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THE CHEVET-SAPHAR TENSOR NORMS FOR OPERATOR SPACES

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Abstract. We introduce the operator space versions of the Chevet-Saphar tensor norms, and show that they share many properties with their Banach-space counterparts. Most importantly, these tensor norms are in operator space duality with the completely p-summing maps of G. Pisier. Our approach, based on the algebraic idea of a tensor contraction, complements previous results on duality for completely p-summing maps due to E. Effros and Z.-J. Ruan, and M. Junge. We also obtain an operator-space version of the Chevet-Persson-Saphar inequalities and give several applications of it, including a completely isomorphic characterization of quotients of subspaces of ultrapowers of the Schatten p-class.

1. Introduction

Within the theory of Banach spaces, (absolutely) p-summing operators have had a very fruitful history. Although already present in the works of A. Grothendieck in the 50's [Gro53], it wasn't until the appearance of A. Pietsch's seminal paper [Pie67] that their systematic study properly started. Shortly afterwards, the breakthrough paper of J. Lindenstrauss and A. Pełczyński [LP68] showed what a powerful tool they are, even for studying problems that at first sight don't seem to have anything to do with them. Over the years, they have been used to prove numerous results not only within the realm of Banach spaces, but also reaching into other areas like harmonic analysis, probability and operator theory.

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An excellent reference for the general theory of p-summing operators is the book [DJT95].

The p-summing operators between two fixed Banach spaces form a new Banach space, so it is natural to study the corresponding duality. There are two known ways of understanding it: one as nuclear operators, and the other via the Chevet-Saphar norms on a tensor product. These norms were introduced independently by S. Chevet [Che69] and P. Saphar [Sap70], as generalizations of earlier work of Saphar [Sap65].

In the theory of operator spaces, p-summing operators are replaced by the completely p-summing maps of Pisier [Pis98]. The corresponding duality, in terms of completely nuclear operators, has been studied in [Jun96]. As far as the author knows the Chevet-Saphar tensor norms for operator spaces have not been studied in the literature, with just one exception. The duality for completely 1-summing maps was studied via tensor products, albeit with a different formulation, by E. Effros and Z.-J. Ruan in [ER94, Sec. 5].

In this paper we define the Chevet-Saphar tensor norms for operator spaces and show that they are in duality with the completely p-summing maps, together with other basic properties that are parallel to those in the Banach space case (Section 3). We also prove operator space versions of other known properties of the Chevet-Saphar tensor norms, including: (1) A noncommutative version of the Chevet-Persson-Saphar inequalities (Section 4); (2) A Fubini-type result for tensor products of Schatten p-classes (Section 5.1); (3) Identification of the uniform operator space tensor norms that are "closest" to the norm on $S_p \otimes E$ induced by $S_p[E]$, in the spirit of the work of Y. Gordon and P. Saphar [GS77] (Section 5.2); (4) Completely isomorphic characterizations of quotients of subspaces of ultraproducts of S_p (Section 5.3); (5) New tensor-product characterizations of completely p-summing maps (Section 5.4).

Our approach is based on the algebraic concept of a tensor contraction, well-known to specialists in the theory of tensor products of Banach spaces (see, e.g., [DF93, Sec. 29] and [Pel82]). This algebraic point of view allows us to obtain rather elegant proofs of several of the properties of these operator space Chevet-Saphar tensor norms, including the aforementioned duality result.

2. Notation and preliminaries

2.1. **Banach spaces.** Our Banach space notation follows closely that of [Rya02]. The letters X and Y will always denote Banach spaces, and we write $X = Y$ to indicate that X and Y are isometrically isomorphic. We denote the injective and projective tensor products of Banach spaces by $\otimes_\varepsilon$ and $\otimes_\pi$, respectively. By a Banach space tensor norm α we mean an assignment of a Banach space $X \otimes_\alpha Y$ to each pair (X, Y) of Banach spaces, in such a way that $X \otimes_\alpha Y$ is the algebraic tensor product $X \otimes Y$ together with a norm on $X \otimes Y$ that we write as α or $\|\cdot\|_\alpha$, and such that $\varepsilon(u) \le \alpha(u) \le \pi(u)$ for every $u \in X \otimes Y$. Moreover, a Banach space tensor norm α is called uniform if additionally for any Banach spaces X_1, X_2, Y_1, Y_2, and bounded linear maps $S : X_1 \to X_2$, $T : Y_1 \to Y_2$ one has

$$\|S \otimes_\alpha T : X_1 \otimes_\alpha Y_1 \to X_2 \otimes_\alpha Y_2\| \le \|S\| \cdot \|T\|$$

where $S \otimes_\alpha T$ is given by

$$(S \otimes_\alpha T)(x \otimes y) = Sx \otimes Ty, \quad x \in X_1, y \in Y_1.$$

For Banach space tensor norms α and β and a constant c, we write "$\alpha \le c\beta$ on $X \otimes Y$" to indicate that the identity map $X \otimes_\beta Y \to X \otimes_\alpha Y$ has norm at most c. If no reference to spaces is made, we mean that the inequality holds for any pair of Banach spaces. Finally, $\Pi_p(X, Y)$ denotes the space of p-summing operators from X to Y.

2.2. **Operator spaces.** We only assume familiarity with the basic theory of operator spaces; the books [Pis03] and [ER00] are excellent references for that. Our notation follows closely that from [Pis98, Pis03]. The letters E, F and G will always denote operator spaces, and we write $E = F$ to indicate that E and F are completely isometrically isomorphic (this is the same notation we use for isometric isomorphism of Banach spaces, but what we mean will always be clear from the context). We denote the minimal and projective operator space tensor products by $\otimes_{\min}$ and $\otimes_{\mathrm{proj}}$, respectively. By an operator space cross norm α we mean an assignment of an operator space $E \otimes_\alpha F$ to each pair (E, F) of operator spaces, in such a way that $E \otimes_\alpha F$ is the algebraic tensor product $E \otimes F$ together with a matricial norm structure on $E \otimes F$ that we write as α_n or $\|\cdot\|_{\alpha_n}$, and such that

$$\alpha_{nm}(x \otimes y) = \|x\|_{M_n(E)} \cdot \|y\|_{M_m(F)} \text{ for every } x \in M_n(E), y \in M_m(F).$$

This implies that the identity map $E \otimes_{\mathrm{proj}} F \to E \otimes_\alpha F$ is completely contractive [BP91, Thm. 5.5]. If in addition the identity map $E \otimes_\alpha F \to E \otimes_{\min} F$ is also completely contractive, we say that α is an operator space tensor norm. Moreover, an operator space tensor norm α is called uniform if additionally for any operator spaces E_1, E_2, F_1, F_2, the map

$$\begin{gathered}\otimes_\alpha : CB(E_1, E_2) \times CB(F_1, F_2) \to CB(E_1 \otimes_\alpha E_2, F_1 \otimes_\alpha F_2) \\ (S, T) \to S \otimes_\alpha T\end{gathered}$$

is jointly completely contractive, where $S \otimes_\alpha T : E_1 \otimes_\alpha F_1 \to E_2 \otimes_\alpha F_2$ is given by

$$(S \otimes_\alpha T)(x \otimes y) = Sx \otimes Ty, \quad x \in E_1, y \in F_1$$

(note that this definition is slightly less restrictive than that of [BP91, Def. 5.9]). For operator space tensor norms α and β and a constant c, we write "$\alpha \le c\beta$ on $E \otimes F$" to indicate that the identity map $E \otimes_\beta F \to E \otimes_\alpha F$ has cb-norm at most c. If no reference to spaces is made, we mean that the inequality holds for any pair of operator spaces. A linear map $Q : E \to F$ between operator spaces is called a complete 1-quotient (resp. complete quotient) if it is onto and the associated map from $E/\ker(Q)$ to F is a completely isometric isomorphism (resp. complete isomorphism). These maps are called complete metric surjections (resp. complete surjections) in [Pis03, Sec. 2.4], where it is proved that a linear map $u : E \to F$ is a complete 1-quotient if and only if its adjoint $u^* : F^* \to E^*$ is a completely isometric embedding. Note that if a linear map $Q : E \to F$ between operator spaces is a complete 1-quotient, then for every $X \in M_N(F)$ we have

$$\|X\|_{M_N(F)} = \inf \left\{ \|Y\|_{M_N(E)} \ : \ Y \in M_N(E),\ (Id_{M_N} \otimes Q)(Y) = X \right\}.$$

An operator space tensor norm α is called completely left projective (resp. completely right projective) if for any operator space G and any complete 1-quotient $Q : E \to F$, the map $Q \otimes Id_G : E \otimes_\alpha G \to F \otimes_\alpha G$ (resp. $Id_G \otimes Q : G \otimes_\alpha E \to G \otimes_\alpha F$) is a complete 1-quotient as well.

For an operator space E, a Hilbert space K and $1 \le p \le \infty$, let us define the spaces S_p, $S_p[E]$ and $S_p(K)$. For $1 < p < \infty$, S_p (resp. $S_p(K)$) denotes the space of Schatten class operators in ℓ_2 (resp. on K). In the case $p = \infty$, we denote by S_∞ (resp. $S_p(K)$) the space of all compact operators on ℓ_2 (resp. on K) with the operator space structure inherited from $B(\ell_2)$ (resp. $B(K)$). We define $S_\infty[E]$ as the minimal operator space tensor product of S_∞ and E, and $S_1[E]$ as the operator space projective tensor product of S_1 and E. In the case $1 < p < \infty$, $S_p[E]$ is defined via complex interpolation between $S_\infty[E]$ and $S_1[E]$. For $1 < p \le \infty$, the dual of $S_p[E]$ can be canonically identified with $S_{p'}[E^*]$ where $1/p + 1/p' = 1$.

Let E, F be operator spaces and $u : E \to F$ a linear map. For $1 \le p \le \infty$, we will say that u is *completely p-summing* if the mapping

$$I_{S_p} \otimes u : S_p \otimes_{\min} E \to S_p[F]$$

is bounded, and we denote its norm by $\pi_p^o(u)$. By a result of Pisier [Pis98, Corollary 5.5], in the case $1 \le p < \infty$ we in fact have that the cb-norm and the norm of the map $I_{S_p} \otimes u$ are equal. As a consequence, we can endow $\Pi_p^o(E, F)$ with an operator space structure inherited from $CB(S_p \otimes_{\min} E, S_p[F])$ [Pis98, p. 55]. For notational convenience, we will use the convention $\pi^o_\infty(\cdot) = \|\cdot\|_{\mathrm{cb}}$. Completely p-summing maps satisfy the ideal property (that is, $\pi^o_p(uvw) \leq \|u\|_{\mathrm{cb}}\, \pi^o_p(v)\, \|w\|_{\mathrm{cb}}$ whenever the composition makes sense), and being completely p-summing is a local property: the completely p-summing norm of $u : E \to F$ is equal to the supremum of the completely p-summing norms of the restrictions of u to finite-dimensional operator subspaces of E. Completely p-summing maps admit a factorization analogous to the Pietsch factorization for p-summing maps [Pis98, Thm. 5.1], but we won't need it in the sequel. Something we will need, that follows from this factorization, is the monotonicity of p-summing norms: if $1 \leq p \leq q$ and u is completely p-summing, then u is completely q-summing and moreover $\pi^o_q(u) \leq \pi^o_p(u)$. In fact, more is true: the inclusion $\Pi^o_p(E,F) \to \Pi^o_q(E,F)$ is a completely contractive map.

3. The Chevet-Saphar tensor norms: definition and elementary properties

Let us recall first the Banach space case (see [Rya02, Chap. 6] for more information). Given Banach spaces X and Y, the right Chevet-Saphar p-norm for $u \in X \otimes Y$ is defined by

$$\|u\|_{d_p} = \inf \left\{ \|(x_j)\|_{\ell_{p'} \otimes_\varepsilon X} \|(y_j)\|_{\ell_p[Y]} \,:\, u = \sum_j x_j \otimes y_j \right\}$$

and the left one by

$$\|u\|_{g_p} = \inf \left\{ \|(x_j)\|_{\ell_p[X]} \|(y_j)\|_{\ell_{p'} \otimes_\varepsilon Y} \,:\, u = \sum_j x_j \otimes y_j \right\}.$$

As a mnemonic device, it is sometimes helpful to know that the notation derives from the French words *droite* and *gauche*. These tensor norms are in trace duality with p-summing operators in the following way

$$(X \otimes_{d_p} Y)^* = \Pi_{p'}(X, Y^*), \qquad (X \otimes_{g_p} Y)^* = \Pi_{p'}(Y, X^*).$$

The proof follows easily from the definitions, see [Rya02, Prop. 6.11] for the details.

Let us now move on to the operator space realm. If we are merely interested in defining a norm on $E \otimes F$ that is in trace duality with $\Pi^o_{p'}(E, F^*)$ purely at the Banach space level, it is sufficient to define the right Chevet-Saphar p-norm of $u \in E \otimes F$ by

$$\|u\|_{d_p^o} = \inf \left\{ \|(x_{ij})\|_{S_{p'} \otimes_{\min} E} \, \|(y_{ij})\|_{S_p[F]} \, : \, u = \sum_{ij} x_{ij} \otimes y_{ij} \right\}$$

and similarly for the left one (which we naturally denote by g_p^o). The duality follows easily by mimicking the proof of the Banach space case (it should be mentioned that a closely related tensor norm and the corresponding Banach-space duality were studied by Pisier in [Pis98, Sec. 7.2]). Once this duality has been established, we can *a posteriori* endow $E \otimes_{d_p^o} F$ (resp. $E \otimes_{g_p^o} F$) with an operator space structure using the duality and the operator space structure of $\Pi_{p'}^o(E, F^*)$ (resp. $\Pi_{p'}^o(F, E^*)$).

In this paper, however, we choose a different approach by explicitly defining an operator space structure on $E \otimes F$ that not only will automatically provide us with the full operator space duality between Chevet-Saphar tensor norms and completely p-summing maps, but will also allow us to prove several other properties of the Chevet-Saphar tensor norms in the operator space category. We believe that this way to tackle the issue is more transparent, being less hands-on and more conceptual. As mentioned in the introduction it is based on the idea of tensor contractions. The tensor contraction approach to the Chevet-Saphar tensor norms is of course not new in the Banach space case (see, e.g. [DF93, Ex. 12.11]), but it is not often discussed (for example, it is absent in [Rya02]). The basic idea goes as follows: the 4-linear map $X \times \ell_{p'} \times \ell_p \times Y \to X \otimes Y$ defined by

$$(x, \xi, \eta, y) \mapsto \langle \xi, \eta \rangle_{\ell_{p'}, \ell_p} x \otimes y$$

can be linearized to a so-called *tensor contraction* $X \otimes \ell_{p'} \otimes \ell_p \otimes Y \to X \otimes Y$. It turns out that when it is considered as a map

$$(X \otimes_\varepsilon \ell_{p'}) \otimes_\pi (\ell_p[Y]) \to X \otimes_{d_p} Y$$

it is in fact a Banach space quotient map (the obvious analogous statement holds for g_p).

Similarly, for operator spaces E and F the 4-linear map $E \times S_{p'} \times S_p \times F \to E \otimes F$ defined by

$$(x, \xi, \eta, y) \mapsto \langle \xi, \eta \rangle_{S_{p'}, S_p} x \otimes y$$

can be linearized to a tensor contraction $E \otimes S_{p'} \otimes S_p \otimes F \to E \otimes F$. We define the right Chevet-Saphar p-operator space structure on the tensor product $E \otimes F$

as the one making the tensor contraction

$$\kappa : (E \otimes_{\min} S_{p'}) \otimes_{\text{proj}} S_p[F] \to E \otimes_{d_p^o} F$$

into a complete 1-quotient: that is, for $X \in M_N(E \otimes F)$ we have

$$\|X\|_{M_N(E\otimes_{d_p^o} F)} = \inf\left\{\|Y\|_{M_N((E\otimes_{\min} S_{p'})\otimes_{\text{proj}} S_p[F])} \;:\; (Id_{M_N} \otimes \kappa)(Y) = X\right\}.$$

Similarly, by definition, we have a complete 1-quotient

$$S_p[E] \otimes_{\text{proj}} (S_{p'} \otimes_{\min} F) \to E \otimes_{g_p^o} F.$$

Now the duality result can be obtained effortlessly:

Theorem 3.1. *For any operator spaces E and F and $1 \le p \le \infty$, $(E \otimes_{d_p^o} F)^*$ is completely isometric to $\Pi^o_{p'}(E, F^*)$.*

Proof. The case $p = 1$ follows from the fact that $d_1^o = \text{proj}$, see Theorem 3.7. Now let us assume that $1 < p \le \infty$. First, recall that the operator space structure in $\Pi^o_{p'}(E, F^*)$ is precisely the one inherited from $CB\big(S_{p'} \otimes_{\min} E, S_{p'}[F^*]\big)$ (see [Pis98, p. 55]), so there is a completely isometric inclusion

$$\Pi^o_{p'}(E, F^*) \hookrightarrow CB\big(S_{p'} \otimes_{\min} E, S_{p'}[F^*]\big).$$

Since the adjoint of a complete 1-quotient is a complete isometry, we also have a completely isometric inclusion

$$(3.1)\quad (E \otimes_{d_p^o} F)^* \hookrightarrow \big((E \otimes_{\min} S_{p'}) \otimes_{\text{proj}} S_p[F]\big)^* = CB\big(E \otimes_{\min} S_{p'}, S_p[F]^*\big) = CB\big(S_{p'} \otimes_{\min} E, S_{p'}[F^*]\big).$$

Therefore, all we need to do is prove that $(E \otimes_{d_p^o} F)^*$ and $\Pi^o_{p'}(E, F^*)$ are equal as subsets of $CB\big(S_{p'} \otimes_{\min} E, S_{p'}[F^*]\big)$. Let $\tilde{T} : E \otimes_{d_p^o} F \to \mathbb{C}$ be a bounded linear map. It is easy to see that the associated linear map $T : E \to F^*$ given by $\langle Tx, y\rangle = \tilde{T}(x \otimes y)$ for every $x \in E$, $y \in F$ is well-defined and bounded (in particular this follows from Proposition 3.5). Equation (3.1) shows that T is completely p'-summing, and the completely p'-summing norm of T is equal to the norm of $\tilde{T}$. For the reverse containment, let $T : E \to F^*$ be a completely p'-summing map. Since the tensor contraction $\kappa : (E \otimes_{\min} S_{p'}) \otimes_{\text{proj}} S_p[F] \to E \otimes_{d_p^o} F$ is a complete 1-quotient, the cb-norm of the map $\tilde{T} : E \otimes_{d_p^o} F \to \mathbb{C}$ corresponding to T is the same as the cb-norm of $\tilde{T} \circ \kappa : (E \otimes_{\min} S_{p'}) \otimes_{\text{proj}} S_p[F] \to \mathbb{C}$. By the properties of the projective tensor product, this cb-norm is not changed if we consider the corresponding map going from $S_{p'} \otimes_{\min} E$ to $CB(S_p[F], \mathbb{C}) = S_{p'}[F^*]$, that is, the cb-norm of $Id_{S_{p'} \otimes T} : S_{p'} \otimes_{\min} E \to S_{p'}[F^*]$. The latter is precisely the completely p'-summing norm of T, so we are done. $\square$

Before continuing our study of the operator space Chevet-Saphar tensor norms, let us prove a general fact about tensor contractions that will be used repeatedly in the sequel.

Lemma 3.2. *Let α be a uniform operator space tensor norm. Then for any operator spaces E, F and G the tensor contractions*

$$\tau : (E \otimes_{\min} G^*) \otimes_{\text{proj}} (G \otimes_\alpha F) \to E \otimes_\alpha F$$
$$\sigma : (E \otimes_{\min} G) \otimes_{\text{proj}} (G^* \otimes_\alpha F) \to E \otimes_\alpha F$$

are completely contractive.

Proof. We will only prove this for τ, the proof for σ is analogous. Because of the complete isometry between

$CB\big((E\otimes_{\min}G^*)\otimes_{\text{proj}}(G\otimes_\alpha F), E\otimes_\alpha F\big)$ and $CB\big(E\otimes_{\min}G^*, CB(G\otimes_\alpha F, E\otimes_\alpha F)\big)$

[Pis03, Thm. 4.1], it suffices to show that the cb-norm of τ is at most one as an element of the latter space. But the operator space structure of $E \otimes_{\min} G^*$ is precisely the one inherited from $CB(G, E)$, so the desired result follows immediately from the definition of uniform operator space tensor norm. □

Remark 3.3. A quick examination of the preceding proof shows that Lemma 3.2 does not require the full strength of the definition of uniform operator space tensor norm, but we have chosen to state the result in this way for convenience.

We now prove several other properties of the operator space Chevet-Saphar tensor norms, reminiscent of those in the Banach space case (cf. [Rya02, Props. 6.6, 6.7])

Proposition 3.4. *d_p^o is completely right projective, and g_p^o is completely left projective.*

Proof. Recall that the projective tensor norm for operator spaces is completely projective [Pis03, Chap. 4], and note that if $Q : E \to F$ is a complete projection, then so is $I_{S_p} \otimes Q : S_p[E] \to S_p[F]$ (this is proved as in [Pis98, Cor. 1.2], using the fact that the Haagerup tensor product is completely projective [Pis03, Cor. 5.7]). The result follows immediately. □

Proposition 3.5. *d_p^o and g_p^o are uniform operator space tensor norms.*

Proof. We will only prove this for d_p^o, the proof for g_p^o is analogous. First, let us observe that d_p^o is an operator space tensor norm, that is, it lies in between the minimal and projective ones.

Let E and F be two operator spaces. Since the identity map $S_p[F] \to S_p \otimes_{\min} F$ is completely contractive, so is $(E \otimes_{\min} S_{p'}) \otimes_{\text{proj}} S_p[F] \to (E \otimes_{\min} S_{p'}) \otimes_{\text{proj}} (S_p \otimes_{\min} F)$. The tensor contraction $(E \otimes_{\min} S_{p'}) \otimes_{\text{proj}} (S_p \otimes_{\min} F) \to E \otimes_{\min} F$ is completely contractive by Lemma 3.2, so by composing the two previous maps we obtain that the tensor contraction $(E \otimes_{\min} S_{p'}) \otimes_{\text{proj}} S_p[F] \to E \otimes_{\min} F$ is completely contractive. By the properties of complete quotients [Pis03, Prop. 2.4.1], the identity $E \otimes_{d_p^o} F \to E \otimes_{\min} F$ is then completely contractive. Now let $x \in M_n(E)$ and $y \in M_m(F)$. From the previous argument, since min is an operator space cross norm, we already have that

$$\|x\|_{M_n(E)} \cdot \|y\|_{M_m(F)} = \|x \otimes y\|_{M_{mn}(E \otimes_{\min} F)} \le \|x \otimes y\|_{M_{mn}(E \otimes_{d_p^o} F)}$$

Now, let $f \in S_{p'}$, $g \in S_p$ be of norm one and such that $\langle f, g \rangle = 1$. Then

$$\|x\|_{M_n(E)} = \|x \otimes f\|_{M_n(S_{p'} \otimes_{\min} E)}, \|y\|_{M_m(F)} = \|g \otimes y\|_{M_m(S_p[F])}$$

Since the tensor contraction maps $x \otimes f \otimes g \otimes y$ to $x \otimes y$, we have that

$$\|x \otimes y\|_{M_{mn}(E \otimes_{d_p^o} F)} \le \|x\|_{M_n(E)} \cdot \|y\|_{M_m(F)}.$$

We already had the reverse inequality, so we conclude that d_p^o is an operator space cross norm. (As mentioned in the introduction, this implies the inequality $d_p^o \le \text{proj}$.)

Now, let us check that it is uniform. Let E_1, E_2, F_1, F_2 be operator spaces. By uniformity of the minimal tensor product and [Pis98, Cor. 1.2], we have completely contractive inclusions

$$CB(E_1, E_2) \to CB(E_1 \otimes_{\min} S_{p'}, E_2 \otimes_{\min} S_{p'}),\ CB(F_1, F_2) \to CB\big(S_p[F_1], S_p[F_2]\big),$$

by tensoring with the identity of $S_{p'}$ and S_p, respectively. On the other hand, by the uniformity of the projective tensor product we have a jointly completely contractive inclusion

$$\begin{aligned} &CB(S_{p'} \otimes_{\min} E_1, S_{p'} \otimes_{\min} E_2) \times CB(S_p[F_1], S_p[F_2]) \\ &\qquad \to CB\big((E_1 \otimes_{\min} S_{p'}) \otimes_{\text{proj}} S_p[F_1], (E_2 \otimes_{\min} S_{p'}) \otimes_{\text{proj}} S_p[F_2]\big) \end{aligned}$$

By composition we have then a jointly completely contractive inclusion

$$\begin{aligned} &CB(E_1, E_2) \times CB(F_1, F_2) \\ &\qquad \to CB\big((E_1 \otimes_{\min} S_{p'}) \otimes_{\text{proj}} S_p[F_1], (E_2 \otimes_{\min} S_{p'}) \otimes_{\text{proj}} S_p[F_2]\big) \end{aligned}$$

Composing now with the tensor contraction $(E_2 \otimes_{\min} S_{p'}) \otimes_{\text{proj}} S_p[F_2] \to E_2 \otimes_{d_p^o} F_2$ gives a completely contractive inclusion

$$CB(E_1, E_2) \times CB(F_1, F_2) \to CB\big((E_1 \otimes_{\min} S_{p'}) \otimes_{\text{proj}} S_p[F_1], E_2 \otimes_{d_p^o} F_2\big)$$

and finally, using the properties of complete quotients [Pis03, Prop. 2.4.1], we obtain the required jointly completely contractive map

$$CB(E_1, E_2) \times CB(F_1, F_2) \to CB(E_1 \otimes_{d_p^o} F_1, E_2 \otimes_{d_p^o} F_2).$$

□

Proposition 3.6. *If* $1 \le p \le q \le \infty$ *then* $d_p^o \ge d_q^o$ *and* $g_p^o \ge g_q^o$.

Proof. This can be deduced from the duality Theorem 3.1, and the fact that, since $1 \le q' \le p' \le \infty$, the inclusion $\Pi_{q'}^o(E, F^*) \to \Pi_{p'}^o(E, F^*)$ is completely contractive. □

Just as in the Banach space case (and even the Lipschitz case, see [CD11] for more on the Chevet-Saphar norms on the "tensor product" of a metric space and a normed space), when $p = 1$ or $p = \infty$ there are alternative formulas for the Chevet-Saphar tensor norms. The formula for d_∞^o below is the one that appeared in [ER94, Sec. 5].

Theorem 3.7. *Let* E , F *be operator spaces.*

(a) $E \otimes_{d_1^o} F$ *is completely isometric to* $E \otimes_{\text{proj}} F$.

(b) For $u \in M_N(E \otimes F)$,

$$\|u\|_{M_N(E \otimes_{d_\infty^o} F)} = \inf \Big\{ \|t\|_{M_N(E \otimes_{\min} S_1^m)} : t \in M_N(E \otimes S_1^m), u = (Id_{M_N} \otimes Id_E \otimes \gamma)(t) \Big\} \tag{3.2}$$

where the infimum is taken over all $\gamma : S_1^m \to F$ *with* $\|\gamma\|_{\text{cb}} \le 1$. *Moreover, if we are given a complete* 1*-quotient* $q : S_1(H) \to F$, *then the* d_∞^o *norm is just the quotient norm* $E \otimes_{\min} S_1(H) \to E \otimes F$.

Proof. (a) We already know that the identity $E \otimes_{\text{proj}} F \to E \otimes_{d_1^o} F$ is completely contractive, from Proposition 3.5. Now consider the identity $E \otimes_{d_1^o} F \to E \otimes_{\text{proj}} F$. By the properties of complete quotients [Pis03, Prop. 2.4.1], we can instead consider the tensor contraction

$$(E \otimes_{\min} S_\infty) \otimes_{\text{proj}} S_1[F] \to E \otimes_{\text{proj}} F.$$

without changing the cb-norm. Since $S_1[F] = S_1 \otimes_{\text{proj}} F$ by definition, this map is a complete contraction by Lemma 3.2.

(b) This follows immediately from Theorem 3.1 and [ER94, Cor. 5.5], since both d^o_∞ and the expression on the right-hand side of (3.2) define operator space structures on $E \otimes F$ that are in duality with $\Pi^o_1(E, F^*)$ in exactly the same way. □

4. The Chevet-Persson-Saphar inequalities

Given a Banach space X there is a very natural way to define a norm Δ_p on $L_p \otimes X$, the one induced from $L_p[X]$. It can be easily checked that $\varepsilon \le \Delta_p \le \pi$ on $L_p \otimes X$. Thus, even though Δ_p is not a Banach space tensor norm (it is not defined for every pair of Banach spaces), it would appear to be a decent substitute if we restrict the first space to be an L_p space. Unfortunately, Δ_p fails to have the crucial property of being uniform: if $S : L_p(\mu) \to L_p(\nu)$ and $T : X \to Y$ are continuous, $S \otimes T : L_p(\mu) \otimes_{\Delta_p} X \to L_p(\nu) \otimes_{\Delta_p} Y$ is not always continuous. For example, if S is the Fourier transform on $L_2(\mathbb{R})$ and T is the identity on a Banach space X, the continuity of $S \otimes T$ is equivalent to X being isomorphic to a Hilbert space [Kwa72a]. The Chevet-Persson-Saphar inequalities [Che69, Per69, Sap72] show that not everything is lost, since at least Δ_p turns out to be "sandwiched" between two uniform Banach space tensor norms. The precise statement is that for any Banach space X

$$d^*_{p'} \le d_p \le \Delta_p \le g^*_{p'} \le g_p \quad \text{on } L_p \otimes X,$$

where the norms $g^*_{p'}$ and $d^*_{p'}$ are the ones induced by the inclusions

$$X \otimes_{g^*_p} Y \hookrightarrow (X^* \otimes_{d_p} Y^*)^* \qquad X \otimes_{d^*_p} Y \hookrightarrow (X^* \otimes_{g_p} Y^*)^*.$$

The reader familiar with the general theory of tensor products for Banach spaces will immediately notice that we are cheating a little bit here. The general definition of the associate norm α^* for a given tensor norm α is more involved, but we have chosen to avoid the unpleasant technicalities here and use this as our definition. Readers interested in the details are encouraged to consult [DF93, Sec. 15 and 21].

We now proceed to prove a noncommutative version. First we define Δ^o_p to be the operator space structure on $S_p \otimes E$ inherited from $S_p[E]$, and $g^{o*}_{p'}$ and $d^{o*}_{p'}$ by the inclusions

$$E \otimes_{g^{o*}_p} F \hookrightarrow (E^* \otimes_{d^o_p} F^*)^* \qquad E \otimes_{d^{o*}_p} F \hookrightarrow (E^* \otimes_{g^o_p} F^*)^*.$$

First we start with a lemma:

Lemma 4.1. *For any* $1 \le p \le \infty$, $d^{o*}_{p'} \le d^o_p$ *and* $g^{o*}_{p'} \le g^o_p$.

Proof. Note that it suffices to prove the first inequality, since the second one follows by transposition. What we are trying to prove is that the identity map

$$E \otimes_{d_p^o} F \to (E^* \otimes_{g_{p'}^o} F^*)^*$$

is completely contractive. By the properties of the projective tensor product [Pis03, Thm. 4.1], the corresponding map

$$(E \otimes_{d_p^o} F) \otimes_{\mathrm{proj}} (E^* \otimes_{g_{p'}^o} F^*) \to \mathbb{C}$$

has the same cb-norm. By the properties of complete quotients [Pis03, Prop. 2.4.1], we can consider instead the cb-norm of

$$(E \otimes_{\min} S_{p'}) \otimes_{\mathrm{proj}} S_p[F] \otimes_{\mathrm{proj}} S_{p'}[E^*] \otimes_{\mathrm{proj}} (S_p \otimes_{\min} F^*) \to \mathbb{C}. \tag{4.1}$$

Since the projective tensor product is commutative, it will suffice to show that the tensor contractions

$$(S_{p'} \otimes_{\min} E) \otimes_{\mathrm{proj}} S_{p'}[E^*] \to S_{p'}[S_{p'}], \qquad (S_p \otimes_{\min} F^*) \otimes_{\mathrm{proj}} S_p[F] \to S_p[S_p] \tag{4.2}$$

are completely contractive, because then the map in (4.1) will be completely contractive by the duality between $S_{p'}[S_{p'}]$ and $S_p[S_p]$. We will only prove that the first map in (4.2) is completely contractive, the proof for the other one is clearly analogous. First, by [Pis03, Thm. 4.1] we can instead consider the norm of this tensor contraction as an element in

$$CB\big(S_{p'} \otimes_{\min} E, CB(S_{p'}[E^*], S_{p'}[S_{p'}])\big)$$

But this is completely contractive from the inclusions

$$S_{p'} \otimes_{\min} E \hookrightarrow CB(E^*, S_{p'}) \hookrightarrow CB\big(S_{p'}[E^*], S_{p'}[S_{p'}]\big)$$

(the first one follows from the definition of the minimal tensor product, and the second from [Pis03, Cor. 1.2]). □

Let us now prove the general theorem.

Theorem 4.2. *For any operator space E, and $1 < p < \infty$, we have*

$$d_{p'}^{o*} \le d_p^o \le \Delta_p^o \le g_{p'}^{o*} \le g_p^o \quad \text{on } S_p \otimes E,$$

Proof. Note that the leftmost and rightmost inequalities follow from Lemma 4.1, so only the two inequalities in the middle need to be proved.

(I) $d_p^o \le \Delta_p^o$: By a standard approximation argument, it will suffice to show that the identity $S_p^n[E] \to S_p^n \otimes_{d_p^o} E$ is completely contractive for every n. For that purpose, consider first the map $\varphi : S_p^n[E] \to (S_p^n \otimes_{\min} S_{p'}) \otimes_{\mathrm{proj}} S_p[E]$ given by $u \mapsto I_{S^n_{p'}} \otimes u$ where $I_{S^n_{p'}}$ is the natural inclusion $S^n_{p'} \hookrightarrow S_{p'}$ considered as an element of $S^n_p \otimes_{\min} S_{p'}$. Observe that φ is completely contractive, and that u is the image of $\varphi(u)$ under the tensor contraction that defines the operator space structure on $S^n_p \otimes_{d^o_p} E$. Therefore, the identity $S^n_p[E] \to S^n_p \otimes_{d^o_p} E$ is completely contractive

(II) $\Delta^o_p \le g^{o*}_{p'}$: From part (I), the identity $S_{p'}[E^*] \to S_{p'} \otimes_{d^o_{p'}} E^*$ is completely contractive. Therefore, so is the identity $S_{p'}[E^*] \to (S_{p'} \otimes_{d^o_{p'}} E^*)^{**}$. This is precisely the dual formulation of the identity $S_p \otimes_{g^{o*}_{p'}} E \to S_p[E]$ being completely contractive. ☐

Remark 4.3. Our noncommutative version of the Chevet-Persson-Saphar inequalities replaces L_p by S_p, but it is natural to wonder whether one could use instead a general noncommutative L_p space. The results in Section 5.3 will shed some light on this issue, see Remark 5.6 for the discussion of this point. Something that can actually be done, and that we will use below, is that S_p can be replaced by $S_p(H)$ for a general Hilbert space H.

5. Applications

5.1. **A Fubini-type result.** It is well known that the Hilbertian tensor product of Hilbert spaces "behaves well" with respect to integration, in the sense that for any measures μ and ν we have that $L_2(\mu) \otimes_2 L_2(\nu) = L_2(\mu \times \nu)$. An enthusiastic student coming across the theory of L_p spaces might wish for a tensor norm that reproduces this same result for values of p other than 2. Chevet [Che69, §4, Cor. 2] proved that this is in fact the case for the right Chevet-Saphar tensor norm: $L_p(\mu) \otimes_{d_p} L_p(\nu) = L_p(\mu \times \nu)$ for any measures μ and ν. The following is an operator space version of that result.

Corollary 5.1. *For any Hilbert spaces H and K, and $1 < p < \infty$, $S_p(H) \otimes_{d^o_p} S_p(K)$ is completely isometric to $S_p(H \otimes_2 K)$.*

Proof. From the Chevet-Persson-Saphar inequalities (in the more general form indicated in Remark 4.3) and [Pis98, Thm. 1.9] we have the following commutative diagram of complete contractions

$$\begin{array}{ccccc} S_p(H) \otimes_{g^o_p} S_p(K) & \longrightarrow & S_p(H; S_p(K)) = S_p(H \otimes_2 K) & \longrightarrow & S_p(H) \otimes_{d^o_p} S_p(K) \\ \uparrow & & & & \downarrow \\ S_p(H) \otimes_{d^o_p} S_p(K) & \longleftarrow & S_p(K; S_p(H)) = S_p(H \otimes_2 K) & \longleftarrow & S_p(K) \otimes_{g^o_p} S_p(H) \end{array}$$

where the vertical arrows are the complete isometries given by flipping. Note that all the maps involved are canonical (either the identity or a flip, and the paths between the two copies of $S_p(H \otimes_2 K)$ are just the identity map. The desired result follows immediately. $\square$

5.2. **The operator space tensor norms that are "closest" to the natural norm on $S_p \otimes E$.** Gordon and Saphar [GS77] proved that the inequality $d_p \leq \Delta_p \leq g^*_{p'}$ is the best possible in the following sense:

Theorem 5.2. *For a Banach space X, $1 < p < \infty$, $c \geq 1$ and a uniform Banach space tensor norm α,*

(a) $\alpha \leq c\Delta_p$ if and only if $\alpha \leq cd_p$ on $L_p \otimes X$.
(b) $\Delta_p \leq c\alpha$ if and only if $g^*_{p'} \leq c\alpha$ on $L_p \otimes X$.

This is usually interpreted as saying that d_p and $g^*_{p'}$ are the uniform tensor norms that are "closest" to Δ_p. We now prove an operator space version of Theorem 5.2. The proof is modeled after that of [DF93, Prop. 15.11]. Recall that Δ^o_p is the operator space structure on $S_p \otimes E$ induced by $S_p[E]$.

Theorem 5.3. *For an operator space E, $1 < p < \infty$, $c \geq 1$ and a uniform operator space tensor norm α,*

(a) $\alpha \leq c\Delta^o_p \Leftrightarrow \alpha \leq cd^o_p$ on $S_p \otimes E$.
(b) $\Delta^o_p \leq c\alpha \Leftrightarrow g^{o*}_{p'} \leq c\alpha$ on $S_p \otimes E$.

Proof. The right-to-left implications are contained in the Chevet-Persson-Saphar inequalities, so we'll only prove the left-to-right ones.

Let us start with (a). The following diagram is clearly commutative, where the horizontal maps are the tensor contractions:

$$\begin{array}{ccc} (S_p \otimes_{\min} S_{p'}) \otimes_{\text{proj}} S_p[E] & \longrightarrow & S_p \otimes_{d^o_p} E \\ \big\downarrow {\scriptstyle id} & & \big\downarrow {\scriptstyle id} \\ (S_p \otimes_{\min} S_{p'}) \otimes_{\text{proj}} (S_p \otimes_\alpha E) & \longrightarrow & S_p \otimes_\alpha E \end{array}$$

The top map is completely contractive by the definition of d^o_p, whereas the bottom map is completely contractive from Lemma 3.2. The left vertical map has cb-norm at most c by assumption, so it follows that the cb-norm of the right vertical map is also at most c. This is the desired conclusion.

The proof for part (b) is quite similar, but based on the diagram

$$\begin{array}{ccc} (S_p \otimes_{\min} S_{p'}) \otimes_{\text{proj}} (S_p \otimes_\alpha E) & \longrightarrow & S_p \otimes_\alpha E \\ \Big\downarrow {\scriptstyle id} & & \Big\downarrow {\scriptstyle id} \\ (S_p \otimes_{\min} S_{p'}) \otimes_{\text{proj}} (S_p \otimes_{g^{o*}_{p'}} E) & \longrightarrow & S_p[E]. \end{array}$$

In order for the proof to work, all we need is to check that the bottom horizontal tensor contraction is completely contractive. That follows easily from the following facts: (1) $S_p \otimes_{g^{o*}_{p'}} E \hookrightarrow S_p[E]$ is completely contractive; (2) We can calculate the cb-norm of the tensor contraction in $CB\big(S_p \otimes_{\min} S_{p'}, CB(S_p \otimes_{g^{o*}_{p'}} E, S_p[E])\big)$ instead; (3) $S_p \otimes_{\min} S_{p'}$ embeds completely isometrically in $CB(S_p, S_p)$. □

5.3. **Quotients of subspaces of ultraproducts of S_p.** The Chevet-Saphar tensor norms can be used to characterize isomorphically several families of Banach spaces, including: subspaces of L_p spaces, quotients of L_p spaces, and subspaces of quotients of L_p spaces. Characterizations of such families were first given by Kwapień [Kwa72b] in terms of operator ideals, and subsequently stated in the language of tensor norms by Gordon and Saphar [GS77]. An excellent exposition of these results can be found in [DF93, Sec. 25.10]. An example of such results is the following:

Theorem 5.4. *Let X be a Banach space, $1 < p < \infty$, μ_0 a measure such that $L_p(\mu_0)$ is infinite-dimensional, and $c \geq 1$ a constant. The following are equivalent:*
(a) X is c-isomorphic to a quotient of a subspace of an $L_p(\nu)$ space.
(b) $d_p \leq g^*_{p'} \leq c d_p$ on $L_p(\mu_0) \otimes X$.
(c) $\Delta_p \leq g^*_{p'} \leq c\Delta_p$ on $L_p(\mu_0) \otimes X$.
(d) $d_p \leq \Delta_p \leq c d_p$ on $L_p(\mu_0) \otimes X$.
(e) There exists a uniform Banach space tensor norm α which is c-equivalent to Δ_p on $L_p(\mu_0) \otimes X$.

Our next result is a noncommutative version of Theorem 5.4.

Theorem 5.5. *Let E be an operator space, $1 < p < \infty$, and $c \geq 1$ a constant. The following are equivalent:*
(a) E is c-completely isomorphic to a quotient of a subspace of an ultrapower of S_p.
(b) $d^o_p \leq g^{o*}_{p'} \leq c d^o_p$ on $S_p \otimes E$.
(c) $\Delta^o_p \leq g^{o*}_{p'} \leq c\Delta^o_p$ on $S_p \otimes E$.
(d) $d^o_p \leq \Delta^o_p \leq c d^o_p$ on $S_p \otimes E$.

(e) There exists a uniform operator space tensor norm α which is c-completely equivalent to Δ_p^o on $S_p \otimes E$.

Remark 5.6. Although very similar in form, a significant difference between the two previous theorems must be emphasized. In the noncommutative version we do not immediately get a characterization of subspaces of quotients of general noncommutative L_p spaces, we are restricted to ultrapowers of S_p. Such ultrapowers are known to be noncommutative L_p spaces [Ray02], but it is not known whether all noncommutative L_p spaces are of this form (the author is indebted to G. Pisier for pointing out the latter fact). This indicates a possible reason why completely p-summing operators have not been as widely used as their commutative cousins.

Proof of Theorem 5.5. The noncommutative Chevet-Persson-Saphar inequalities (Theorem 4.2) show that (b) implies (c), and also (b) implies (d). If in addition we use Theorem 5.3, we have that (c) implies (d) [1], and also that (d) implies (b). The same kind of argument shows that (e) implies (d), and it is obvious that (d) implies (e). Therefore, (b), (c), (d) and (e) are equivalent.

In order to show the equivalence with (a), we will make use of the following result of Pisier [Pis98, Thm. 7.2.7]: an operator space E is c-completely isomorphic to a subspace of a quotient of an ultraproduct of S_p if and only if for any finite-rank linear map $v : S_p \to S_p$ we have

$$\|v \otimes I_E : S_p[E] \to S_p[E]\| \leq C \|v\|_{\mathrm{cb}}. \tag{5.1}$$

Suppose that (d) holds. Then the composition of the tensor contraction $(S_p \otimes_{\min} S_{p'}) \otimes_{\mathrm{proj}} S_p[E] \to S_p \otimes_{d_p^o} E$ followed by the identity map $S_p \otimes_{d_p^o} E \to S_p[E]$ has cb-norm at most c. By the properties of the projective tensor product [Pis03, Thm. 4.1], this map has the same norm in

$$CB(S_p \otimes_{\min} S_{p'}, CB\big(S_p[E], S_p[E]\big)).$$

Since $S_p \otimes_{\min} S_{p'}$ has the operator space structure inherited from $CB(S_p, S_p)$, we obtain a condition formally stronger than (5.1), and thus conclude that (a) holds.

To show that (a) implies (d), it will suffice to prove that if E is a quotient of a subspace of an ultraproduct of S_p, then the identity map $S_p \otimes_{d_p^o} E \to S_p[E]$ is completely contractive.

(I) Suppose that E is actually an ultrapower of S_p. By the usual arguments, we can instead look at the norm of the associated map in

$$CB(S_p \otimes_{\min} S_{p'}, CB\big(S_p[E], S_p[E]\big)).$$

[1] At this point we need to observe that $g_{p'}^{o*}$ is an uniform operator space tensor norm, which follows easily from the corresponding property for d_p^o.

Now the result follows from [Pis98, Lemma 5.4], which can be restated as saying that Δ_p^o commutes with ultraproducts.

(II) If F is a subspace of E then $S_p[F]$ is completely isometrically a subspace of $S_p[E]$, so once again we can get the conclusion by looking at the associated map in

$$CB\big(S_p \otimes_{\min} S_{p'}, CB\big(S_p[F], S_p[F]\big)\big).$$

(III) Finally, passing to quotients is merely a formality. If $q : E \to F$ is a complete 1-quotient, the diagram

$$\begin{array}{ccc} S_p \otimes_{d_p^o} E & \xrightarrow{id} & S_p[E] \\ \Big\downarrow{\scriptstyle id\otimes q} & & \Big\downarrow{\scriptstyle id\otimes q} \\ S_p \otimes_{d_p^o} F & \xrightarrow{id} & S_p[F] \end{array}$$

shows that if $S_p \otimes_{d_p^o} E \to S_p[E]$ is completely contractive, so is $S_p \otimes_{d_p^o} F \to S_p[F]$. $\square$

5.4. **Characterizations of completely p-summing maps by tensoring with the identity.** Recall that a linear map $u : E :\to F$ is completely p-summing if $Id_{S_p} \otimes u$ is continuous from $S_p \otimes_{\min} E$ to $S_p[F]$. In this section we prove other similar characterizations, with two changes: using uniform operator space tensor norms in the codomain; and tensoring with the identity on spaces other than S_p.

The first one is just what one gets from duality.

Proposition 5.7. *Let $1 \leq p \leq \infty$. The map $u : E \to F$ is completely p-summing if and only if for any operator space G (or only $G = F^*$) the map*

$$u \otimes Id_G : E \otimes_{d_{p'}^o} G \to F \otimes_{\text{proj}} G$$

is bounded. In this case,

$$\pi_p^o(u) = \left\| u \otimes Id_{F^*} : E \otimes_{d_{p'}^o} F^* \to F \otimes_{\text{proj}} F^* \right\| \geq \left\| u \otimes Id_G : E \otimes_{d_{p'}^o} G \to F \otimes_{\text{proj}} G \right\|.$$

Proof. Suppose that u is completely p summing, and let G be any operator space. By the properties of complete quotients [Pis03, Prop. 2.4.1], the cb-norm of $u \otimes Id_G$ is the same as that of the tensor contraction $\kappa : (E \otimes_{\min} S_p) \otimes_{\text{proj}} S_{p'}[G] \to E \otimes_{d_{p'}^o} G$ followed by $u \otimes Id_G$. But $(u \otimes Id_G) \circ \kappa : (E \otimes_{\min} S_p) \otimes_{\text{proj}} S_{p'}[G] \to F \otimes_{proj} G$ can be expressed as the composition

$$(E \otimes_{\min} S_p) \otimes_{\text{proj}} S_{p'}[G] \xrightarrow{(u\otimes Id_{S_p})\otimes(Id_{S_{p'}[G]})} S_p[F] \otimes_{\text{proj}} S_{p'}[G] \xrightarrow{C'} F \otimes_{\text{proj}} G.$$

The first map above has cb-norm at most $\pi_p^o(u)$ by hypothesis. The second one has cb-norm one, and is in fact a complete 1-quotient (this is proved in [Pis98, Prop. 1.15] at the Banach space level, and it's easy to adapt the proof to see that in fact this holds at the operator space level). Thus,

$$\pi_p^o(u) \geq \left\| u \otimes Id_G : E \otimes_{d_{p'}^o} G \to F \otimes_{\text{proj}} G \right\|_{\text{cb}} \geq \left\| u \otimes Id_G E \otimes_{d_{p'}^o} G \to F \otimes_{\text{proj}} G \right\|.$$

Now assume that $\left\| u \otimes Id_{F^*} : E \otimes_{d_{p'}^o} F^* \to F \otimes_{proj} F^* \right\|_{\text{cb}} = C$. Since Δ_p^o respects subspaces, $\pi_p^o(u) = \pi_p^o(j_F \circ u)$ where $j_F : F \to F^{**}$ is the canonical complete isometry. By duality between $\Pi_p^o(E, F^{**})$ and $E \otimes_{d_{p'}^o} F^*$, it's clear that

$$\pi_p^o(j_F \circ u) = \left\| u \otimes Id_{F^*} : E \otimes_{d_{p'}^o} F^* \to F \otimes_{\text{proj}} F^* \right\|$$

and we are done. □

The next characterization is more interesting, since it makes use of the Chevet-Persson-Saphar inequalities.

Proposition 5.8. *The map $u : E \to F$ is completely p-summing if and only if for any operator space G (or only $G = S_p$) the map*

$$I_G \otimes u : G \otimes_{\min} E \to G \otimes_{g_{p'}^{o*}} F$$

is bounded. In this case,

$$\pi_p^o(u) = \left\| I_{S_p} \otimes u : S_p \otimes_{\min} E \to S_p \otimes_{g_{p'}^{o*}} F \right\| \geq \left\| I_G \otimes u : G \otimes_{\min} E \to G \otimes_{g_{p'}^{o*}} F \right\|.$$

Proof. For the "only if" implication, suppose that $u : E \to F$ is completely p-summing and G is an operator space. By the definition of $g_{p'}^*$, we need to show that

$$I_G \otimes u : G \otimes_{\min} E \to (G^* \otimes_{d_{p'}^o} F^*)^* = \Pi_p^o(G^*, F^{**})$$

is bounded. To that end, let $z \in G \otimes_{\min} E$, and consider it as an element in $CB(G^*, E)$. Then $(I_G \otimes u)(z)$ corresponds to the composition $u \circ z$, and therefore

$$\pi_p^o\big((I_G \otimes u)(z)\big) \leq \pi_p^o(u) \left\| z \right\|_{\text{cb}},$$

showing that

$$\left\| I_G \otimes u : G \otimes_{\min} E \to (G^* \otimes_{d_{p'}^o} F^*)^* \right\| \leq \pi_p^o(u)$$

as needed.

Now for the "if" part we will use the Chevet-Persson-Saphar inequalities. Since both

$$I_{S_p} \otimes u : S_p \otimes_{\min} E \to S_p \otimes_{g^{o*}_{p'}} F \quad \text{and} \quad I_{S_p} \otimes I_F : S_p \otimes_{g^{o*}_{p'}} F \to S_p[F]$$

are bounded, so is

$$I_{S_p} \otimes u : S_p \otimes_{\min} E \to S_p[F],$$

showing that u is completely p-summing and moreover

$$\pi^o_p(u) \leq \left\| I_{S_p} \otimes u : S_p \otimes_{\min} E \to S_p \otimes_{g^{o*}_{p'}} F \right\|.$$

The reverse inequality now follows from the first part. □

Remark 5.9. Both propositions 5.7 and 5.8 remain true if we replace "bounded" with "completely bounded".

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