) \left[\int_{B_{X^\#}} \left|f(x) - f(x')\right|^p d\nu(f)\right]^{1/p},$$

so we have condition (b) with $C = \pi^L_p(j_\mu \circ T) \leq \mathfrak{m}^L_{q,p}(T)\pi^L_q(j_\mu) \leq \mathfrak{m}^L_{q,p}(T)$.

$(b) \Rightarrow (c)$: By homogeneity, we may assume without loss of generality that $\sum_{k=1}^n \operatorname{Lip}(g_k)^q = 1$. Then $\mu := \sum_{k=1}^n \operatorname{Lip}(g_k)^q \delta_{g_k/\operatorname{Lip}(g_k)}$ (where δ_g is the Dirac measure at $g \in Y^\#$) is a probability measure on $B_{Y^\#}$, so there exists a corresponding ν as in (b). Therefore,

$$\sum_{j=1}^m \left[\sum_{k=1}^n \left|g_k(Tx_j) - g_k(Tx'_j)\right|^q\right]^{p/q} = \sum_{j=1}^m \left[\int_{B_{Y^\#}} \left|g(Tx_j) - g(Tx'_j)\right|^q d\mu(g)\right]^{p/q}$$
$$\leq C^p \sum_{j=1}^m \int_{B_{X^\#}} |f(x) - f(x')|^p \, d\nu(f) \leq C^p \sup_{f \in B_{X^\#}} \sum_{j=1}^m |f(x) - f(x')|^p,$$

so we have (c) with the same constant C.

$(c) \Rightarrow (d)$: Condition (c) means that all finitely supported probability measures μ on $B_{Y^\#}$ already satisfy (4.1). Since the set of all finitely supported probability measures on $B_{Y^\#}$ is $\sigma\big(C(B_{Y^\#})^*, C(B_{Y^\#})\big)$-dense in the set of all probability measures on $B_{Y^\#}$, it follows that inequality (4.1) holds for all probability measures μ on $B_{Y^\#}$.

$(d) \Rightarrow (a)$: Now let $S : Y \to Z$ be Lipschitz q-summing. Appealing to the domination theorem again, there is a measure μ on $B_{Y^\#}$ such that for all $y, y' \in Y$,

$$d_Z\big(Sy, Sy\big)^p \leq \pi_q^L(S)^p \left[\int_{B_{Y^\#}} \left|g(y) - g(y')\right|^q d\mu(g)\right]^{p/q}.$$

Fix $x_1, \dots, x_m, x'_1, \dots, x'_m \in X$. Then, from the previous inequality,

$$\left[\sum_{j=1}^m d_Z\big(S(Tx_j), S(Tx'_j)\big)^p\right]^{1/p}$$
$$\leq \pi_q^L(S) \left[\sum_{j=1}^m \left[\int_{B_{Y^\#}} \left|g(Tx_j) - g(Tx'_j)\right|^q d\mu(g)\right]^{p/q}\right]^{1/p},$$

which together with (4.1) implies that

$$\left[\sum_{j=1}^m d_Z\big(STx_j, STx'_j\big)^p\right]^{1/p} \leq C\pi_q^L(S) \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^m \left|f(x) - f(x')\right|^p\right]^{1/p},$$

so $S \circ T$ is Lipschitz p-summing and $\pi_p^L(S \circ T) \leq C\pi_q^L(S)$. Therefore, T is Lipschitz (q,p)-mixing and $\mathfrak{m}_{q,p}^L(T) \leq C$. □

4.2. **Lipschitz (q,p)-mixed sequences.** Linear (q,p)-mixing operators were given such a name by Pietsch [Pie80] because a linear operator is linearly (q,p)-mixing if and only if it maps every weakly p-summable sequence into a (q,p)-mixed sequence, i.e. one that can be expressed as the pointwise product of a weakly q-summable sequence and an r-summable scalar sequence where $1/p = 1/q + 1/r$. The analogous result in the nonlinear case will follow from Theorem 4.1 as soon as we find an appropiate nonlinear counterpart of (q,p)-mixing sequences. We will use Ky Fan's minimax lemma as stated in [Pie80, Lemma E.4.2]. A collection of real-valued functions $\mathscr{A}$ defined on a set K is called *concave* if given $\Phi_1, \dots, \Phi_n \in \mathscr{A}$ and $\alpha_1, \dots, \alpha_n \geq 0$ such that $\sum_{j=1}^n \alpha_j = 1$, there is $\Phi \in \mathscr{A}$ satisfying $\Phi(x) \geq \sum_{j=1}^n \alpha_j \Phi_j(x)$ for all $x \in K$. Now we prove a result analogous to [Pie80, Thm. 16.4.3] (credited mostly to [Mau74]).

Proposition 4.2. *Let $1 \leq p < q < \infty$ and $1/p = 1/q + 1/r$. Then, for any points $x_1, \ldots, x_n, x'_1, \ldots, x'_n$ in X,*

(4.2)
$$\sup\left\{\left[\sum_{j=1}^{n}\left[\int_{B_{X^\#}}\left|f(x_j)-f(x'_j)\right|^q d\mu(f)\right]^{p/q}\right]^{1/p} : \mu \text{ is a probability on } B_{X^\#}\right\}$$
$$= \inf\left\{\left[\sum_{j=1}^{n}\lambda_j^r\right]^{1/r} \sup_{f\in B_{X^\#}}\left[\sum_{j=1}^{n}\lambda_j^{-q}\left|f(x_j)-f(x'_j)\right|^q\right]^{1/q} : \lambda_j > 0\right\}.$$

Proof. Define σ to be the supremum on the left-hand side of (4.2) (noting that it is finite). Let $u = r/p$ and $v = q/p$, so that $1/u + 1/v = 1$. We now consider the compact, convex subset

$$K = \left\{\xi = (\xi_j)_{j=1}^n : \sum_{j=1}^{n}\xi_j^u \leq \sigma^p \text{ and } \xi_j \geq 0\right\}$$

of ℓ_u^n. For $\varepsilon > 0$ and μ a probability on $B_{X^\#}$, observe that the equation

$$\Phi(\xi) = \sum_{j=1}^{n}(\xi_j+\varepsilon)^{-v}\int_{B_{X^\#}}\left|f(x_j)-f(x'_j)\right|^q d\mu(f)$$

defines a continuous convex function Φ on K. Take the special vector $\xi \in \mathbb{R}^n$ with

$$\xi_j = \left(\int_{B_{X^\#}}\left|f(x_j)-f(x'_j)\right|^q d\mu(f)\right)^{1/(uv)}.$$

Then $\xi \in K$ and $\Phi(\xi) \leq \sigma^p$. Since the collection $\mathscr{A}$ of all functions Φ obtained in this way is concave, by Ky Fan's lemma we can find $\xi^0 \in K$ such that $\Phi(\xi^0) \leq \sigma^p$ for all $\Phi \in \mathscr{A}$ simultaneously. In particular, considering the Dirac measure δ_f at a function $f \in B_{X^\#}$ we obtain

$$\sum_{j=1}^{n}(\xi_j^0+\varepsilon)^{-v}\left|f(x_j)-f(x'_j)\right|^q \leq \sigma^p.$$

Set $\lambda_j(\varepsilon) := (\xi_j^0+\varepsilon)^{1/p}$. Then

$$\lim_{\varepsilon\downarrow 0}\left[\sum_{j=1}^{n}\lambda_j(\varepsilon)^r\right]^{1/r} = \left[\sum_{j=1}^{n}\xi_j^{r/p}\right]^{1/r} = \left[\sum_{j=1}^{n}\xi_j^u\right]^{1/r} \leq \sigma^{p/r} = \sigma^{1/u}$$

and, for $f \in B_{X^\#}$,

$$\left[\sum_{j=1}^{n}\lambda_j(\varepsilon)^{-q}\left|f(x_j)-f(x'_j)\right|^q\right]^{1/q} =$$
$$\left[\sum_{j=1}^{n}(\xi_j^0+\varepsilon)^{-v}\left|f(x_j)-f(x'_j)\right|^q\right]^{1/q} \leq \sigma^{p/q} = \sigma^{1/v}.$$

Therefore, the right-hand side of (4.2) is less than or equal to the left-hand side.

Conversely, let $\lambda_j > 0$ be arbitrary. Then, by Hölder's inequality for any probability measure μ on $B_{X^\#}$ we have

$$\left[\sum_{j=1}^n \left[\int_{B_{X^\#}} \left|f(x_j) - f(x_j')\right|^q d\mu(f)\right]^{p/q}\right]^{1/p}$$
$$= \left[\sum_{j=1}^n \left[\lambda_j \left(\int_{B_{X^\#}} \lambda_j^{-q} \left|f(x_j) - f(x_j')\right|^q d\mu(f)\right)^{1/q}\right]^p\right]^{1/p}$$
$$\leq \left[\sum_{j=1}^n \lambda_j^r\right]^{1/r} \left(\sum_{j=1}^n \int_{B_{X^\#}} \lambda_j^{-q} \left|f(x_j) - f(x_j')\right|^q d\mu(f)\right)^{1/q}$$
$$= \left[\sum_{j=1}^n \lambda_j^r\right]^{1/r} \left(\int_{B_{X^\#}} \sum_{j=1}^n \lambda_j^{-q} \left|f(x_j) - f(x_j')\right|^q d\mu(f)\right)^{1/q}$$
$$\leq \left[\sum_{j=1}^n \lambda_j^r\right]^{1/r} \sup_{f \in B_{X^\#}} \left(\sum_{j=1}^n \lambda_j^{-q} \left|f(x_j) - f(x_j')\right|^q\right)^{1/q}. \qquad \square$$

Together, Theorem 4.1 and Proposition 4.2 immediately give us another characterization of Lipschitz (q,p)-mixing operators, stated below.

Corollary 4.3. *Let $1 \leq p < q < \infty$ and $1/p = 1/q + 1/r$. A Lipschitz map $T : X \to Y$ is (q,p)-mixing if and only if there exists a constant C such that for all $x_1, \dots, x_n, x_1', \dots, x_n' \in X$,*

$$\inf\left\{\left[\sum_{j=1}^n \lambda_j^r\right]^{1/r} \sup_{g \in B_{Y^\#}} \left[\sum_{j=1}^n \lambda_j^{-q} \left|g(Tx_j) - g(Tx_j')\right|^q\right]^{1/q} : \lambda_j > 0\right\}$$
$$\leq C \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^n \left|f(x_j) - f(x_j')\right|^p\right]^{1/p}.$$

In this case, $\mathfrak{m}^L_{q,p}(T)$ is equal to the infimum of such constants C.

4.3. **Chevet-Saphar spaces.** The expression on the right-hand side of (4.2) looks reminiscent of the Chevet-Saphar norms introduced in [CD]. This section is devoted to a characterization of Lipschitz (q,p)-mixing operators in terms of such norms. Let us recall the pertinent definitions first.

An *E-valued molecule on X* is a finitely supported function $m : X \to E$ such that $\sum_{x \in X} m(x) = 0$. The space of E-valued molecules on X, denoted $\mathcal{M}(X,E)$, is clearly a vector space under pointwise addition. Given $x, x' \in X$, define $m_{xx'} := \chi_{\{x\}} - \chi_{\{x'\}}$. The simplest nonzero molecules, i.e. those of the form $vm_{xx'}$ for some $x, x' \in X$ and $v \in E$, are called *atoms*. Note that any molecule may be expressed (in a nonunique way) as a finite sum of atoms. The p-th Chevet-Saphar norm of a molecule m is given by

$$cs_p(m) := \inf \left\{ \left(\sum_j \lambda_j^p \|v_j\|^p \right)^{1/p} \sup_{f \in B_{X^\#}} \left(\sum_j \lambda_j^{-p'} \left| f(x_j) - f(x'_j) \right| \right)^{1/p'} \right.$$

$$\left. : \ m = \sum_j v_j m_{x_j x'_j}, \lambda_j > 0 \right\}.$$

The space of E-valued molecules on E, endowed with the norm $cs_p(\cdot)$, is denoted by $\mathcal{CS}_p(X,E)$. There is a canonical way of inducing a pairing between E-valued molecules on X and functions from X to E^*: given $m \in \mathcal{M}(X,E)$ and a function $T : X \to E^*$, define $\langle T, m \rangle := \sum_{x \in X} \langle T(x), m(x) \rangle$. If we know an expression of the molecule as a sum of atoms, say $m = \sum_j v_j m_{x_j x'_j}$, then $\langle T, m \rangle = \sum_j \langle T x_j - T x'_j, v_j \rangle$. The main theorem in [CD] states that with this pairing, the dual space of $\mathcal{CS}_p(X,E)$ is canonically identified with the space of Lipschitz p'-summing operators from X into E^*. Also from [CD], recall that for any Banach space E a Lipschitz map $T : X \to Y$ naturally induces a well-defined linear map $T_E : \mathcal{M}(X,E) \to \mathcal{M}(Y,E)$ given by

$$T_E\left(\sum_{j=1}^n v_j m_{x_j x'_j} \right) = \sum_{j=1}^n v_j m_{T x_j T x'_j}.$$

Now we come to the third characterization of Lipschitz (q,p)-mixing operators.

Theorem 4.4. *Let $T : X \to Y$ be a Lipschitz map. The following are equivalent:*
(a) T is Lipschitz (q,p)-mixing.
(b) For every Banach space G (or only $G = \ell_{q'}$), the operator

$$T_G : \mathcal{CS}_{p'}(X,G) \to \mathcal{CS}_{q'}(Y,G)$$

is continuous.

In this case,

$$\mathfrak{m}^L_{q,p}(T) = \left\| T_{\ell_{q'}} : \mathcal{CS}_{p'}(X,\ell_{q'}) \to \mathcal{CS}_{q'}(Y,\ell_{q'}) \right\| \geq \| T_G : \mathcal{CS}_{p'}(X,G) \to \mathcal{CS}_{q'}(Y,G) \|.$$

Proof. First, suppose that T is Lipschitz (q,p)-mixing. Let $\varphi \in \left(\mathcal{CS}_{q'}(Y,G)\right)^*$ with $\|\varphi\| \leq 1$. Since $\left(\mathcal{CS}_{q'}(Y,G)\right)^* \equiv \Pi^L_q(Y,G^*)$, we can identify φ with a map $L_\varphi \in \Pi^L_q(Y,G^*)$ with $\pi^L_q(L_\varphi) = \|\varphi\| \leq 1$. Let $m = \sum v_j m_{x_j x'_j} \in \mathcal{M}(X,G)$. Then $T_G(m) = \sum v_j m_{T x_j T x'_j}$, so

$$\langle \varphi, T_G(m) \rangle = \sum_j \langle L_\varphi(T x_j) - L_\varphi(T x'_j), v_j \rangle = \langle L_\varphi \circ T, m \rangle,$$

and thus

$$\begin{aligned} \left| \langle \varphi, T_G(m) \rangle \right| = \left| \langle L_\varphi \circ T, m \rangle \right| &\leq \pi^L_p(L_\varphi \circ T) cs_{p'}(m) \\ &\leq \pi^L_q(L_\varphi) \mathfrak{m}^L_{q,p}(T) cs_{p'}(m) \leq \mathfrak{m}^L_{q,p}(T) cs_{p'}(m). \end{aligned}$$

Taking the supremum over all such φ we obtain, $cs_{q'}\left(T_G(m)\right) \leq \mathfrak{m}^L_{q,p}(T) cs_{p'}(m)$; i.e. $T_G : \mathcal{CS}_{p'}(X,G) \to \mathcal{CS}_{q'}(Y,G)$ is continuous and $\|T_G\| \leq \mathfrak{m}^L_{q,p}(T)$.

Now, suppose that $T_{\ell_{q'}} : \mathcal{CS}_{p'}(X, \ell_{q'}) \to \mathcal{CS}_{q'}(Y, \ell_{q'})$ is continuous and has norm C, and let $S : Y \to \ell_q$ be a q-summing operator. Let m be an $\ell_{q'}$-valued molecule on X, say $m = \sum_j v_j m_{x_j x'_j}$ with $v_j \in \ell_{q'}$ and $x_j, x'_j \in X$. Then

$$\langle S \circ T, m\rangle = \sum_j \langle v_j, STx_j - STx'_j\rangle = \Big\langle S, \sum_j v_j m_{Tx_j Tx'_j} \Big\rangle = \langle S, T_{\ell_{q'}}(m)\rangle.$$

By the duality between the Lipschitz q-summing norm and the q'-Chevet-Saphar norm, together with the boundedness of $T_{\ell_{q'}}$,

$$\big|\langle S \circ T, m\rangle\big| = \big|\langle S, T_{\ell_{q'}}(m)\rangle\big| \leq \pi_q^L(S) cs_{q'}\big(T_{\ell_{q'}}(m)\big) \leq \pi_q^L(S) \cdot C \cdot cs_{p'}(m).$$

Taking the supremum over all m with $cs_{p'}(m) \leq 1$ and invoking the duality between the Lipschitz p-summing norm and the p'-Chevet-Saphar norm, we conclude that $\pi_p^L(S \circ T) \leq C\pi_q^L(S)$. By the remarks in Section 3, we conclude that T is Lipschitz (q, p)-mixing with $\mathfrak{m}^L_{q,p}(T) \leq C$. □

Of course, the space $\ell_{q'}$ in the preceding theorem may be replaced by any other infinite-dimensional $L_{q'}$ space.

5. Applications

5.1. **The Lipschitz $(2, 1)$-mixing constant of the identity on a tree.** As already mentioned in Section 3, Farmer and Johnson [FJ09] proved a nonlinear Grothendieck inequality which, in our language, means that the identity on a metric tree is Lipschitz $(2, 1)$-mixing with constant at most Grothendieck's constant. While both their proof and the one given in [CZ] make explicit use of the lifting property for trees, using Theorem 4.1 we can reobtain the same bound without explicitly appealing to the lifting property.

Lemma 5.1. *When T is an unweighted graph-theoretic tree on $n+1$ points and H is a Hilbert space,* $\mathrm{Lip}(T, H)$ *is isometric to* $\ell_\infty^n(H)$.

Proof. From [CD, Sec. 4.2], $\mathcal{CS}_1(T, H)$ is isometric to $\ell_1^n(H)$ in a natural way. By the duality result [CD, Thm. 4.3], $\mathrm{Lip}(T, H)$ is then isometric to $\ell_\infty^n(H)$. □

Proposition 5.2. *Let T be a finite unweighted graph-theoretic tree. Then the identity on T is Lipschitz $(2, 1)$-mixing with constant at most K_G.*

Proof. Let $x_1, \dots, x_m, x'_1, \dots, x'_m \in T$ and let μ be a probability measure on $B_{T^\#}$. Note that

$$\sup_{f \in B_{T^\#}} \sum_{j=1}^m \big|f(x_j) - f(x'_j)\big|$$

is the norm of the linear operator A from $T^\#$ to ℓ_1^m given by $f \mapsto \big(f(x_j) - f(x'_j)\big)_{j=1}^m$. By Lemma 5.1, $T^\#$ can be identified with ℓ_∞^N for some N, so the operator A under consideration goes from ℓ_∞^N to ℓ_1^m. The classical Grothendieck inequality gives us

$$\big\|A : \ell_\infty^N(L_2(\mu)) \to \ell_1^m(L_2(\mu))\big\| \leq K_G \big\|A : \ell_\infty^N \to \ell_1^m\big\|.$$

But another application of Lemma 5.1 reveals that $\ell_\infty^N(L_2(\mu))$ can be identified with the space of Lipschitz functions from T to $L_2(\mu)$, so in fact one has

$$\sup_{\mathrm{Lip}(F:T \to L_2(\mu)) \leq 1} \sum_{j=1}^m \big\|F(x_j) - F(x'_j)\big\|_{L_2(\mu)} \leq K_G \sup_{f \in B_{T^\#}} \sum_{j=1}^m \big|f(x_j) - f(x'_j)\big|.$$

In particular, consider the pointwise evaluation $\delta : T \to L_2(\mu)$. For any $x, x' \in T$ we have

$$\|\delta(x) - \delta(x')\|_{L_2(\mu)} = \left[\int_{g \in B_{T^\#}} \left| g(x) - g(x') \right|^2 d\mu(g) \right]^{1/2} \leq d(x, x');$$

hence $\mathrm{Lip}(\delta : T \to L_2(\mu)) \leq 1$ and thus

$$\sum_{j=1}^{m} \left[\int_{B_{T^\#}} \left| g(x_j) - g(x_j') \right|^2 d\mu(g) \right]^{1/2} \leq K_G \sup_{f \in B_{T^\#}} \sum_{j=1}^{m} \left| f(x_j) - f(x_j') \right|.$$

By Theorem 4.1, we conclude that the identity on T is Lipschitz $(2,1)$-summing with constant at most K_G. □

5.2. **An "interpolation style" theorem.** As it so often happens with many constants associated to mappings, it is not easy to calculate the Lipschitz (q,p)-mixing constant of a specific map. The following "interpolation style" theorem is based on [Puh77, Lemma 5] and gives useful bounds that are sufficient in some cases.

Theorem 5.3. *Let $1 \leq p, q, r \leq \infty$ with $1/r + 1/q = 1/p$. Then every Lipschitz p-summing map $T : X \to Y$ is Lipschitz (q,p)-mixing and satisfies*

$$\mathfrak{m}^L_{q,p}(T) \leq \pi^L_p(T)^{p/r} \, \mathrm{Lip}(T)^{p/q}.$$

Proof. The fact that T is (q,p)-mixing is obvious from the ideal property of Lipschitz p-summing operators. Now, let $x_1, \ldots, x_n, x_1', \ldots, x_n' \in X$. For any probability measure μ on $B_{Y^\#}$, from the pointwise inequality $|g(y) - g(y')| \leq \mathrm{Lip}(g) \cdot d(y,y')$ for any $y, y' \in Y$ and $g \in Y^\#$ we have that

$$\left[\sum_{j=1}^{n} \left(\int_{B_{Y^\#}} \left| g(Tx_j) - g(Tx_j') \right|^q d\mu(g) \right)^{p/q} \right]^{1/p} \tag{5.1}$$
$$\leq \left[\sum_{j=1}^{n} \left(\int_{B_{Y^\#}} \left| g(Tx_j) - g(Tx_j') \right|^p d\mu(g) \right)^{p/q} d(Tx_j, Tx_j')^{(q-p)p/q} \right]^{1/p}.$$

Noting that $(q-p)r/q = p$, Hölder's inequality lets us bound the latter expression by

$$\left[\sum_{j=1}^{n} \int_{B_{Y^\#}} \left| g(Tx_j) - g(Tx_j') \right|^p d\mu(g) \right]^{1/q} \left[\sum_{j=1}^{n} d(Tx_j, Tx_j')^p \right]^{1/r}. \tag{5.2}$$

On the one hand, the fact that T is Lipschitz p-summing means that

$$\left[\sum_{j=1}^{n} d(Tx_j, Tx_j')^p \right]^{1/r} \leq \pi^L_p(T)^{p/r} \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^{n} \left| f(x_j) - f(x_j') \right|^p \right]^{1/r}, \tag{5.3}$$

whereas on the other hand, a simple pointwise estimate gives

$$\left[\sum_{j=1}^{n}\int_{B_{Y^\#}}\left|g(Tx_j)-g(Tx_j')\right|^p d\mu(g)\right]^{1/q} \leq \operatorname{Lip}(T)^{p/q}\sup_{f\in B_{X^\#}}\left[\sum_{j=1}^{n}\left|f(x_j)-f(x_j')\right|^p\right]^{1/q}. \tag{5.4}$$

Bringing (5.1), (5.2), (5.3) and (5.4) together we have

$$\left[\sum_{j=1}^{n}\left(\int_{B_{Y^\#}}\left|g(Tx_j)-g(Tx_j')\right|^q d\mu(g)\right)^{p/q}\right]^{1/p} \leq \pi_p^L(T)^{p/r}\operatorname{Lip}(T)^{p/q}\sup_{f\in B_{X^\#}}\left[\sum_{j=1}^{n}\left|f(x_j)-f(x_j')\right|^p\right]^{1/p}$$

and thus the desired conclusion follows from Theorem 4.1. □

5.2.1. *The identity on a finite discrete metric space.* Denote by D_n the discrete metric space on n points. Theorem 5.3 allows us to explicitly evaluate the (q,p)-mixing norm of the identity on D_n. In fact, if $1\leq p\leq q\leq\infty$, then the Lipschitz (q,p)-mixing norm of the identity on D_n is equal to $(2-2/n)^{1/p-1/q}$. To see it, let $1\leq r\leq\infty$ satisfy $1/r+1/q=1/p$. From [FJ09] we have that $\pi_s^L(id_{D_n}) = (2-2/n)^{1/s}$ for any $s\in[1,\infty]$, and therefore

$$\mathfrak{m}_{q,p}^L(id_{D_n}) \geq \frac{\pi_p^L(id_{D_n}\circ id_{D_n})}{\pi_q^L(id_{D_n})} = \frac{(2-2/n)^{1/p}}{(2-2/n)^{1/q}} = (2-2/n)^{1/p-1/q}.$$

On the other hand, from Theorem 5.3,

$$\mathfrak{m}_{q,p}^L(id_{D_n}) \leq \pi_p^L(id_{D_n})^{p/r}\operatorname{Lip}(id_{D_n})^{p/q} = (2-2/n)^{1/r}\cdot 1 = (2-2/n)^{1/p-1/q}$$

and thus $\mathfrak{m}_{q,p}^L(id_{D_n}) = (2-2/n)^{1/p-1/q}$. Let us observe what this means: for every metric space X and any $T: D_n\to X$, $\pi_p^L(T)\leq(2-2/n)^{1/p-1/q}\pi_q^L(T)$ and this inequality is sharp.

5.2.2. *Reversed inequalities between Lipschitz p-summing norms.* The next result goes along the same theme: using Theorem 5.3 together with known estimates for Lipschitz p-summing norms.

Theorem 5.4. *(a) For any $n\in\mathbb{N}$ and $1\leq p\leq q$,*

$$\mathfrak{m}_{q,p}^L(id_{\ell_2^n}) \leq c_{p,n}^{p/q-1} \qquad \textit{where} \qquad c_{p,n} = \left[\int_{S_{n-1}}|x_1|^p d\lambda(x)\right]^{1/p},$$

λ being the normalized rotation invariant measure on S_{n-1}. Hence, $\pi_p^L(T)\leq c_{p,n}^{p/q-1}\pi_q^L(T)$ for any Lipschitz map $T:\ell_2^n\to Y$.

(b) For any finite-dimensional normed space E and $2\leq q$,

$$\mathfrak{m}_{q,2}^L(id_E) \leq \left[\dim(E)\right]^{1/2-1/q}.$$

Hence, $\pi_2^L(T)\leq\left[\dim(E)\right]^{1/2-1/q}\pi_q^L(T)$ for any Lipschitz map $T:E\to Y$.

(c) *There exists a universal constant C so that for any finite metric space X on n points and $1 \le q$,*

$$\mathfrak{m}^L_{q,1}(id_X) \le C^{1/q'}\left[\log n\right]^{1/q'}.$$

Hence, $\pi^L_1(T) \le C^{1/q'}\left[\log n\right]^{1/q'}\pi^L_q(T)$ for any Lipschitz map $T : X \to Y$.

Proof. Everything follows from Theorem 5.3, together with the fact that the Lipschitz p-summing norm and the linear p-summing norm of a linear operator between Banach spaces coincide ([FJ09, Theorem 2]), and also the following estimates on p-summing norms:

(a) $\pi_p(id_{\ell_2^n}) = c_{p,n}^{-1}$ (see, for instance, [TJ89, Theorem 10.3]).
(b) $\pi_2(id_E) = \left[\dim(E)\right]^{1/2}$ for any finite-dimensional space E (see, for instance [TJ89, Proposition 9.11]).
(c) $\pi_1(id_X) \le C \log n$, essentially proved in [Bou85] as mentioned in [FJ09]. □

5.3. **The general "interpolation style" theorem.** Theorem 5.3 is in fact a particular case of the following more general theorem.

Theorem 5.5. *Let $0 < \theta < 1$ and $1 \le p \le q_0, q_1 \le \infty$. Define $1/q := (1-\theta)/q_0 + \theta/q_1$. For a Lipschitz map $T : X \to Y$,*

$$\mathfrak{m}^L_{q,p}(T) \le \mathfrak{m}^L_{q_0,p}(T)^{1-\theta}\mathfrak{m}^L_{q_1,p}(T)^{\theta}.$$

Proof. Set $1/r := 1/p - 1/q$, $1/r_0 := 1/p - 1/q_0$ and $1/r_1 := 1/p - 1/q_1$. Note that $1/r := (1-\theta)/r_0 + \theta/r_1$. Let $x_1, \dots, x_n, x'_1, \dots, x'_n \in X$. Given $\varepsilon > 0$, from Corollary 4.3 for each $k = 0, 1$ there exist $\lambda_{j,k} > 0$, $1 \le j \le n$ such that

$$\left[\sum_{j=1}^n \lambda_{j,k}^{r_k}\right]^{1/r_k} \sup_{g \in B_{Y^\#}} \left[\sum_{j=1}^n \lambda_{j,k}^{-q_k}\left|g(Tx_j) - g(Tx'_j)\right|^{q_k}\right]^{1/q_k}$$
$$\le (1+\varepsilon)\mathfrak{m}^L_{q_k,p}(T) \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^m \left|f(x_j) - f(x'_j)\right|^p\right]^{1/p}.$$

Moreover, dividing by the appropriate constant we may assume that in fact

$$\left[\sum_{j=1}^n \lambda_{j,k}^{r_k}\right]^{1/r_k} \le (1+\varepsilon)\mathfrak{m}^L_{q_k,p}(T) \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^m \left|f(x_j) - f(x'_j)\right|^p\right]^{1/p}$$
$$\text{and} \quad \sup_{g \in B_{Y^\#}} \left[\sum_{j=1}^n \lambda_{j,k}^{-q_k}\left|g(Tx_j) - g(Tx'_j)\right|^{q_k}\right]^{1/q_k} \le 1.$$

For $1 \le j \le n$, set $\lambda_j = \lambda_{j,0}^{1-\theta}\lambda_{j,1}^{\theta}$. Then, by Hölder's inequality,

$$\left[\sum_{j=1}^n \lambda_j^r\right]^{1/r} \le \left[\sum_{j=1}^n \lambda_{j,0}^{r_0}\right]^{(1-\theta)/r_0} \cdot \left[\sum_{j=1}^n \lambda_{j,1}^{r_1}\right]^{\theta/r_1}$$
$$\le (1+\varepsilon)\mathfrak{m}^L_{q_0,p}(T)^{1-\theta}\mathfrak{m}^L_{q_1,p}(T)^{\theta} \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^m \left|f(x_j) - f(x'_j)\right|^p\right]^{1/p}.$$

On the other hand, it follows from

$$\lambda_j^{-1}\big|f(x_j)-f(x_j')\big| = \lambda_{j,0}^{-(1-\theta)}\big|f(x_j)-f(x_j')\big|^{1-\theta}\lambda_{j,1}^{-\theta}\big|f(x_j)-f(x_j')\big|^{\theta}$$

that

$$\sup_{g\in B_{Y^\#}}\left[\sum_{j=1}^n \lambda_j^{-q}\big|g(Tx_j)-g(Tx_j')\big|^q\right]^{1/q} \le \prod_{k=0,1}\sup_{g\in B_{Y^\#}}\left[\sum_{j=1}^n \lambda_{j,k}^{-q_k}\big|g(Tx_j)-g(Tx_j')\big|^{q_k}\right]^{1/q_k} \le 1.$$

Therefore, using the other direction of Corollary 4.3,

$$\mathfrak{m}_{q,p}^L(T) \le (1+\varepsilon)\mathfrak{m}_{q_0,p}^L(T)^{1-\theta}\mathfrak{m}_{q_1,p}^L(T)^{\theta},$$

and by letting $\varepsilon \downarrow 0$, the proof is finished. □

For $q > p \ge 1$, we say that a metric space X is (q,p)-mixing if the identity on X is (q,p)-mixing. The following lemma shows that the class of (q,p)-mixing spaces does not depend on p. This result is basically the nonlinear extrapolation theorem of Chen and Zheng [CZ, Thm. 2.2], presented in a different language.

Corollary 5.6. *Let X be a metric space and $1 \le p_0 < p_1 < q$. Then X is (q,p_0)-mixing if and only if it is (q,p_1)-mixing. Moreover,*

$$\mathfrak{m}_{q,p_1}^L(id_X) \le \mathfrak{m}_{q,p_0}^L(id_X) \le \mathfrak{m}_{q,p_1}^L(id_X)^{1/\theta},$$

where θ is defined by $1/p_1 = (1-\theta)/q + \theta/p_0$.

Proof. The monotonicity property for (q,p)-mixing constants from Section 3 gives $\mathfrak{m}_{q,p_1}^L(id_X) \le \mathfrak{m}_{q,p_0}^L(id_X)$, whereas the composition property from the same section provides us with the inequality $\mathfrak{m}_{q,p_0}^L(id_X) \le \mathfrak{m}_{q,p_1}^L(id_X)\cdot\mathfrak{m}_{p_1,p_0}^L(id_X)$. Now, from Theorem 5.5,

$$\mathfrak{m}_{p_1,p_0}^L(id_X) \le \mathfrak{m}_{q,p_0}^L(id_X)^{1-\theta}\cdot\mathfrak{m}_{p_0,p_0}^L(id_X)^{\theta} = \mathfrak{m}_{q,p_0}^L(id_X)^{1-\theta}\cdot 1.$$

So we obtain

$$\mathfrak{m}_{q,p_0}^L(id_X) \le \mathfrak{m}_{q,p_1}^L(id_X)\cdot\mathfrak{m}_{q,p_0}^L(id_X)^{1-\theta}$$

from which the result follows. □

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