IDEALS OF EXTENDIBLE LIPSCHITZ MAPS

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Abstract. Given an ideal $\mathcal{A}$ of Lipschitz maps, we study the structure of the collection of all Lipschitz maps $f\colon X \to Y$ between metric spaces X and Y satisfying the following property: not only can f be extended to every metric space containing X, but the extension can be taken to be in the ideal $\mathcal{A}$. We call such maps $\mathcal{A}$-extendible, and show that various properties can be inherited from the class $\mathcal{A}$ to the class of $\mathcal{A}$-extendible maps. These include being an ideal, forming a Banach space and being a dual space. We also prove a characterization of these maps in terms of factorizations through absolute Lipschitz retracts.

Introduction

The extension problem for Lipschitz mappings $f\colon X \to Y$ preserving the Lipschitz constant, either exactly or up to a constant, has been studied by a number of authors. The first results in this matter are due to McShane [23], Lindenstrauss [22] and Jenkins [18]. We refer to the book [3] of Benyamini and Lindenstrauss for a study about this problem.

Our different approach, looking at the spaces of maps that always admit such an extension, is motivated by the study of extendibility of polynomials on Banach spaces, initiated by Kirwan and Ryan in [21] and developed by Carando, Lassalle and Zalduendo in [6, 7, 8, 26]. This extension problem has also been addressed for bilinear and multilinear forms by Galindo, García, Maestre and Mujica [16] and Castillo, García and Jaramillo [9].

A general approach for the case of linear operators was undertaken by Domański in [14]. In several specific important theorems for extension of linear operators, we find that the extension can sometimes be taken to be not only a bounded linear operator but in fact to belong to a particular ideal of linear operators. One first example is the Kwapień-Maurey extension theorem (see [1, Thm. 7.4.4]): if E and F are Banach spaces with, respectively, type 2 and cotype 2, and E_0 is a subspace of E, then any bounded linear map $E_0 \to F$ can be extended to a bounded linear map $E \to F$ that in addition factors through a Hilbert space. A second example is the extension property for 2-summing maps (see [13, Thm. 4.15]): if E, F are Banach spaces and E_0 is a subspace of E, then every 2-summing map $E_0 \to F$ extends to a 2-summing map $E \to F$ (and with the same 2-summing norm).

With this in mind, we consider a similar extension property for Lipschitz maps: we will look at Lipschitz maps $f\colon X \to Y$ that not only can be extended to any metric space containing X, but in fact the extension can be taken to belong to a distinguished ideal $\mathcal{A}$ of Lipschitz maps. Our ideals of Lipschitz maps will be quantitative in nature, every map f in an ideal $\mathcal{A}$ will have an associated constant $\mathcal{A}(f)$ that plays a role similar to a norm. Think, for example, of the Lipschitz constant $\mathrm{Lip}(f)$ for a Lipschitz map f.

In the first section, we look at the basic properties of these spaces of so-called $\mathcal{A}$-extendible maps. We show that the $\mathcal{A}$-extendible maps form themselves an ideal, and each $\mathcal{A}$-extendible map has a "best" $\mathcal{A}$-extendibility constant. Moreover, $\mathcal{A}$-extendible maps can be characterized in terms of a factorization scheme through absolute Lipschitz retracts. In the particular case of Lipschitz extendible maps (that is, when the extension is only required to be Lipschitz), this factorization has the spirit of the Davis–Figiel–Johnson–Pełczyński factorization: any Lipschitz extendible map can be factored through a metric space whose identity map is itself Lipschitz extendible.

The second section is devoted to the case where the maps in $\mathcal{A}$ take values in a Banach space, so that the maps in $\mathcal{A}$ themselves form a Banach space. Once again, this property passes to the $\mathcal{A}$-extendible maps. We give explicit examples, noting that various ideals of Lipschitz maps that have appeared in the literature

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(finite-rank, p-nuclear, p-integral, Pietsch-integral) satisfy extension properties analogous to the extension property for (linear) 2-summing maps mentioned above.

A short, mostly technical third section introduces a restricted version of $\mathcal{A}$-extendibility, where the mappings are only extended to one fixed metric space. While in the second section we consider the case where the maps in $\mathcal{A}$ form themselves a Banach space, in the fourth section we look at the case where they form a dual space. One fundamental example where this happens is the space of Lipschitz maps from a metric space X to a dual Banach space E^* (modulo the constant functions), which was shown by Johnson in [19] to have a canonical predual. This predual can be interpreted as a "tensor product" of sorts between a metric space and a Banach space, or as a vector-valued version of the concept of Lipschitz-free space over a metric space (see [2, 17, 25]). This approach has been studied in generality in [5], and used to find canonical preduals for different ideals of Lipschitz maps taking values in a dual Banach space, for example, Lipschitz p-summing operators [10], Lipschitz operators admitting a Lipschitz factorization through subsets of Hilbert space [11] or Lipschitz Grothendieck-integral operators [4]. The main result of this last section is the fact that when the maps in $\mathcal{A}$ taking values in a dual Banach space have such a tensor-product-like predual, then so does the space of $\mathcal{A}$-extendible maps taking values in the same dual Banach space.

1. Extendible Lipschitz mappings between metric spaces

Let X and Y be metric spaces. We represent by d the distance in any metric space. Let us recall that a mapping $f\colon X \to Y$ is Lipschitz if there exists a constant $\lambda > 0$ such that $d(f(x), f(y)) \le \lambda\, d(x, y)$ for all $x, y \in X$; f will be called λ-Lipschitz if we want to specify the constant. The infimum of such constants λ, that is, the Lipschitz constant of f, will be denoted by $\mathrm{Lip}(f)$. From now on, $\mathrm{Lip}(X, Y)$ stands for the set of all mappings $f\colon X \to Y$ for which $\mathrm{Lip}(f) < \infty$.

Let X and Z be metric spaces. We will say that Z is a metric space containing X if X is isometric to a subset of Z, that is, there exists a mapping $\iota\colon X \to Z$ satisfying $d(\iota(x), \iota(x')) = d(x, x')$ for all $x, x' \in X$. It is said that $\iota\colon X \to Z$ is an isometric embedding. If X and Z are normed spaces, we also will assume that ι is linear.

Definition 1.1. By an ideal of Lipschitz maps $\mathcal{A}$, we mean an assignment, for each pair of metric spaces X and Y, of a family of mappings $\mathcal{A}(X, Y) \subset \mathrm{Lip}(X, Y)$ together with a real-valued function defined on $\mathcal{A}(X, Y)$ that we also denote by $\mathcal{A}$ in an abuse of notation, satisfying the following two properties:

(a) $\mathrm{Lip}(f) \le \mathcal{A}(f)$ for all $f \in \mathcal{A}(X, Y)$.
(b) The ideal property: If $g \in \mathrm{Lip}(U, X)$, $f \in \mathcal{A}(X, Y)$ and $h \in \mathrm{Lip}(Y, V)$, then $hfg \in \mathcal{A}(U, V)$ and $\mathcal{A}(hfg) \le \mathrm{Lip}(h)\mathcal{A}(f)\,\mathrm{Lip}(g)$.

Definition 1.2. Let $\mathcal{A}$ be an ideal of Lipschitz maps, and let X and Y be metric spaces. A Lipschitz mapping $f \in \mathrm{Lip}(X, Y)$ is said to be $\mathcal{A}$-*extendible* if for every metric space Z containing X, there exists an extension of f to a mapping $F \in \mathcal{A}(Z, Y)$. Such an extension will be called an $\mathcal{A}$-extension of f, and we will denote by $\mathcal{A}_e(X, Y)$ the subset of $\mathrm{Lip}(X, Y)$ formed by all $\mathcal{A}$-extendible mappings. In the case of $\mathcal{A} = \mathrm{Lip}$, we will often say Lipschitz extendible instead of Lip-extendible.

Note that, in particular, an $\mathcal{A}$-extendible map must belong to $\mathcal{A}$: simply take $Z = X$ in the definition. The following facts about extensions of Lipschitz mappings can be found in [3]. Let us recall that every metric space X is isometric to a subset of $\ell_\infty(X)$ by the isometric embedding $J_X\colon X \hookrightarrow \ell_\infty(X)$ given by $J_X(x)(y) = d(x, y) - d(x_0, y)$, where x_0 is a fixed point of X. We say that a metric space Y is *Lipschitz* 1-*injective* if for every metric space Z, every subset X of Z and every mapping $f \in \mathrm{Lip}(X, Y)$, there is a mapping $\widetilde{f} \in \mathrm{Lip}(Z, Y)$ extending f with $\mathrm{Lip}(\widetilde{f}) = \mathrm{Lip}(f)$. The real Banach spaces $\ell_\infty(S)$ for a set S and $L_\infty(\mu)$ for a finite measure μ are Lipschitz 1-injective.

Proposition 1.3. *Let X and Y be metric spaces. If $f \in \mathcal{A}_e(X, Y)$ there exists a constant $\lambda > 0$ such that for every metric space Z containing X, there exists $F \in \mathcal{A}(Z, Y)$ which is an extension of f with $\mathcal{A}(F) \le \lambda$.*

Proof. Let $f \in \mathcal{A}_e(X, Y)$. Since J_X isometrically embeds X in $\ell_\infty(X)$, there exists a mapping $F_1 \in \mathcal{A}(\ell_\infty(X), Y)$ extending f. Let Z be a metric space containing X. Since $J_X \in \mathrm{Lip}(X, \ell_\infty(X))$ with $\mathrm{Lip}(J_X) = 1$ and $\ell_\infty(X)$ is Lipschitz 1-injective, J_X can be extended to a mapping $\widetilde{J}_X \in \mathrm{Lip}(Z, \ell_\infty(X))$ with $\mathrm{Lip}(\widetilde{J}_X) = 1$. Then $F = F_1\widetilde{J}_X$ is in $\mathcal{A}(Z, Y)$ with $\mathcal{A}(F) \le \mathcal{A}(F_1)$ and it extends f. Taking $\lambda = \mathcal{A}(F_1)$ concludes the proof. □

Remark 1.4. It follows from the previous proof that if $f \in \mathcal{A}_e(X,Y)$, in the conclusion of Proposition 1.3 we can take λ to be the Lipschitz constant of any $\mathcal{A}$-extension of f through the isometric embedding $J_X : X \hookrightarrow \ell_\infty(X)$.

Definition 1.5. Let X and Y be metric spaces. For each $f \in \mathcal{A}_e(X,Y)$, we define the $\mathcal{A}$-*extendible constant* $\mathcal{A}_e(f)$ of f to be the infimum of all constants $\lambda > 0$ with the property that for every metric space Z containing X, there is an extension $F \in \mathcal{A}(Z,Y)$ of f with $\mathcal{A}(F) \leq \lambda$.

It is not hard to check that if $f \in \mathcal{A}_e(X,Y)$, then

$$\mathcal{A}_e(f) = \sup \{ \inf \{ \mathcal{A}(F) : F \in \mathcal{A}(Z,Y) \text{ extends } f \} : Z \text{ contains } X \},$$

and this supremum is in fact a maximum, achieved at the isometric embedding $J_X : X \hookrightarrow \ell_\infty(X)$.

Let us recall that if X is a subset of Z, then a Lipschitz mapping $r : Z \to X$ is called a *Lipschitz retraction* if it is the identity on X. When such a Lipschitz retraction exists, it is said that X is a *Lipschitz retract* of Z. A metric space X is called an *absolute Lipschitz retract* if it is a Lipschitz retract of every metric space containing it. For each $\lambda \in \mathbb{R}$, a Lipschitz mapping $f : X \to Y$ is said to be λ-Lipschitz if $\mathrm{Lip}(f) \leq \lambda$. λ-Lipschitz retractions, λ-Lipschitz retracts and absolute λ-Lipschitz retracts are defined in a similar way.

There exists a strong connection between Lipschitz retracts and the problem of extension of Lipschitz mappings. For instance, in our terminology, [3, Proposition 1.2 and Remarks] state that given a metric space X and a constant $\lambda \geq 1$, the following conditions are equivalent:

(i) X is an absolute Lipschitz retract (absolute λ-Lipschitz retract).
(ii) For each metric space Y, every Lipschitz mapping $f : Y \to X$ is Lipschitz extendible (to a $\lambda \,\mathrm{Lip}(f)$-Lipschitz mapping).
(iii) For each metric space Y, every Lipschitz mapping $f : X \to Y$ is Lipschitz extendible (to a $\lambda \,\mathrm{Lip}(f)$-Lipschitz mapping).
(iv) The identity map $Id_X : X \to X$ is Lipschitz extendible (with $\mathrm{Lip}_e(Id_X) \leq \lambda$).

It follows from the above discussion that the real Banach spaces $\ell_\infty(S)$ for a set S and $L_\infty(\mu)$ for a finite measure μ are absolute 1-Lipschitz retracts. It is also known by [20] that if K is a compact metric space, then $\mathcal{C}(K)$ is an absolute 2-Lipschitz retract.

We next prove that $\mathcal{A}$-extendible mappings between metric spaces X and Y can be characterized in terms of special factorizations through an absolute λ-Lipschitz retract.

Proposition 1.6. *Let $\mathcal{A}$ be an ideal of Lipschitz maps, X and Y be metric spaces and $\lambda, \lambda_0 \geq 1$. The following are equivalent:*

(i) f is $\mathcal{A}$-extendible, and $\mathcal{A}_e(f) \leq \lambda$.
(ii) f has an $\mathcal{A}$-extension, with $\mathcal{A}$-constant at most λ, through the isometric embedding $J_X : X \hookrightarrow \ell_\infty(X)$.
(iii) f has an $\mathcal{A}$-extension F_0 through some isometric embedding $j : X \hookrightarrow X_0$, where X_0 is an absolute λ_0-Lipschitz retract, and $\lambda_0 \mathcal{A}(F_0) \leq \lambda$.
(iv) f admits a factorization $f = f_2 \circ f_1$ where $f_1 : X \to X_0$ is a Lipschitz map, $f_2 \in \mathcal{A}(X_0, Y)$, X_0 is an absolute λ_0-Lipschitz retract and $\lambda_0 \,\mathrm{Lip}(f_1) \mathcal{A}(f_2) \leq \lambda$.

Proof. Clearly, (i) ⇒ (ii) ⇒ (iii) ⇒ (iv). To prove the remaining implication, suppose that f admits a factorization as in (iv). Let Z be a metric space containing X. Since X_0 is an absolute λ_0-Lipschitz retract, f_1 extends to a Lipschitz mapping $F_1 : Z \to X_0$ with $\mathrm{Lip}(F_1) \leq \lambda_0 \,\mathrm{Lip}(f_1)$. Then $F = f_2 \circ F_1 : Z \to Y$ is an $\mathcal{A}$-extension of f to Z with $\mathcal{A}(F) \leq \mathcal{A}(f_2) \lambda_0 \,\mathrm{Lip}(f_1)$ as desired. □

Remark 1.7. The well-known Davis–Figiel–Johnson–Pełczyński factorization theorem states that weakly compact linear mappings between Banach spaces are precisely those which factor through a reflexive space, that is, a space whose identity map is itself weakly compact. In the case of Lipschitz extendible maps, Proposition 1.6 has exactly the same flavor, since it shows that Lipschitz extendible maps are precisely those which factor through a metric space whose identity map is Lipschitz extendible (that is, an absolute Lipschitz retract).

Remark 1.8. It is also worth noting that (under mild assumptions) Lipschitz extendible maps are a true generalization of an analogous linear concept. To wit, we say that a bounded linear map $u : E \to F$ between Banach spaces is *linearly extendible* if whenever $\iota : E \to G$ is a linear isometric embedding of E into a Banach space G, u admits a bounded linear extension $U : G \to F$. It is clear that such a map is Lipschitz extendible, since it admits a bounded linear (and thus Lipschitz) factorization through the linear isometric embedding $j_E\colon E \to \ell_\infty(B_{E^*})$ given by $e \mapsto (e^*(e))_{e^*\in B_{E^*}}$. Now let us assume that the linear map $u\colon E \to F$ is Lipschitz extendible, and that F is a dual Banach space (or more generally, a Banach space complemented in its bidual). By the Lipschitz extendibility of u, we can find a Lipschitz extension $\hat{u}\colon \ell_\infty(B_{E^*}) \to F$ of u, and then by [3, Thm. 7.2] we obtain a linear extension $U\colon \ell_\infty(B_{E^*}) \to F$ of u. Therefore, u is linearly extendible.

We will use Proposition 1.6 in Section 2.1 to give examples of $\mathcal{A}$-extendible maps, but first we note some of their general properties.

Proposition 1.9. *Let X, Y, U and V be metric spaces.*

(i) Ideal property: If $g \in \mathrm{Lip}(U, X)$, $f \in \mathcal{A}_e(X, Y)$ and $h \in \mathrm{Lip}(Y, V)$, then $hfg \in \mathcal{A}_e(U, V)$ and $\mathcal{A}_e(hfg) \le \mathrm{Lip}(h)\,\mathrm{Lip}_e(f)\,\mathrm{Lip}(g)$.

(ii) Isometric property: If $f \in \mathrm{Lip}(X, Y)$ and $\iota\colon Y \to V$ is a surjective isometry, then $f \in \mathcal{A}_e(X, Y)$ if and only if $\iota f \in \mathcal{A}_e(X, V)$. In this case, $\mathcal{A}_e(\iota f) = \mathcal{A}_e(f)$.

Proof. By Proposition 1.6, f admits a factorization $f = f_2 \circ f_1$ where $f_1\colon X \to X_0$ is a Lipschitz map, $f_2 \in \mathcal{A}(X_0, Y)$, X_0 is an absolute λ_0-Lipschitz retract and $\lambda_0\,\mathrm{Lip}(f_1)\mathcal{A}(f_2) \le \mathcal{A}_e(f)$. Note that $(hf_2)\circ(f_1 g)$ provides the same sort of factorization for hfg, and since

$$\mathcal{A}(hf_2)\lambda_0\,\mathrm{Lip}(f_1 g) \le \mathrm{Lip}(h)\mathcal{A}(f_2)\lambda_0\,\mathrm{Lip}(f_1)\,\mathrm{Lip}(g) \le \mathrm{Lip}(h)\mathcal{A}_e(f)\,\mathrm{Lip}(g)$$

it follows (by Proposition 1.6 again) that $hfg \in \mathcal{A}_e(U, V)$ and $\mathcal{A}_e(hfg) \le \mathrm{Lip}(h)\,\mathrm{Lip}_e(f)\,\mathrm{Lip}(g)$.

(ii) follows easily by using (i). Indeed, if $f \in \mathcal{A}_e(X, Y)$, then $\iota f \in \mathcal{A}_e(X, V)$ and $\mathcal{A}_e(\iota f) \le \mathrm{Lip}(\iota)\mathcal{A}_e(f) = \mathcal{A}_e(f)$ by (i). Conversely, if $\iota f \in \mathcal{A}_e(X, V)$, then $f = \iota^{-1}(\iota f) \in \mathcal{A}_e(X, Y)$ and $\mathcal{A}_e(f) = \mathcal{A}_e(\iota^{-1}(\iota f)) \le \mathrm{Lip}(\iota^{-1})\mathcal{A}_e(\iota f) = \mathcal{A}_e(\iota f)$ by (i). □

2. Extendible Banach-valued Lipschitz ideals

For many ideals of Lipschitz maps $\mathcal{A}$ of interest, the Banach-valued case has a richer structure. It is often the case that when X is a metric space and E is a Banach space, $\mathcal{A}(X, E)$ has a natural Banach space structure itself. In this situation, the set of $\mathcal{A}$-extendible mappings will itself have a natural structure of a Banach space with the $\mathcal{A}$-extendibility constant as norm.

Let us recall some concepts and results that will be needed in this section. A pointed metric space X is a metric space with a designated special point or base point, denoted always by 0. If X is a normed space, the base point will be its origin. Given pointed metric spaces X and Y, let us denote by $\mathrm{Lip}_0(X, Y)$ the set of all Lipschitz mappings f from X into Y such that $f(0) = 0$. If E is a real Banach space then $\mathrm{Lip}_0(X, E)$ is a Banach space norm given by the Lipschitz constant Lip. The elements of $\mathrm{Lip}_0(X, E)$ are also referred to as Lipschitz operators. The space $\mathrm{Lip}_0(X, \mathbb{R})$ is known as the *Lipschitz dual* of X and denoted frequently by $X^\#$.

Given Banach spaces E and F, we denote by $\mathcal{L}(E, F)$ the Banach space of all bounded linear operators from E into F with the usual norm. As usual, B_E, E^* and κ_E will denote the closed unit ball of E, the topological dual of E and the canonical isometric embedding from E into E^{**}, respectively.

Given a pointed metric space X and a Banach space E, for any $g \in X^\#$ and $e \in E$, we denote by $g \cdot e$ the mapping from X to E defined by $(g\cdot e)(x) = g(x)e$ for all $x \in X$. It is easy to prove that $g \cdot e \in \mathrm{Lip}_0(X, E)$ and $\mathrm{Lip}(g \cdot e) = \mathrm{Lip}(g)\,\|e\|$.

Definition 2.1. By a Banach ideal of Lipschitz maps $\mathcal{A}$, we mean an assignment, for each metric space X and each Banach space E, of a linear subspace $\mathcal{A}(X, E) \subset \mathrm{Lip}_0(X, E)$ together with a real-valued function defined on $\mathcal{A}(X, E)$ that we also denote by $\mathcal{A}$ in an abuse of notation, satisfying the following properties:

(i) $\mathrm{Lip}(f) \le \mathcal{A}(f)$ for all $f \in \mathcal{A}(X, E)$.
(ii) $\mathcal{A}$ is a Banach space norm on $\mathcal{A}(X, E)$.
(iii) If $g \in X^\#$ and $e \in E$, then $g \cdot e \in \mathcal{A}(X, E)$ with $\mathcal{A}(g \cdot e) = \mathrm{Lip}(g)\,\|e\|$.
(iv) If Y is a pointed metric space, F is a Banach space, $h \in \mathrm{Lip}_0(Y, X)$, $f \in \mathcal{A}(X, E)$ and $T \in \mathcal{L}(E, F)$, then $Tfh \in \mathcal{A}(Y, F)$ and $\mathcal{A}(Tfh) \le \|T\|\,\mathrm{Lip}_e(f)\,\mathrm{Lip}(h)$.

Remark 2.2. Even though our definition of $\mathcal{A}$-extendible mappings (Definition 1.2) was given for a full ideal of Lipschitz maps, it is clear that we can consider a corresponding notion for a Banach ideal of Lipschitz maps (and also use the symbol $\mathcal{A}_e$ to denote them). As mentioned at the beginning of the section, it turns out that in this case those will themselves form a Banach ideal of Lipschitz maps. Before proceeding, let us note that all the content of Section 1 can effortlessly be adapted to the context of Banach ideals of Lipschitz maps. The only adjustment needed would be in the ideal property of Proposition 1.9, where the Lipschitz map h has to be replaced by a bounded linear map between Banach spaces.

Theorem 2.3. *If $\mathcal{A}$ a Banach ideal of Lipschitz maps, then so is $\mathcal{A}_e$.*

Proof. Note that since the maps in $\mathcal{A}$ are required to be in Lip_0, then the $\mathcal{A}_e$-mappings automatically are in Lip_0 as well.

(i) The very definition of $\mathcal{A}_e(f)$ ensures that $\mathrm{Lip}(f) \le \mathcal{A}_e(f)$.

(ii) Let $f, g \in \mathcal{A}_e(X,E)$ and $\alpha \in \mathbb{K}$. Clearly, $\mathcal{A}_e(f) \ge 0$ and $\mathcal{A}_e(\mathbf{0}) = 0$ where $\mathbf{0}$ denotes the function constantly equal to $0 \in E$ on X. Moreover, $f = \mathbf{0}$ if $\mathcal{A}_e(f) = 0$ by using (i). Let Z be a metric space containing X. Then there are extensions $F, G \in \mathcal{A}(Z,E)$ of f and g, respectively, with $\mathcal{A}(F) \le \mathcal{A}_e(f)$ and $\mathcal{A}(G) \le \mathcal{A}_e(g)$. Clearly, $F+G$ and αF are $\mathcal{A}$-extensions to Z of $f+g$ and αf, respectively. Therefore $f+g, \alpha f \in \mathcal{A}_e(X,E)$. Since $\mathcal{A}(F+G) \le \mathcal{A}(F) + \mathcal{A}(G) \le \mathcal{A}_e(f) + \mathcal{A}_e(g)$, it follows that $\mathcal{A}_e(f+g) \le \mathcal{A}_e(f) + \mathcal{A}_e(g)$. On the other hand, $\mathcal{A}(\alpha F) = |\alpha|\, \mathcal{A}(F) \le |\alpha|\, \mathcal{A}_e(f)$ and therefore $\mathcal{A}_e(\alpha f) \le |\alpha|\, \mathcal{A}_e(f)$. Conversely, if $\alpha \ne 0$, this last inequality gives $\mathcal{A}_e(f) \le |\alpha|^{-1} \mathcal{A}_e(\alpha f)$ and so $|\alpha|\, \mathcal{A}_e(f) \le \mathcal{A}_e(\alpha f)$.

We have just proved that $\mathcal{A}_e$ is a norm on $\mathcal{A}_e(X,E)$. To prove that it is complete, it suffices to check that every absolutely convergent series is convergent. Accordingly, let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}_e(X,E)$ such that $\sum_{n=1}^{\infty} \mathcal{A}_e(f_n) < \infty$. Then, by (i), $\sum_{n\in\mathbb{N}} f_n$ converges in $\mathrm{Lip}_0(X,E)$ to a Lipschitz operator $f \in \mathrm{Lip}_0(X,E)$. Let Z be a metric space containing X. For each $n \in \mathbb{N}$, find an extension $F_n \in \mathcal{A}(Z,E)$ of f_n, with $\mathcal{A}(F_n) \le \mathcal{A}_e(f_n)$. This implies that the series $\sum_{n\in\mathbb{N}} F_n$ converges in $\mathcal{A}(Z,E)$ to a map $F \in \mathcal{A}(Z,E)$. Furthermore, $\mathcal{A}(F) \le \sum_{n=1}^{\infty} \mathcal{A}_e(f_n)$. Clearly, F is an $\mathcal{A}$-extension of f, and hence $f \in \mathcal{A}_e(X,E)$ with $\mathcal{A}_e(f) \le \sum_{n=1}^{\infty} \mathcal{A}_e(f_n)$. Applying the same reasoning to $f - \sum_{n=1}^{k} f_n$, we have $\mathcal{A}_e(f - \sum_{n=1}^{k} f_n) \le \sum_{n=k+1}^{\infty} \mathcal{A}_e(f_n)$, and so $\sum_{n=1}^{\infty} f_n$ converges to f in $\mathcal{A}_e(X,E)$.

(iii) Let $g \in X^{\#}$ and $e \in E$. Let Z be a metric space containing X. Since $\mathbb{R}$ is Lipschitz 1-injective, there exists $\widetilde{g} \in Z^{\#}$ extending g such that $\mathrm{Lip}(\widetilde{g}) = \mathrm{Lip}(g)$. Note that $\widetilde{g} \cdot e$ is in $\mathcal{A}(Z,E)$ with $\mathcal{A}(\widetilde{g} \cdot e) = \mathrm{Lip}(\widetilde{g})\, \|e\|$, and it clearly extends $g \cdot e$. Hence $g \cdot e \in \mathcal{A}_e(X,E)$ and $\mathcal{A}_e(g \cdot e) \le \mathrm{Lip}(g)\, \|e\|$. Moreover, $\mathrm{Lip}(g)\, \|e\| \le \mathcal{A}_e(g \cdot e)$ by (i).

(iv) It follows immediately from the corresponding property for $\mathcal{A}$ and the Banach-valued version of Proposition 1.9. □

2.1. **Examples.** Several known ideals of Lipschitz maps can easily be shown to have extendibility properties. For example, recall that a Lipschitz operator $f \in \mathrm{Lip}_0(X,E)$ is said to have *finite rank* if the linear subspace of E generated by $f(X)$ is finite dimensional. The set of all Lipschitz finite-rank operators from X into E is the linear subspace of $\mathrm{Lip}_0(X,E)$ spanned by the Lipschitz operators $g \cdot e$ with $g \in X^{\#}$ and $e \in E$ and it has the ideal property (see [4, Proposition 1.1]). Since $\mathbb{R}$ is an absolute Lipschitz retract, we obtain the following.

Proposition 2.4. *Let X and Z be pointed metric spaces with $X \subset Z$ and let E be a Banach space. Then every Lipschitz finite-rank operator $f : X \to E$ admits a Lipschitz finite-rank extension $F : Z \to E$.*

A mapping $f \in \mathrm{Lip}_0(X,E)$ is called a *Lipschitz nuclear operator* [4, Definition 4] if there exist bounded sequences $\{g_n\}_{n\in\mathbb{N}}$ in $X^{\#}$ and $\{e_n\}_{n\in\mathbb{N}}$ in E such that $f(x) = \sum_{n=1}^{\infty} g_n(x) e_n$ for all $x \in X$, satisfying $\sum_{n=1}^{\infty} \mathrm{Lip}(g_n)\, \|e_n\| < \infty$. The Lipschitz nuclear norm of f is

$$\mathrm{Lip}_N(f) = \inf \sum_{n=1}^{\infty} \mathrm{Lip}(g_n)\, \|e_n\|,$$

the infimum being taken over all such representations of f. It is straightforward to see that every Lipschitz nuclear operator f from X into E is Lipschitz extendible and $\mathrm{Lip}_e(f) \le \mathrm{Lip}_N(f)$; in fact, since every one of the $g_n \in X^{\#}$ can be extended to an element of $Z^{\#}$ with the same Lipschitz constant, we have the following.

Proposition 2.5. *Let X and Z be pointed metric spaces with $X \subset Z$ and let E be a Banach space. Then every Lipschitz nuclear operator $f : X \to E$ is Lip_N-extendible, and $(\mathrm{Lip}_N)_e(f) \le \mathrm{Lip}_N(f)$.*

As pointed out in Remark 1.7, Proposition 1.6 easily allows us to give examples of Lipschitz extendible mappings between metric spaces, namely any Lipschitz map factorizing through an absolute Lipschitz retract (for instance, the real spaces $\ell_\infty(S)$ for any set S or $L_\infty(\mu)$ for a finite measure μ). In particular, the following known families of Lipschitz mappings are Lipschitz extendible:

(a) The Lipschitz p-nuclear operators with $p \in [1, \infty)$ from a metric space to a Banach space and the Lipschitz 2-dominated operators between Banach spaces from [12].
(b) The Lipschitz p-integral operators with $p \in [1, \infty)$ from a metric space to a dual Banach space from [15].
(c) The Lipschitz Pietsch-integral operators from a metric space to a Banach space from [4].

In fact, just as in Proposition 2.5 above, in all these cases a finer result holds: a map in one of these families $\mathcal{A}$ is in fact $\mathcal{A}$-extendible. To see why, let us show the details for one case. Let us recall that a Lipschitz operator $f \in \mathrm{Lip}_0(X, E)$ is called a *Lipschitz Pietsch-integral* (*P-integral* for short) operator if there exist a probability measure μ, a bounded linear operator $A \in \mathcal{L}(L_1(\mu), E)$ and a Lipschitz operator $b \in \mathrm{Lip}_0(X, L_\infty(\mu))$ giving rise to the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & E \\ {\scriptstyle b}\downarrow & & \uparrow{\scriptstyle A} \\ L_\infty(\mu) & \xrightarrow{I_{\infty,1}} & L_1(\mu) \end{array}$$

where $I_{\infty,1}\colon L_\infty(\mu) \to L_1(\mu)$ is the formal inclusion operator. The triple (A, b, μ) is called a *Lipschitz P-integral factorization* of f. The Lipschitz P-integral norm of f is defined to be $\mathrm{Lip}_{PI}(f) = \inf \|A\| \, \mathrm{Lip}(b)$, where the infimum is extended over all Lipschitz P-integral factorizations of f. This definition together with Proposition 1.6 immediately imply the following.

Proposition 2.6. *Let X and Z be pointed metric spaces with $X \subset Z$ and let E be a Banach space. Then every Lipschitz P-integral operator $f\colon X \to E$ can be extended to a Lipschitz P-integral operator $F\colon Z \to E$, with $\mathrm{Lip}_{PI}(F) \le \mathrm{Lip}_{PI}(f)$. As a consequence, $\mathrm{Lip}_e(f) \le \mathrm{Lip}_{PI}(f)$.*

Since the Lipschitz p-nuclear and Lipschitz p-integral operators are defined via similar factorization schemes through absolute Lipschitz retracts, we have the following analogous results.

Proposition 2.7. *Let X, Z be metric spaces with Z containing X, and E a Banach space. In the following cases, a mapping $f \in \mathcal{A}(X, E)$ can be extended to $F \in \mathcal{A}(Z, E)$ with $\mathcal{A}(F) \le \mathcal{A}(f)$. In particular, $\mathrm{Lip}_e(f) \le \mathcal{A}(f)$.*

(a) $\mathcal{A}$ = the Lipschitz p-nuclear operators with $p \in [1, \infty)$ from [12].
(b) $\mathcal{A}$ = the Lipschitz p-integral operators with $p \in [1, \infty)$ from [15], if in addition E is a dual Banach space.

3. Extendibility to a fixed space

We now turn our attention to a restricted notion of $\mathcal{A}$-extendibility similar in spirit to [6, Section 2]: extendibility to a fixed space. It is a small adaptation of Definitions 1.2 and 1.5.

Definition 3.1. Let X, Y and Z be metric spaces with $X \subset Z$, and $\mathcal{A}$ an ideal of Lipschitz maps. A Lipschitz map $f\colon X \to Y$ is said to be $(\mathcal{A}, Z)$-*extendible if there exists an extension of f to a mapping $F \in \mathcal{A}(Z, Y)$. We denote by $\mathcal{A}_{e,Z}(X, Y)$ the subset of* $\mathrm{Lip}(X, Y)$ *formed by all $(\mathcal{A}, Z)$-extendible mappings. For $f \in \mathcal{A}_{e,Z}(X, Y)$, we denote*

$$\mathcal{A}_{e,Z}(f) = \inf\{\mathcal{A}(F) \colon F\colon Z \to Y \text{ is an } \mathcal{A}\text{-extension of } f\}.$$

Analogous definition and notation can be considered for Banach ideals of Lipschitz maps. Similarly to $\mathcal{A}$-extendible maps, the $(\mathcal{A}, Z)$-extendible ones also satisfy nice properties in the spirit of Proposition 1.9. The proof is essentially the same, so it is omitted.

Proposition 3.2. *Let X, Y and V be metric spaces.*

(i) Ideal property: If $f \in \mathcal{A}_{e,Z}(X, Y)$ and $h \in \mathrm{Lip}(Y, V)$, then $hf \in \mathcal{A}_{e,Z}(X, V)$ and $\mathcal{A}_{e,Z}(hf) \le \mathrm{Lip}(h)\mathcal{A}_{e,Z}(f)$.
(ii) Isometric property: If $f \in \mathcal{A}_{e,Z}(X, Y)$ and $\iota\colon Y \to V$ is a surjective isometry, then $f \in \mathcal{A}_{e,Z}(X, Y)$ if and only if $\iota f \in \mathcal{A}_{e,Z}(X, V)$. In this case, $\mathcal{A}_{e,Z}(\iota f) = \mathcal{A}_{e,Z}(f)$.

Moreover, when $\mathcal{A}$ is a Banach ideal of Lipschitz maps, then the Lipschitz $(\mathcal{A}, Z)$-extendible maps taking values in a fixed Banach space form a Banach space. We omit the proof since it is essentially the same as that for Theorem 2.3.

Proposition 3.3. *Let $\mathcal{A}$ be a Banach ideal of Lipschitz maps, X and Z pointed metric spaces with Z containing X, and E a Banach space. The set $\mathcal{A}_{e,Z}(X,E)$ is a linear subspace of $\mathrm{Lip}_0(X,E)$ satisfying the following properties:*

(i) $\mathrm{Lip}(f) \le \mathcal{A}_{e,Z}(f)$ *for all* $f \in \mathcal{A}_{e,Z}(X,E)$.
(ii) $\mathcal{A}_{e,Z}$ *is a Banach space norm on* $\mathcal{A}_{e,Z}(X,E)$.
(iii) *If* $g \in X^{\#}$ *and* $e \in E$, *then* $g \cdot e \in \mathrm{Lip}_{0e,Z}(X,E)$ *with* $\mathcal{A}_{e,Z}(g\cdot e) = \mathrm{Lip}(g)\,\|e\|$.
(iv) *If F is a Banach space,* $f \in \mathcal{A}_{e,Z}(X,E)$ *and* $T \in \mathcal{L}(E,F)$, *then* $Tf \in \mathcal{A}_{e,Z}(X,F)$ *and* $\mathcal{A}_{e,Z}(Tf) \le \|T\|\,\mathcal{A}_{e,Z}(f)$.

4. Duality for $\mathcal{A}$-extendible operators

We can paraphrase the results from Section 2 as saying that whenever $\mathcal{A}(X,E)$ is a Banach space, so is $\mathcal{A}_e(X,E)$ (where X is a pointed metric space and E is a Banach space). In several interesting cases, it is furthermore known that $\mathcal{A}(X,E^*)$ is a dual space: the basic example is that of $\mathrm{Lip}_0(X,E^*)$ [19, Theorem 4.1], and various other examples are known for other ideals of Lipschitz maps: Lipschitz p-summing maps [10], maps admitting a Lipschitz factorization through a subset of a Hilbert space [11], Lipschitz Grothendieck-integral maps [4]. The common technique in these cases has been to use an object playing the role of a sort of "tensor product" between a pointed metric space and a Banach space, and that is precisely what we will use to achieve our next goal: to show that if $\mathcal{A}(X,E^*)$ has always a canonical predual presented as one such "tensor product", then so does $\mathcal{A}_e(X,E^*)$. In particular, we obtain that the space $\mathrm{Lip}_{0e}(X,E^*)$ is a dual space.

The following concepts and facts can be found in [5]. The *Lipschitz tensor product* $X \boxtimes E$ of a pointed metric space X and a Banach space E is the linear span of all linear functionals $\delta_{(x,y)} \boxtimes e$ on $\mathrm{Lip}_0(X,E^*)$ of the form

$$\left(\delta_{(x,y)} \boxtimes e\right)(f) = \langle f(x) - f(y), e\rangle$$

for $(x,y) \in X^2$ and $e \in E$. A norm α on $X \boxtimes E$ is called a *Lipschitz cross-norm* if

$$\alpha\left(\delta_{(x,y)} \boxtimes e\right) = d(x,y)\,\|e\|$$

for all $(x,y) \in X^2$ and $e \in E$. We denote by $X \boxtimes_\alpha E$ the linear space $X \boxtimes E$ equipped with the norm α, and by $X \widehat{\boxtimes}_\alpha E$ the completion of $X \boxtimes_\alpha E$. For each $u \in X \boxtimes E$, the *Lipschitz projective* norm π is defined on $X \boxtimes E$ as

$$\pi(u) = \inf\left\{\sum_{i=1}^n d(x_i,y_i)\,\|e_i\| : u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i\right\},$$

where the infimum is taken over all such representations of u. Sometimes, as in the case of π, we will use the same symbol to denote a family of norms defined on different Lipschitz tensor products. If we need to clarify exactly what spaces are being considered at a particular moment, we will write them as subindices: for example, $\pi_{X,E}$ for the Lipschitz projective norm on $X \boxtimes E$. Note that, using the triangle inequality, if α is a Lipschitz cross-norm on $X \boxtimes E$, then $\alpha \le \pi_{X,E}$.

Given $h \in \mathrm{Lip}_0(X,Y)$ and $T \in \mathcal{L}(E,F)$, define the linear operator $h \boxtimes T\colon X \boxtimes E \to Y \boxtimes F$ by

$$(h \boxtimes T)\left(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i\right) = \sum_{i=1}^n \delta_{(h(x_i),h(y_i))} \boxtimes T(e_i).$$

The mapping $f \mapsto \Lambda_{X,E}(f)$ from $\mathrm{Lip}_0(X,E^*)$ into $(X \boxtimes_\pi E)^*$, defined by

$$\Lambda_{X,E}(f)(u) = \sum_{i=1}^n \langle f(x_i) - f(y_i), e_i\rangle$$

for $f \in \mathrm{Lip}_0(X,E^*)$ and $u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes_\pi E$, is an isometric isomorphism. More generally, we will say that a Lipschitz cross-norm α on $X \boxtimes E$ (or the space $X \boxtimes_\alpha E$) is *in canonical duality* with a space of Lipschitz maps $\mathcal{A}(X,E^*) \subset \mathrm{Lip}_0(X,E^*)$ if the map $\Lambda_{X,E}$ is an isometric isomorphism between $\mathcal{A}(X,E^*)$ and $(X \boxtimes_\alpha E)^*$. If we only say that α is in canonical duality with $\mathcal{A}$ without mentioning specific spaces, we mean that the canonical duality holds for any choice of spaces.

We start by studying the duality for $(\mathcal{A}, Z)$-extendible maps. Specifically, we show that if a Lipschitz cross-norm on $Z \boxtimes E$ is in canonical duality with $\mathcal{A}(Z, E^*)$, then we can find a Lipschitz cross-norm in canonical duality with $\mathcal{A}_{e,Z}(X, E^*)$ for any X contained in Z. The approach is similar to [21, 6].

Definition 4.1. Let X, Z be pointed metric spaces, $\iota\colon X \to Z$ a base-point preserving isometric embedding, E a Banach space and $\mathcal{A}$ a Banach ideal of Lipschitz maps. Suppose that α is a Lipschitz cross-norm on $Z \boxtimes E$ in canonical duality with $\mathcal{A}(Z, E^*)$. For every $u \in X \boxtimes E$, define

$$\eta_{\alpha,Z}(u) = \alpha_{Z,E}\big((\iota \boxtimes id_E)(u)\big).$$

In other words, the norm $\eta_{\alpha,Z}$ on $X \boxtimes E$ is the norm induced by the (algebraic) embedding

$$\iota \boxtimes id_E\colon X \boxtimes E \to Z \boxtimes_\alpha E.$$

It is quite clear from the definition that $\eta_{\alpha,Z}$ is a Lipschitz cross-norm on $X \boxtimes E$ (because so is α on $Z \boxtimes E$).

The basic duality result is the following:

Theorem 4.2. *Let X, Z be pointed metric spaces, $\iota\colon X \to Z$ a base-point preserving isometric embedding, E a Banach space, and $\mathcal{A}$ a Banach ideal of Lipschitz maps. Suppose that α is a Lipschitz cross-norm on $Z \boxtimes E$ in canonical duality with $\mathcal{A}(Z, E^*)$. Then $X \boxtimes_{\eta_{\alpha,Z}} E$ is in canonical duality with $\mathcal{A}_{e,Z}(X, E^*)$*

Proof. Let $f \in \mathcal{A}_{e,Z}(X, E^*)$, and consider $u \in X \boxtimes E$. By definition of $(\mathcal{A}, Z)$-extendibility, there exists $F \in \mathcal{A}(Z, E^*)$ extending f. Note that since F extends f, then $\Lambda_{X,E}(f)(u) = \Lambda_{Z,E}(F)\big((\iota \boxtimes id_E)(u)\big)$. By assumption, $(Z \boxtimes_\alpha E)^*$ is isometrically isomorphic to $\mathcal{A}(Z, E^*)$ via the canonical identification $\Lambda_{Z,E}$, so it follows that

$$|\Lambda_{X,E}(f)(u)| = |\Lambda_{Z,E}(F)\left((\iota \boxtimes id_E)(u)\right)| \leq \mathcal{A}(F)\alpha_{Z,E}((\iota \boxtimes id_E)(u)) = \mathcal{A}(F)\eta_{\alpha,Z}(u).$$

Taking the infimum over all such extensions F, we arrive at $|\Lambda(f)(u)| \leq \mathcal{A}_{e,Z}(f)\eta_{\alpha,Z}(u)$ and thus $\Lambda_{X,E}(f)$ belongs to $(X \boxtimes_{\eta_{\alpha,Z}} E)^*$ and $\|\Lambda_{X,E}(f)\| \leq \mathcal{A}_{e,Z}(f)$.

Let $\varphi \in (X \boxtimes_{\eta_{\alpha,Z}} E)^*$ be a bounded linear functional. Since $\eta_{\alpha,Z} \leq \pi$, it follows that $\varphi \in (X \boxtimes_\pi E)^*$ and thus $\varphi = \Lambda_{X,E}(f)$ for some $f \in \mathrm{Lip}_0(X, E^*)$. By the Hahn–Banach theorem, seeing $X \boxtimes_{\eta_{\alpha,Z}} E$ as a subspace of $Z \boxtimes_\alpha E$, φ extends to a bounded linear functional $\Phi \in (Z \boxtimes_\alpha E)^*$ of norm $\|\varphi\|$, which in turn can be written as $\Phi = \Lambda_{Z,E}(F)$ for a map $F \in \mathcal{A}(Z, E^*)$ with $\mathcal{A}(F) = \|\Phi\|$. Note that F extends f since

$$\langle F(x), e\rangle = \Phi(\delta_{(x,0)} \boxtimes e) = \varphi(\delta_{(x,0)} \boxtimes e) = \langle f(x), e\rangle\,,$$

for any $x \in X$ and $e \in E$. Thus, $f \in \mathcal{A}_{e,Z}(X, E^*)$ and $\mathcal{A}_{e,Z}(f) \leq \|\Lambda_{X,E}(f)\|$. This concludes the proof. □

The duality result for $\mathcal{A}_e(X, E^*)$ now follows easily.

Corollary 4.3. *Let X be a pointed metric space, E a Banach space, and α a Lipschitz cross-norm on $X \boxtimes E$ in canonical duality with a Banach ideal of Lipschitz maps $\mathcal{A}(X, E^*)$. Then $X \boxtimes_\eta E$ is in canonical duality with $\mathcal{A}_e(X, E^*)$, where $\eta\colon = \eta_{\alpha,\ell_\infty(X)}$.*

Proof. The equivalence between (i) and (ii) in Theorem 1.6 shows that $\mathcal{A}_e(X, E^*)$ and $\mathcal{A}_{e,\ell_\infty(X)}(X, E^*)$ are isometrically isomorphic. The desired result then follows immediately from Theorem 4.2. □

Remark 4.4. Though it is convenient to use the specific embedding $J_X\colon X \to \ell_\infty(X)$ in Corollary 4.3, it is by no means necessary. Except for minor adjustments, essentially the same proof as that of Theorem 4.2 shows directly that under the hypotheses of Corollary 4.3, the Lipschitz cross-norm on $X \boxtimes E$,

$$\eta(u) = \inf\big\{\alpha_{Z,E}\big((\iota \boxtimes id_E)(u)\big) \mid \iota\colon X \to Z \text{ is a base-point preserving isometric embedding}\big\},$$

is in canonical duality with $\mathcal{A}_e(X, E^*)$.

Let us single out an important case of Corollary 4.3, namely the case of Lipschitz extendible maps.

Corollary 4.5. *Let X be a pointed metric space and E a Banach space. Then $X \boxtimes_\eta E$ is in canonical duality with $\mathrm{Lip}_{0e}(X, E^*)$, where $\eta\colon = \eta_{\pi,\ell_\infty(X)}$.*

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